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POSITIVELY HOMOGENEOUS EQUATIONS IN THE PLANE

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Abstract. We prove the multiplicity of periodic solutions to second order ordinary differential equations in $\mathbb{R}^2$ with nonlinearities crossing the two first eigenvalues of the differential operator.

1. **Introduction.** We look for periodic solutions of the following system in $\mathbb{R}^2$:

$$u'' - Au + \mu u^+ - \nu u^- = v + h(t) . \tag{1}$$

Here, $h : \mathbb{R} \to \mathbb{R}^2$ is a continuous T−periodic function, $v = (v_1, v_2) \in \mathbb{R}^2$ is a vector to be fixed later, μ and ν are real numbers, and A is a 2×2 matrix. Given $u = (u_1, u_2) \in \mathbb{R}^2$, we use the notation $u^+ = (u_1^+, u_2^+)$ and $u^- = (u_1^-, u_2^-)$, where u_1^+ and u_1^- denote the positive and negative part of u_1, respectively, and similarly for u_2.

This equation can be seen as a natural extension to two degrees of freedom of the scalar equation modeling an asymmetric oscillator (just assume u, v and h are scalar and $A = 0$). There is a large literature on this scalar case studying the associated boundary value problems and the dynamics (see [3, 6, 9, 2, 5, 10] and the references therein). Equations like (1) in more than one degree of freedom appear after discretization in space of certain partial differential equations (see [9] in this context), and it is also possible to create mechanical machines with several springs and stops (as in [7]) which are modeled by these equations. In view of the many phenomena that appear in the scalar equation, we think that also the study of the vector-valued case can be of interest. In this paper we analyze the equation in the plane, using only elementary tools. First we discuss the existence of equilibria of the autonomous equation (i.e. $h = 0$), and we are led to the study of

$$F(u) := -Au + \mu u^+ - \nu u^- = v . \tag{2}$$

This is a piecewise linear equation in the plane and we study the number of (constant) solutions depending on the position of the eigenvalues of A with respect to μ and ν. Our approach is related in some sense to [11]. From the study of equation

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(2) and a perturbation argument we obtain some results on the number of periodic solutions of (1) when v has a sufficiently large norm.

In order to illustrate our result, let A be symmetric, $\sigma(A) = \{\lambda_1, \lambda_2\}$, i.e. λ_1 and λ_2 be the two (possibly equal) real eigenvalues of A, and assume

$$\nu < \lambda_1 \leq \lambda_2 < \mu < \lambda_1 + \left(\frac{2\pi}{T}\right)^2 . \tag{3}$$

This is a condition of the type "jumping of eigenvalues". Indeed, if $A = 0$, in (1) we have two uncoupled equations, and with assumption (3) we are in the classical Ambrosetti-Prodi situation [1], leading, for $v_1 > 0$ and $v_2 > 0$ sufficiently large, to the existence of two T-periodic solutions for each equation, and hence to four T-periodic solutions to equation (1). We will show that, for a general symmetric matrix A, there is an open cone $\overset{\circ}{S}$ contained in the first quadrant of $\mathbb{R}^2$, such that, if v belongs to $\overset{\circ}{S}$ and has sufficiently large norm, equation (1) has exactly four T-periodic solutions.

Our results can be extended to a large class of boundary value problems. On the other hand, it is not obvious how to generalize them to systems in $\mathbb{R}^N$, when the nonlinearity crosses at least N eigenvalues of the differential operator $Lu = u'' - Au$, with periodic boundary conditions. One would like to conjecture the existence of at least 2^N periodic solutions. On the other hand, one can see analogies with a conjecture by Lazer and McKenna [8], related to Dirichlet problems, which was disproved by Dancer [4], showing that, in some cases, one can not expect more than four solutions. We do not know the answer.

2. A piecewise linear operator in $\mathbb{R}^2$.

Consider four nonsingular 2×2 matrices

$$B_j = \begin{pmatrix} b_{11}^{(j)} & b_{12}^{(j)} \\ b_{21}^{(j)} & b_{22}^{(j)} \end{pmatrix} ,$$

with $j = 1, ..., 4$. We make the following

Assumption H. *The matrices B_j are such that a continuous function $F : \mathbb{R}^2 \to \mathbb{R}^2$ is well defined by*

$$F(u) = B_j u , \quad \textit{whenever} \quad u \in Q_j , \tag{4}$$

for $j = 1, ..., 4$ (here, Q_j is the closed j-th quadrant of $\mathbb{R}^2$).

We look for solutions of the equation

$$Fu = v ,$$

where $v \in \mathbb{R}^2$. To this aim, we want to have some information on the four cones $F(Q_j)$. Being the matrices B_j nonsingular, these cones are convex, and they can neither be a half-plane nor reduce to a half-line.

For convenience of notations, for any integer k we define

$$Q_k = Q_{k+4} .$$

Proposition 2.1. *Let Assumption H hold and assume that*

$$(-1)^{j-1} \det B_j > 0 , \tag{5}$$

for every $j = 1, ..., 4$. Then, one of the following four situations must occur:

$$(S_k) \qquad F(Q_k) = F(Q_{k+1}) \cap F(Q_{k+3}) \subset F(Q_{k+1}) \cup F(Q_{k+3}) = F(Q_{k+2}) ,$$

for some $k = 1, ..., 4$.

Proof: Consider the curve obtained as the image of the unit circle through the piecewise linear operator F; set

$$F(\cos t, \sin t) = (\rho(t)\cos\theta(t), \rho(t)\sin\theta(t)) .$$

Since $F^{-1}(0) = \{0\}$, for every t one has $\rho(t) > 0$ and, either $\cos\theta(t) \neq 0$, or $\sin\theta(t) \neq 0$. Let t be such that $(\cos t, \sin t) \in Q_j$, for some j, and assume for instance that $\cos\theta(t) \neq 0$. Being

$$\rho(t)\cos\theta(t) = b_{11}^{(j)}\cos t + b_{12}^{(j)}\sin t , \quad \rho(t)\sin\theta(t) = b_{21}^{(j)}\cos t + b_{22}^{(j)}\sin t ,$$

one has that

$$\tan\theta(t) = \frac{b_{21}^{(j)}\cos t + b_{22}^{(j)}\sin t}{b_{11}^{(j)}\cos t + b_{12}^{(j)}\sin t} .$$

Differentiating both sides,

$$\frac{\theta'(t)}{(\cos\theta(t))^2} = \frac{\det B_j}{(b_{11}^{(j)}\cos t + b_{12}^{(j)}\sin t)^2} ,$$

so that

$$\theta'(t) \begin{cases} > 0 & \text{if } t \in \left]0, \frac{\pi}{2}\right[\cup \left]\pi, \frac{3\pi}{2}\right[, \\ < 0 & \text{if } t \in \left]\frac{\pi}{2}, \pi\right[\cup \left]\frac{3\pi}{2}, 2\pi\right[. \end{cases}$$

The same conclusion is reached if $\sin\theta(t) \neq 0$, considering $\cot\theta(t)$ instead of $\tan\theta(t)$. Set $\theta_k = \theta(k\pi/2)$, for $k = 0, ..., 4$. We thus have

$$\theta_0 < \theta_1 , \quad \theta_2 < \theta_1 , \quad \theta_2 < \theta_3 , \quad \theta_4 < \theta_3 .$$

Just from the definition of $\theta(t)$, we know that, for every $t \in \mathbb{R}$,

$$\theta(t + 2\pi) = \theta(t) + 2N\pi ,$$

for some integer N (the degree of F). We are going to prove that $N = 0$. From the convexity of the regions $F(Q_i)$ we have

$$|\theta_{j+1} - \theta_j| < \pi \qquad (j = 0, ..., 4) .$$

Hence, being $2N\pi = \theta_4 - \theta_0$,

$$-2\pi < (\theta_4 - \theta_3) + (\theta_2 - \theta_1) < 2N\pi < (\theta_3 - \theta_2) + (\theta_1 - \theta_0) < 2\pi ,$$

so that N must be zero. The proof is concluded by the following elementary fact.

Proposition 2.2. *Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and 2π–periodic, and let*

$$\tau_0 < \tau_1 < \tau_2 < \tau_3 < \tau_0 + 2\pi .$$

Assume f to be increasing on the intervals $I_1 = [\tau_0, \tau_1]$ and $I_3 = [\tau_2, \tau_3]$ and decreasing on $I_2 = [\tau_1, \tau_2]$ and $I_4 = [\tau_3, \tau_0 + 2\pi]$. Then, there exists a $j = 1, ..., 4$ for which

$$f(I_j) = f(I_{j+1}) \cap f(I_{j+3}) , \quad f(I_{j+2}) = f(I_{j+1}) \cup f(I_{j+3})$$

(with the convention, as before, that $I_k = I_{k+4}$).

3. Jumping nonlinearities in $\mathbb{R}^2$. Let us consider the 2×2 matrix

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} ,$$

and assume throughout that

$$a_{12}a_{21} > 0 . \tag{6}$$

Notice that, in this case, the eigenvalues of A, given by the formula

$$\frac{1}{2}\left[a_{11} + a_{22} \pm \sqrt{(a_{11} - a_{22})^2 + 4a_{12}a_{21}}\right] ,$$

are real and distinct. We denote them by λ_1, λ_2, with $\lambda_1 < \lambda_2$. Observe that, since

$$(a_{11} - \lambda_1)(a_{22} - \lambda_1) = a_{12}a_{21} > 0 ,$$
$$(a_{11} - \lambda_2)(a_{22} - \lambda_2) = a_{12}a_{21} > 0 ,$$

and

$$\lambda_1 < \frac{1}{2}(a_{11} + a_{22}) < \lambda_2 ,$$

it has to be that

$$\lambda_1 < a_{11} < \lambda_2 \quad \text{and} \quad \lambda_1 < a_{22} < \lambda_2 . \tag{7}$$

Consider two real numbers μ, ν such that

$$\nu < \lambda_1 < \lambda_2 < \mu , \tag{8}$$

and define the four matrices

$$B_1 = \begin{pmatrix} \mu - a_{11} & -a_{12} \\ -a_{21} & \mu - a_{22} \end{pmatrix} , \quad B_2 = \begin{pmatrix} \nu - a_{11} & -a_{12} \\ -a_{21} & \mu - a_{22} \end{pmatrix} ,$$

$$B_3 = \begin{pmatrix} \nu - a_{11} & -a_{12} \\ -a_{21} & \nu - a_{22} \end{pmatrix} , \quad B_4 = \begin{pmatrix} \mu - a_{11} & -a_{12} \\ -a_{21} & \nu - a_{22} \end{pmatrix} .$$

The operator defined by (4) is then given by

$$Fu = -Au + \mu u^+ - \nu u^- .$$

By (8), we have

$$\det B_1 = (\mu - \lambda_1)(\mu - \lambda_2) > 0 ,$$
$$\det B_3 = (\nu - \lambda_1)(\nu - \lambda_2) > 0 .$$

On the other hand, (6), (7) and (8) imply

$$\det B_2 = (\nu - a_{11})(\mu - a_{22}) - a_{12}a_{21} < 0 ,$$
$$\det B_4 = (\mu - a_{11})(\nu - a_{22}) - a_{12}a_{21} < 0 .$$

So, condition (5) of Proposition 2.1 is satisfied. We have the following four target points:

$$F(1,0) = (\mu - a_{11}, -a_{21}) , \qquad F(0,1) = (-a_{12}, \mu - a_{22}) ,$$
$$F(-1,0) = (a_{11} - \nu, a_{21}) , \qquad F(0,-1) = (a_{12}, a_{22} - \nu) .$$

We thus have two possibilities. Either, $a_{12} < 0$ and $a_{21} < 0$. In this case, we are in situation (S_1) with strict inclusion. We have

$$F(Q_1) = \left\{(x,y) : \frac{a_{21}}{a_{11} - \mu}x \le y \le \frac{a_{22} - \mu}{a_{12}}x\right\} ,$$

$$F(Q_3) = \left\{(x,y) : y \ge \frac{a_{21}}{a_{11} - \nu}x \text{ and } y \ge \frac{a_{22} - \nu}{a_{12}}x\right\} .$$

Or, $a_{12} > 0$ and $a_{21} > 0$. In this case, we are in situation (S_3) with strict inclusion. We have

$$F(Q_3) = \left\{(x,y) : \frac{a_{21}}{a_{11}-\nu}x \le y \le \frac{a_{22}-\nu}{a_{12}}x\right\} ,$$

$$F(Q_1) = \left\{(x,y) : y \ge \frac{a_{21}}{a_{11}-\mu}x \text{ and } y \ge \frac{a_{22}-\mu}{a_{12}}x\right\} .$$

Remark 1. Assume for instance that $a_{12} > 0$ and $a_{21} > 0$. If we consider the equation

$$F(u) = v ,$$

with $v = (x,y) \in \mathbb{R}^2\backslash\{(0,0)\}$, we see that there are:

- four solutions if

$$\frac{a_{21}}{a_{11}-\nu}x < y < \frac{a_{22}-\nu}{a_{12}}x ;$$

- three solutions if

$$y = \frac{a_{21}}{a_{11}-\nu}x > 0 \quad \text{or} \quad y = \frac{a_{22}-\nu}{a_{12}}x > 0 ;$$

- two solutions if

$$\frac{a_{21}}{a_{11}-\mu}x < y < \frac{a_{21}}{a_{11}-\nu}x \quad \text{or} \quad y > \max\left\{\frac{a_{22}-\mu}{a_{12}}x , \frac{a_{22}-\nu}{a_{12}}x\right\} ;$$

- one solution if

$$y = \frac{a_{21}}{a_{11}-\mu}x < 0 \quad \text{or} \quad y = \frac{a_{22}-\mu}{a_{12}}x > 0 .$$

- no solutions otherwise.

The situation in the case $a_{12} < 0$ and $a_{21} < 0$ can be described in an analogous way.

Remark 2. Notice that situations (S_2) and (S_4) cannot occur in this setting, because of assumption (6). As an illustration of what can happen when (6) does not hold, consider for example the matrix

$$A = \begin{pmatrix} 2 & 1 \\ -1 & -2 \end{pmatrix} .$$

In this case, $\lambda_1 = -\sqrt{3}$ and $\lambda_2 = \sqrt{3}$. Take, for example, $\mu = -\nu$ in such a way that (8) is verified, i.e. $\mu > \sqrt{3}$. The same considerations made above tell us that $\det B_1 > 0$ and $\det B_3 > 0$. On the other hand,

$$\det B_2 = 1 - (2+\mu)^2 , \quad \det B_4 = 1 - (2-\mu)^2 .$$

So, $\det B_2 < 0$, but it can be seen that, if $\sqrt{3} < \mu < 3$, then $\det B_4$ is positive. In this case, condition (5) does not hold. However, if $\mu > 3$, then (5) holds and one sees that we are in situation (S_4) of Proposition 2.1.

4. Periodic solutions of coupled asymmetric oscillators. Let Assumption H hold and consider the equation

$$u'' + F(u) = v + h(t) , \tag{9}$$

where $h : \mathbb{R} \to \mathbb{R}^2$ is continuous and T−periodic and $v \in \mathbb{R}^2$. Assume (5) of Proposition 2.1, and denote by $\mathcal{S}$ the smaller cone appearing in either situations $(S_1) - (S_4)$. The points of $\mathring{\mathcal{S}}$, the interior of $\mathcal{S}$, have four counter-images by the operator F, one in each quadrant of $\mathbb{R}^2$. More precisely, taking $v \in \overset{\circ}{\mathcal{S}}$, there are four points $u^{(j)} \in \mathbb{R}^2$, such that

$$u^{(j)} \in \mathcal{Q}_j \quad \text{and} \quad F(u^{(j)}) = v \quad (j = 1, ..., 4) . \tag{10}$$

In this section, we will prove that, under a suitable non-resonance assumption, each of the points $u^{(j)}$ produces a T-periodic solution of (9) when $v \in \overset{\circ}{\mathcal{S}}$ has large norm.

Theorem 4.1. *Let Assumption H and (5) hold. If*

$$\sigma(B_j) \cap \left\{ \left(\frac{2\pi k}{T}\right)^2 : k = 0, 1, 2, ... \right\} = \emptyset , \tag{11}$$

for $j = 1, ..., 4$, then, taking $\tilde{v} \in \overset{\circ}{\mathcal{S}}$ with $||\tilde{v}|| = 1$, there is a $\rho_0 > 0$ such that, if $v = \rho\tilde{v}$ with $\rho \geq \rho_0$, then equation (9) has at least four T-periodic solutions.

Proof: Dividing equation (9) by $\rho = ||v||$, setting $w(t) = u(t)/||v||$ and $\tilde{v} = v/||v||$, one has the equivalent equation

$$w'' + F(w) = \tilde{v} + \rho^{-1}h(t) . \tag{12}$$

Moreover, setting $w^{(j)} = u^{(j)}/||v||$, by (10) and the fact that F is positively homogeneous one has

$$w^{(j)} \in \mathcal{Q}_j \quad \text{and} \quad F(w^{(j)}) = \tilde{v} \quad (j = 1, ..., 4) . \tag{13}$$

Let us restrict our attention to the cone

$$\Omega_j = \{w \in C^2([0, T], \mathcal{Q}_j) : w(0) - w(T) = w'(0) - w'(T) = 0\} ,$$

and consider, for $\varepsilon \geq 0$, the problem

$$w'' + B_j w = \tilde{v} + \varepsilon h(t) , \quad w \in \Omega_j . \tag{14}$$

By assumption (11), the equation $w'' + B_j w = 0$ has no nontrivial T-periodic solutions. As a consequence, if $\varepsilon = 0$, problem (14) has a unique solution $w(t)$, so that, by (13), $w(t) \equiv w^{(j)}$. By the nonresonance condition (11), the periodic solutions of (14) depend continuously on ε. Then, for ε small enough, the unique T-periodic solution of (14) lies in the quadrant $\mathcal{Q}_j$ and so it is also a solution of (12) with $\varepsilon = \rho^{-1}$. This proves that, if ρ is sufficiently large, equation (12) has a solution in each Ω_j $(j = 1, ..., 4)$, i.e. at least four solutions, each of which takes values in a different quadrant of $\mathbb{R}^2$.

5. The symmetric case. In this section, we assume A to be symmetric, i.e. $a_{12} = a_{21}$. Hence, unless A is diagonal, (6) holds and we are in the setting of section 3. We assume (8) and define consequently the matrices B_j, so that

$$F(u) = -Au + \mu u^+ - \nu u^- .$$

In this setting, the open cone $\overset{\circ}{\mathcal{S}}$ is as follows. If $a_{12} = a_{21} < 0$, then $\mathcal{S} = F(\mathcal{Q}_1)$ so that

$$\overset{\circ}{\mathcal{S}} = \left\{ (x, y) : \frac{a_{21}}{a_{11} - \mu} x < y < \frac{a_{22} - \mu}{a_{12}} x \right\} .$$

If $a_{12} = a_{21} > 0$, then $\mathcal{S} = F(\mathcal{Q}_3)$ so that

$$\overset{\circ}{\mathcal{S}} = \left\{ (x, y) : \frac{a_{21}}{a_{11} - \nu} x < y < \frac{a_{22} - \nu}{a_{12}} x \right\} .$$

Noticing that

$$B_3 \leq B_2 \leq B_1 , \quad B_3 \leq B_4 \leq B_1 ,$$

assumption (11) is satisfied if, for instance, $\mu - \lambda_1$, the greatest eigenvalue of B_1, is less than $(2\pi/T)^2$. In this case, moreover, we are able to prove the following exact multiplicity result.

Theorem 5.1. *Assume A is symmetric and (3) holds. Taking $\tilde{v} \in \mathring{S}$ with $||\tilde{v}|| = 1$, there is a $\rho_1 > 0$ such that, if $v = \rho\tilde{v}$ with $\rho \geq \rho_1$, then equation (1) has exactly four $T-$periodic solutions.*

Remark 3. In the above statement we only consider v such that the equation $F(u) = v$ has four solutions. Referring to Remark 1, we could as well concentrate in the cases when there are two solutions, or no solution at all, with similar statements.

Proof: The existence follows from Theorem 4.1. The rest of the proof is divided into three steps.

Step 1. We first prove that the only T-periodic solutions of

$$u'' - Au + \mu u^+ - \nu u^- = \chi\,, \tag{15}$$

for any $\chi \in \mathbb{R}^2$, are constants. Indeed, let $u(t) = (u_1(t), u_2(t))$ be a T-periodic solution of (15). From the equation one sees that u'' is Lipschitz continuous and so $u \in W^{3,\infty}$. Thus, $\xi = u'$ has two derivatives in the weak sense. Moreover,

$$\frac{d}{dt}u_i^+ = (sign^+ u_i)u_i'\,, \quad \frac{d}{dt}u_i^- = (sign^- u_i)u_i'\,,$$

where, for $\alpha \in \mathbb{R}$, we set

$$sign^+\alpha = \begin{cases} 1 & \text{if } \alpha > 0 \\ 0 & \text{if } \alpha \leq 0 \end{cases}, \qquad sign^-\alpha = \begin{cases} 1 & \text{if } \alpha < 0 \\ 0 & \text{if } \alpha \geq 0 \end{cases}.$$

Since $u_i' = 0$ almost everywhere on the set $\{t : u_i(t) = 0\}$, taking derivatives in equation (15) we have that

$$\xi'' + (M(t) - A)\xi = 0\,, \tag{16}$$

almost everywhere, where

$$M(t) = \begin{pmatrix} m_1(t) & 0 \\ 0 & m_2(t) \end{pmatrix}, \quad \text{with} \quad m_i(t) = \begin{cases} \mu & \text{if } u_i(t) > 0\,, \\ \nu & \text{if } u_i(t) < 0\,, \end{cases}$$

for $i = 1, 2$. Since $\int_0^T \xi = 0$ and $M(t) \leq \mu I$, multiplying equation (16) by ξ and integrating, by Wirtinger inequality we have

$$\left(\frac{2\pi}{T}\right)^2 \int_0^T |\xi|^2 \leq \int_0^T |\xi'|^2 \leq \int_0^T \langle (M(t) - A)\xi, \xi\rangle \leq (\mu - \lambda_1)\int_0^T |\xi|^2\,.$$

Hence, by (3), ξ must be identically zero, so that u must be constant.

Step 2. Like in the proof of Theorem 4.1, we consider the equivalent equation

$$w'' + F(w) = \tilde{v} + \rho^{-1}h(t)\,, \tag{17}$$

and prove that there are constants $R > 0$ and $C > 0$ such that, if $\rho \geq R$, then any T-periodic solution of (17) is such that $||w||_{C^1} \leq C$. By contradiction, let (ρ_n) and (w_n) be such that $\rho_n \to \infty$, $||w_n||_{C^1} \to \infty$ and

$$w_n'' + F(w_n) = \tilde{v} + \rho_n^{-1}h(t)\,. \tag{18}$$

Define $z_n = w_n/||w_n||_{C^1}$. It satisfies

$$z_n'' + F(z_n) = \frac{\tilde{v} + \rho_n^{-1}h(t)}{||w_n||_{C^1}}\,.$$

Since $||z_n||_{C^1} = 1$, it follows from the equation that $(||z_n||_{C^2})$ is bounded, and we can find a subsequence converging in the C^1−norm to some z, with $||z||_{C^1} = 1$ and $z'' + F(z) = 0$. By step 1, z must be constant and, being $F^{-1}(0) = \{0\}$, it must be that $z = 0$, a contradiction.

Step 3. We will prove that, if ρ is large enough, all the T-periodic solutions of (17) necessarily lie in one quadrant of $\mathbb{R}^2$, and hence satisfy a linear equation. Since there is uniqueness of solutions for these linear equations, the proof will then be complete. By contradiction, assume that, for a sequence (ρ_n), with $\rho_n \to \infty$, there is a corresponding sequence (w_n) for which (18) holds, and $\{w_n(t) : t \in [0, T]\}$ is not contained in a single quadrant, for every n. By step 2, there is a constant $C > 0$ for which $||w_n||_{C^1} \leq C$, and it follows from the equation that (w_n) is bounded in the C^2−norm, as well. Hence, there is a subsequence (w_{n_k}) converging in the C^1−norm to some function w satisfying $w'' + F(w) = \tilde{v}$. By step 1, w must be constant, and it coincides with one of the $w^{(j)}$'s satisfying (13). Hence, (w_{n_k}) converges uniformly to $w^{(j)}$ and, if k is large enough, $\{w_{n_k}(t) : t \in [0, T]\}$ lies entirely in one quadrant, in contradiction with the assumption.

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REFERENCES

[1] Ambrosetti, A. and Prodi, G. (1972): On the inversion of some differential mappings with singularities between Banach spaces. *Ann. Mat. Pura Appl.* **93**, 231-247.
[2] Alonso, J.M. and Ortega, R. (1998): Roots of unity and unbounded motions of an asymmetric oscillator. *J. Differential Equations* 143, 201-220.
[3] Dancer, E.N. (1976): Boundary-value problems for weakly nonlinear ordinary differential equations. *Bull. Austral. Math. Soc.* 15, 321-328.
[4] Dancer, E.N. (1989): A counter-example to Lazer-McKenna conjecture. *Nonlinear Anal.* 13, 19-22.
[5] Fabry, C. and Fonda, A. (1998): Nonlinear resonance in asymmetric oscillators. *J. Differential Equations* 147, 58-78.
[6] Fučík, S. (1980): *Solvability of Nonlinear Equations and Boundary Value Problems.* Reidel.
[7] Den Hartog, J.P. (1985): *Mechanical Vibrations.* Dover.
[8] Lazer, A.C. and McKenna, P.J. (1981): On the number of solutions of a Dirichlet problem. *J. Math. Anal. Appl.* 84, 282-294.
[9] Lazer, A.C. and McKenna, P.J. (1990): Large-amplitude periodic oscillations in suspension bridges: Some new connections with nonlinear analysis. *SIAM Review* 32, 537-578.
[10] Liu, B. (1999): Boundedness in asymmetric oscillators. *J. Math. Anal. Appl.* 231, 355-373.
[11] Švarc, R. (1987): Some combinatorial results about the operators with jumping nonlinearities. *Comm. Math. Univ. Carolinae* 28, 707-733.

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