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Bifurcations from infinity in asymmetric nonlinear oscillators*

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Abstract. We study the existence of large-amplitude periodic or almost periodic solutions of second order differential equations with asymmetric nonlinearities, when the system is close to "nonlinear resonance".

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1 Introduction

In the first part of this paper we consider periodic problems of the type

$$\begin{cases} x'' + \mu x^+ - \nu x^- = g(t, x, x', \varepsilon) \\ x(t + 2\pi) = x(t), \end{cases} \qquad (P_{\mu,\nu,\varepsilon})$$

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where $x^+ = \max\{x, 0\}$ is the positive part of x, and $x^- = \max\{-x, 0\}$ is its negative part, μ, ν are real numbers satisfying

$$\mu > 0, \quad \nu > 0, \quad \frac{1}{\sqrt{\mu}} + \frac{1}{\sqrt{\nu}} = \frac{2}{n}, \tag{1}$$

for some positive integer n, and ε is a small parameter. The pair (μ, ν) thus belongs to the n-th Fučík curve for the homogeneous problem

$$\begin{cases} \varphi'' + \mu\varphi^+ - \nu\varphi^- = 0 \\ \varphi(t + 2\pi) = \varphi(t), \end{cases} \tag{$H_{\mu,\nu}$}$$

which then has nontrivial solutions (cf. [7]). We call this situation "nonlinear resonance".

In Section 2, we prove a general result on bifurcation from infinity for problem $(P_{\mu,\nu,\varepsilon})$, when ε is near 0. We develop two direct consequences of this result in Section 3.

As a first consequence, we improve some results of [4] for the problem with a small damping term

$$\begin{cases} x'' + \varepsilon x' + \mu x^+ - \nu x^- = f(t) \\ x(t + 2\pi) = x(t). \end{cases} \tag{$P'_{\mu,\nu,\varepsilon}$}$$

Under some conditions on the 2π-periodic forcing $f(t)$, we prove the existence of large amplitude periodic solutions when ε is a small nonzero parameter. Notice that the undamped problem

$$\begin{cases} x'' + \mu x^+ - \nu x^- = f(t) \\ x(t + 2\pi) = x(t), \end{cases} \tag{$P'_{\mu,\nu,0}$}$$

was considered, in a recent paper, by Alonso and Ortega [1]. They proved that, if $f(t)$ verifies the same kind of conditions, every solution of $(P'_{\mu,\nu,0})$ having large amplitude must be unbounded.

As a second consequence of our general theorem we find new sufficient conditions on the function $f(t)$ for the existence of large amplitude solutions for the undamped problem $(P'_{\mu,\nu,0})$, when the pair (μ, ν) is slightly perturbed away from the Fučík curve. Exploiting these ideas in Section 4, we are able to prove the existence of large amplitude subharmonic solutions when the pair (μ, ν) varies in a dense subset of a neighborhood of a point on the Fučík curve.

In Section 5, we consider the existence of almost periodic solutions for the differential equation in $(P_{\mu,\nu,\varepsilon})$, when the forcing term is assumed to be almost periodic in t. We give conditions under which bifurcation from infinity for almost periodic solutions appears. These results are analogous to those obtained in the first part of the paper for the periodic case. However, the proofs in the periodic case are based on topological degree arguments, while for the almost periodic case we use a general theorem due to Hale [8].

2 Bifurcations from infinity for periodic solutions: a general approach

In the following, we denote by $\mathbb{R}_0, \mathbb{R}^+, \mathbb{R}^-$ the sets of nonzero, positive or negative real numbers, respectively.

For some $\varepsilon_0 > 0$, let $g : \mathbb{R}^3 \times [0, \varepsilon_0] \to \mathbb{R}$ satisfy the following conditions.

- For every $(x, y, \varepsilon) \in \mathbb{R} \times \mathbb{R} \times [0, \varepsilon_0]$, the function $t \mapsto g(t, x, y, \varepsilon)$ is locally integrable and 2π-periodic (in the a.e. sense).
- For almost every $t \in \mathbb{R}$, the map $(x, y, \varepsilon) \mapsto g(t, x, y, \varepsilon)$ is continuous on $\mathbb{R} \times \mathbb{R} \times [0, \varepsilon_0]$ and locally Lipschitz in (x, y), i.e., for every $R > 0$, there is a L^1-function $\ell_R(t)$ (independent of ε) for which, if $|x| + |y| \le R$ and $|x'| + |y'| \le R$, then
$$|g(t, x, y, \varepsilon) - g(t, x', y', \varepsilon)| \le \ell_R(t)(|x - x'| + |y - y'|).$$

Moreover, we make the following

Assumption H. *A function* $G : \mathbb{R} \times \mathbb{R}_0 \times \mathbb{R}_0 \to \mathbb{R}$ *can be well defined by*
$$G(t, x, y) = \lim_{\varepsilon \to 0_+} g\left(t, \frac{x}{\varepsilon}, \frac{y}{\varepsilon}, \varepsilon\right),$$
the above limit being uniform with respect to (t, x, y) *on compact subsets of* $\mathbb{R} \times \mathbb{R}_0 \times \mathbb{R}_0$. *Moreover, the function* G *satisfies the following Carathéodory conditions.*

- *For every* $(x, y) \in \mathbb{R}_0 \times \mathbb{R}_0$, *the function* $t \mapsto G(t, x, y)$ *is measurable and* 2π*-periodic.*
- *For almost every* $t \in \mathbb{R}$, *the map* $(x, y) \mapsto G(t, x, y)$ *is continuous on* $\mathbb{R}_0 \times \mathbb{R}_0$.
- *For every* $R > 0$, *there is a* L^1*-function* $H_R(t)$ *such that*
$$0 < |x| \le R, \quad 0 < |y| \le R \quad \Longrightarrow \quad |G(t, x, y)| \le H_R(t).$$

Assumption H might seem restrictive; it will be seen, however, that it covers some interesting particular cases.

Given $\mu > 0$, $\nu > 0$, we denote by $\varphi_{\mu,\nu}$ the solution of the initial value problem
$$\begin{cases} \varphi'' + \mu\varphi^+ - \nu\varphi^- = 0 \\ \varphi(0) = 0,\ \varphi'(0) = 1. \end{cases} \qquad (I_{\mu,\nu})$$

This solution is periodic with minimal period $\pi(1/\sqrt{\mu}+1/\sqrt{\nu})$ and can be written explicitly:

$$\varphi_{\mu,\nu}(t) = \begin{cases} \frac{1}{\sqrt{\mu}}\sin(\sqrt{\mu}t) & (t \in [0, \frac{\pi}{\sqrt{\mu}}]), \\ -\frac{1}{\sqrt{\nu}}\sin(\sqrt{\nu}(t - \frac{\pi}{\sqrt{\mu}})) & (t \in [\frac{\pi}{\sqrt{\mu}}, \frac{\pi}{\sqrt{\mu}} + \frac{\pi}{\sqrt{\nu}}]). \end{cases}$$

When μ, ν satisfy (1), $\varphi_{\mu,\nu}$ is a $2\pi/n$-periodic function, and it can be seen that the nontrivial solutions of $(H_{\mu,\nu})$ are of the form $\varphi(t) = A\varphi_{\mu,\nu}(t+\theta)$, for some $A > 0$ and $\theta \in [0, 2\pi/n[$ (cf. [7]).

Let us define the function $\Psi_{\mu,\nu} : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^2$ by

$$\begin{aligned} \Psi_{\mu,\nu}(\rho,\theta) \;=\; & \left(\frac{1}{2\pi}\int_0^{2\pi} G(t, \rho\varphi_{\mu,\nu}(t+\theta), \rho\varphi'_{\mu,\nu}(t+\theta))\, \varphi'_{\mu,\nu}(t+\theta)\, dt, \right. \\ & \left. - \frac{1}{2\pi\rho}\int_0^{2\pi} G(t, \rho\varphi_{\mu,\nu}(t+\theta), \rho\varphi'_{\mu,\nu}(t+\theta))\, \varphi_{\mu,\nu}(t+\theta)\, dt \right). \end{aligned}$$

Notice that $\Psi_{\mu,\nu}$ is well defined and continuous on $\mathbb{R}^+ \times \mathbb{R}$, even if $G(t,x,y)$ is not defined for $x = 0$ or $y = 0$.

Theorem 1. *Assume that, for some $\bar{\mu}, \bar{\nu}$ verifying* (1), *there is an open bounded set U, whose closure is contained in $\mathbb{R}^+ \times \mathbb{R}$, for which the Brouwer degree $d_B(\Psi_{\bar{\mu},\bar{\nu}}, U, 0)$ is different from zero. Then, there are $\eta > 0$ and $\varepsilon_1 \in]0, \varepsilon_0]$ such that, if μ, ν verify* (1) *and $|\mu - \bar{\mu}| \le \eta$, $|\nu - \bar{\nu}| \le \eta$ and $0 < \varepsilon \le \varepsilon_1$, then problem $(P_{\mu,\nu,\varepsilon})$ has a solution $x(t) = x(t; \mu, \nu, \varepsilon)$ of the form*

$$(x(t), x'(t)) = \frac{1}{\varepsilon}\rho(t)(\varphi_{\mu,\nu}(t+\theta(t)), \varphi'_{\mu,\nu}(t+\theta(t))),$$

with $(\rho(t), \theta(t)) \in U$ for every $t \in \mathbb{R}$.

Proof. The change of variables

$$x = \frac{1}{\varepsilon}\,\rho\varphi_{\mu,\nu}(t+\theta), \quad x' = \frac{1}{\varepsilon}\,\rho\varphi'_{\mu,\nu}(t+\theta)$$

transforms the differential equation in $(P_{\mu,\nu,\varepsilon})$ into

$$\begin{cases} \rho' = \varepsilon g\left(t, \frac{\rho}{\varepsilon}\,\varphi_{\mu,\nu}(t+\theta), \frac{\rho}{\varepsilon}\,\varphi'_{\mu,\nu}(t+\theta), \varepsilon\right) \varphi'_{\mu,\nu}(t+\theta) \\ \theta' = -\frac{\varepsilon}{\rho}\, g\left(t, \frac{\rho}{\varepsilon}\,\varphi_{\mu,\nu}(t+\theta), \frac{\rho}{\varepsilon}\,\varphi'_{\mu,\nu}(t+\theta), \varepsilon\right) \varphi_{\mu,\nu}(t+\theta), \end{cases} \qquad (\odot)$$

as soon as $\rho > 0$ (we use the fact that $[\varphi'_{\mu,\nu}]^2 - \varphi_{\mu,\nu}\varphi''_{\mu,\nu} = 1$). Given $(\rho_0, \theta_0) \in \mathbb{R}^+ \times \mathbb{R}$, let us denote by $\rho(t; \mu, \nu, \varepsilon; \rho_0, \theta_0), \theta(t; \mu, \nu, \varepsilon; \rho_0, \theta_0)$ the solution of $(\odot)$ with initial point (ρ_0, θ_0) at time $t = 0$. By Assumption H and the locally Lipschitz continuity of g, this solution is unique and, for ε small enough, it is defined on $[0, 2\pi]$. Integrating the differential equations ($\odot$), the periodicity conditions

$$\begin{cases} \rho(2\pi; \mu, \nu, \varepsilon; \rho_0, \theta_0) = \rho(0; \mu, \nu, \varepsilon; \rho_0, \theta_0) \\ \theta(2\pi; \mu, \nu, \varepsilon; \rho_0, \theta_0) = \theta(0; \mu, \nu, \varepsilon; \rho_0, \theta_0) \end{cases}$$

will be satisfied for $\varepsilon > 0$ if $\psi_{\mu,\nu,\varepsilon}(\rho_0, \theta_0) = (0, 0)$, where the function $\psi_{\mu,\nu,\varepsilon} : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^2$ is defined by

$$\begin{aligned} &\psi_{\mu,\nu,\varepsilon}(\rho_0, \theta_0) \\ &\quad = \left(\frac{1}{2\pi} \int_0^{2\pi} g\left(t, \frac{1}{\varepsilon}\rho\varphi_{\mu,\nu}(t+\theta), \frac{1}{\varepsilon}\rho\varphi'_{\mu,\nu}(t+\theta), \varepsilon\right) \varphi'_{\mu,\nu}(t+\theta)\, dt, \right. \\ &\qquad \left. - \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{\rho} g\left(t, \frac{1}{\varepsilon}\rho\varphi_{\mu,\nu}(t+\theta), \frac{1}{\varepsilon}\rho\varphi'_{\mu,\nu}(t+\theta), \varepsilon\right) \varphi_{\mu,\nu}(t+\theta) dt \right) \end{aligned}$$

(in the above integrals, we have written ρ instead of $\rho(t; \mu, \nu, \varepsilon; \rho_0, \theta_0)$, and θ instead of $\theta(t; \mu, \nu, \varepsilon; \rho_0, \theta_0)$). Letting ε tend to zero, one has from ($\odot$) and Assumption H that

$$\rho(t; \mu, \nu, \varepsilon; \rho_0, \theta_0) \longrightarrow \rho_0, \quad \theta(t; \mu, \nu, \varepsilon; \rho_0, \theta_0) \longrightarrow \theta_0,$$

so that, by Assumption H and Lebesgue's dominated convergence theorem, we have

$$\lim_{\varepsilon \to 0_+} \psi_{\mu,\nu,\varepsilon}(\rho_0, \theta_0) = \Psi_{\mu,\nu}(\rho_0, \theta_0)\,.$$

The above limit is uniform with respect to (ρ_0, θ_0) on compact subsets of $\mathbb{R}^+ \times \mathbb{R}$, and it is also uniform with respect to (μ, ν) on compact subsets of the Fučík curve defined by (1).

Now we claim that, for any fixed $\gamma > 0$, there is a $\bar{\eta}_\gamma > 0$ such that, if μ, ν verify (1) and $|\mu - \bar{\mu}| \le \bar{\eta}_\gamma$, $|\nu - \bar{\nu}| \le \bar{\eta}_\gamma$, then

$$\|\Psi_{\mu,\nu}(\rho_0, \theta_0) - \Psi_{\bar{\mu},\bar{\nu}}(\rho_0, \theta_0)\| \le \gamma, \tag{2}$$

for every (ρ_0, θ_0) in a compact subset K of $\mathbb{R}^+ \times \mathbb{R}$. Indeed, the function G is bounded by a L^1-function on a compact subset of its domain. Hence, using the properties of the function $\varphi_{\bar{\mu},\bar{\nu}}$, we can find a neighborhood $A \subset [0, 2\pi]$ of the points where $\varphi_{\bar{\mu},\bar{\nu}}$ or $\varphi'_{\bar{\mu},\bar{\nu}}$ vanish, over which

$$\left| \int_A G(t, \rho_0\varphi_{\mu,\nu}(t+\theta_0), \rho_0\varphi'_{\mu,\nu}(t+\theta_0))\varphi'_{\mu,\nu}(t+\theta_0)\, dt \right| \le \frac{\gamma\pi}{4},$$

$$\left| \frac{1}{\rho_0} \int_A G(t, \rho_0\varphi_{\mu,\nu}(t+\theta_0), \rho_0\varphi'_{\mu,\nu}(t+\theta_0))\varphi_{\mu,\nu}(t+\theta_0)\, dt \right| \le \frac{\gamma\pi}{4},$$

for every (μ, ν) sufficiently near to $(\bar{\mu}, \bar{\nu})$ and $(\rho_0, \theta_0) \in K$. On the other hand, being G continuous in its second and third variables on $\mathbb{R}_0 \times \mathbb{R}_0$, we have that

$$\Big| \int_{[0,2\pi]\setminus A} [G(t, \rho_0\varphi_{\mu,\nu}(t+\theta_0), \rho_0\varphi'_{\mu,\nu}(t+\theta_0))\varphi'_{\mu,\nu}(t+\theta_0)$$

$$- \; G(t, \rho_0\varphi_{\bar{\mu},\bar{\nu}}(t+\theta_0), \rho_0\varphi'_{\bar{\mu},\bar{\nu}}(t+\theta_0))\varphi'_{\bar{\mu},\bar{\nu}}(t+\theta_0)]\, dt \Big| \leq \frac{\gamma\pi}{2},$$

$$\frac{1}{\rho_0}\Big| \int_{[0,2\pi]\setminus A} [G(t, \rho_0\varphi_{\mu,\nu}(t+\theta_0), \rho_0\varphi'_{\mu,\nu}(t+\theta_0))\varphi_{\mu,\nu}(t+\theta_0)$$

$$- \; G(t, \rho_0\varphi_{\bar{\mu},\bar{\nu}}(t+\theta_0), \rho_0\varphi'_{\bar{\mu},\bar{\nu}}(t+\theta_0))\varphi_{\bar{\mu},\bar{\nu}}(t+\theta_0)]\, dt \Big| \leq \frac{\gamma\pi}{2},$$

for every (μ, ν) sufficiently near to $(\bar{\mu}, \bar{\nu})$ and $(\rho_0, \theta_0) \in K$. Putting together the above inequalities and splitting the integrals as

$$\int_0^{2\pi} = \int_A + \int_{[0,2\pi]\setminus A},$$

we get (2).

We have thus proved that, if (ρ_0, θ_0) belongs to a compact subset of $\mathbb{R}^+ \times \mathbb{R}$, for any $\gamma > 0$ there are $\bar{\eta}_\gamma > 0$ and $\bar{\varepsilon}_\gamma > 0$ for which, if μ, ν satisfy (1) and $|\mu - \bar{\mu}| \leq \bar{\eta}_\gamma$, $|\nu - \bar{\nu}| \leq \bar{\eta}_\gamma$ and $0 < \varepsilon < \bar{\varepsilon}_\gamma$, then

$$\|\psi_{\mu,\nu,\varepsilon}(\rho_0, \theta_0) - \Psi_{\bar{\mu},\bar{\nu}}(\rho_0, \theta_0)\| \leq 2\gamma.$$

Being $d_B(\Psi_{\bar{\mu},\bar{\nu}}, U, 0) \neq 0$, even $d_B(\psi_{\mu,\nu,\varepsilon}, U, 0)$ will be different from 0 if γ is chosen small enough. Correspondingly, we find $\eta = \bar{\eta}_\gamma$ and $\varepsilon_1 = \bar{\varepsilon}_\gamma$ with the desired properties, and the proof is complete. □

3 Two particular cases

As illustrations of Theorem 1, particular cases are considered as corollaries below. In the first corollary, we will consider an equation with a small damping term whereas, in Corollary 2, an equation is studied with an asymmetric nonlinear term close to $\bar{\mu}x^+ - \bar{\nu}x^-$, with $(\bar{\mu}, \bar{\nu})$ on the Fučík curve defined by (1).

Assume that the function $f : \mathbb{R}^2 \to \mathbb{R}$ satisfies the following conditions.

- For every $x \in \mathbb{R}$, the function $t \mapsto f(t, x)$ is locally integrable and 2π-periodic (in the a.e. sense).
- For almost every $t \in \mathbb{R}$, the map $x \mapsto f(t, x)$ is locally Lipschitz continuous, i.e., for every $R > 0$ there is a L^1-function $\lambda_R(t)$ for which, if $|x| \leq R$ and $|x'| \leq R$, then
$$|f(t, x) - f(t, x')| \leq \lambda_R(t)|x - x'|.$$

Assume moreover that the following limits exist

$$f_+(t) = \lim_{x\to+\infty} f(t,x), \quad f_-(t) = \lim_{x\to-\infty} f(t,x),$$

uniformly in t. For $\mu > 0,\ \nu > 0$, define the C^1-function

$$\Phi_{\mu,\nu}(\theta) = \frac{1}{2\pi}\int_{\varphi_{\mu,\nu}>0} f_+(t-\theta)\varphi_{\mu,\nu}(t)\,dt + \frac{1}{2\pi}\int_{\varphi_{\mu,\nu}<0} f_-(t-\theta)\varphi_{\mu,\nu}(t)\,dt.$$

Corollary 1. *For some $\bar\mu, \bar\nu$ satisfying* (1), *let θ^* be a simple zero of $\Phi_{\bar\mu,\bar\nu}$. Then, for every $\gamma > 0$ there are $\eta > 0$ and $\varepsilon_1 > 0$ such that, if μ, ν verify* (1) *and $|\mu - \bar\mu| \le \eta$, $|\nu - \bar\nu| \le \eta$, $|\varepsilon| \le \varepsilon_1$ and $\varepsilon\Phi'_{\bar\mu,\bar\nu}(\theta^*) > 0$, then the problem*

$$\begin{cases} x'' + \varepsilon x' + \mu x^+ - \nu x^- = f(t,x) \\ x(t+2\pi) = x(t) \end{cases} \qquad (P^{(1)}_{\mu,\nu,\varepsilon})$$

has a solution $x(t) = x(t;\mu,\nu,\varepsilon)$ such that

$$(x(t), x'(t)) = \frac{1}{|\varepsilon|}\rho(t)(\varphi_{\mu,\nu}(t+\theta(t)), \varphi'_{\mu,\nu}(t+\theta(t))),$$

the functions $\rho(t), \theta(t)$ being such that

$$|\rho(t) - \rho^*| \le \gamma, \quad |\theta(t) - \theta^*| \le \gamma,$$

where $\rho^* := 2|\Phi'_{\bar\mu,\bar\nu}(\theta^*)|$.

Proof. We will consider the case $\varepsilon > 0$ (the case $\varepsilon < 0$ is treated by replacing ε by $-\varepsilon$). We apply Theorem 1 with

$$g(t,x,y,\varepsilon) = f(t,x) - \varepsilon y.$$

It is easy to check that Assumption H is satisfied with

$$G(t,x,y) = \begin{cases} f_+(t) - y & \text{if } x > 0, \\ f_-(t) - y & \text{if } x < 0. \end{cases}$$

Since $\int_0^{2\pi} [\varphi'_{\mu,\nu}]^2 = \pi$ and $\int_0^{2\pi} \varphi_{\mu,\nu}\varphi'_{\mu,\nu} = 0$, we have

$$\Psi_{\mu,\nu}(\rho,\theta) = \left(\Phi'_{\mu,\nu}(\theta) - \frac{\rho}{2}, -\frac{1}{\rho}\Phi_{\mu,\nu}(\theta)\right).$$

Fix $\gamma > 0$. Choose θ_1, θ_2 such that

$$\theta^* - \gamma \le \theta_1 < \theta^* < \theta_2 \le \theta^* + \gamma,$$

$$\Phi_{\bar\mu,\bar\nu}(\theta_1) < 0 < \Phi_{\bar\mu,\bar\nu}(\theta_2),$$

and

$$|\Phi'_{\bar{\mu},\bar{\nu}}(\theta) - \Phi'_{\bar{\mu},\bar{\nu}}(\theta^*)| \leq \min\left\{\frac{\gamma}{2}, \frac{1}{2}\Phi'_{\bar{\mu},\bar{\nu}}(\theta^*)\right\},$$

for every $\theta \in [\theta_1, \theta_2]$. Then choose ρ_1, ρ_2 in $\mathbb{R}^+$ such that

$$\Phi'_{\bar{\mu},\bar{\nu}}(\theta^*) - \gamma \leq \frac{\rho_1}{2} < \Phi'_{\bar{\mu},\bar{\nu}}(\theta) < \frac{\rho_2}{2} \leq \Phi'_{\bar{\mu},\bar{\nu}}(\theta^*) + \gamma,$$

for every $\theta \in [\theta_1, \theta_2]$. By Miranda's Theorem (see [11], p. 178), setting $U =]\rho_1, \rho_2[\times]\theta_1, \theta_2[$, we have that $d_B(\Psi_{\bar{\mu},\bar{\nu}}, U, 0) \neq 0$. By Theorem 1, we have a solution with $\rho(t) \in]\rho_1, \rho_2[$ and $\theta(t) \in]\theta_1, \theta_2[$, and the proof is complete. □

Remark 1. The particular case where f does not depend on x has already been treated in [4], using the implicit function theorem rather than degree arguments. It has been shown in [4] that the (locally unique) solutions which tend to infinity when ε goes to 0 are asymptotically stable for $\varepsilon > 0$, ε "small". This is the same "classical resonance" phenomenon which arises in the linear case $\mu = \nu = n$. Assume now that the function $\Phi_{\bar{\mu},\bar{\nu}}$ only has simple zeros. Then, denoting by $2z$ the number of those in $[0, 2\pi[$, we have z asymptotically stable large amplitude periodic solutions (and the arguments in [10] can then be used to prove the existence of at least other $z - 1$ unstable solutions).

As an example of application, we have

$$\begin{cases} x'' + \varepsilon x' + \mu x^+ - \nu x^- = \cos(zt) \\ x(t + 2\pi) = x(t), \end{cases}$$

where z is a multiple of n, $\varepsilon > 0$ and $\mu \neq \nu$ (cf. [4]). Notice that, when $\mu = \nu$ and $z \neq 1$, large amplitude periodic solutions do not exist.

Remark 2. When f does not depend on x, the function $\Phi_{\mu,\nu}$ is given by

$$\Phi_{\mu,\nu}(\theta) = \frac{1}{2\pi}\int_0^{2\pi} f(t)\varphi_{\mu,\nu}(t + \theta)\, dt.$$

The amplitude of the solution provided by Corollary 1 is of the order of $\rho^*/|\varepsilon|$, where $\rho^* = 2|\Phi'_{\bar{\mu},\bar{\nu}}(\theta^*)|$; by the definition of $\Phi_{\mu,\nu}$, we have

$$\rho^* \leq \frac{1}{\pi}\|f\|_2\|\varphi'_{\bar{\mu},\bar{\nu}}\|_2 = \frac{\|f\|_2}{\sqrt{\pi}},$$

where $\|\cdot\|_2$ denotes the usual norm in L^2. The equality is obtained in that relation when f is a multiple of $\varphi'_{\bar{\mu},\bar{\nu}}$ (notice that only positive multiples are allowed when $\varepsilon > 0$, negative multiples when $\varepsilon < 0$, in order to maintain the condition $\varepsilon\Phi'_{\bar{\mu},\bar{\nu}}(\theta^*) > 0$). It is remarkable that the (asymptotic) amplitude of the response of the oscillator to a forcing term f is maximal when f is a multiple of the derivative of the free oscillations. Moreover, the coefficient ρ^* is independent of $\bar{\mu}, \bar{\nu}$. The maximal amplitude is then asymptotically between $\|f\|_2/(|\varepsilon|n\sqrt{\pi})$ and $2\|f\|_2/(|\varepsilon|n\sqrt{\pi})$ (the amplitude of φ itself is equal to $\max\{1/\sqrt{\mu}, 1/\sqrt{\nu}\}$ which, because of (1), lies between $1/n$ and $2/n$).

We will now consider an equation with no friction term and perturb the coefficients away from the Fučík curve.

Corollary 2. *For some $\bar{\mu}, \bar{\nu}$ satisfying* (1)*, assume that $\Phi_{\bar{\mu},\bar{\nu}}$ is nonconstant and that there is a local minimum or maximum with value $\sigma^* \neq 0$. Then, for every $\gamma > 0$ there are $\eta > 0$ and $\varepsilon_1 > 0$ such that, if μ, ν verify* (1) *and $|\mu - \bar{\mu}| \leq \eta$, $|\nu - \bar{\nu}| \leq \eta$, $|\varepsilon| \leq \varepsilon_1$ and $\varepsilon\sigma^* > 0$, then the problem*

$$\begin{cases} x'' + (\mu + \varepsilon)x^+ - (\nu + \varepsilon)x^- = f(t, x) \\ x(t + 2\pi) = x(t) \end{cases} \qquad (P^{(2)}_{\mu,\nu,\varepsilon})$$

has a solution $x(t) = x(t; \mu, \nu, \varepsilon)$ such that

$$(x(t), x'(t)) = \frac{1}{|\varepsilon|}\rho(t)(\varphi_{\mu,\nu}(t + \theta(t)), \varphi'_{\mu,\nu}(t + \theta(t))),$$

the functions $\rho(t), \theta(t)$ being such that

$$|\rho(t) - \rho^*| \leq \gamma, \quad |\Phi_{\mu,\nu}(\theta(t)) - \sigma^*| \leq \gamma,$$

where

$$\rho^* := \frac{4|\sigma^*|}{n(\bar{\mu}^{-3/2} + \bar{\nu}^{-3/2})}.$$

Proof. We will consider the case $\varepsilon > 0$, the case $\varepsilon < 0$ being treated similarly. Having

$$g(t, x, y, \varepsilon) = f(t, x) - \varepsilon x,$$

it is easy to check that Assumption H is verified with

$$G(t, x, y) = \begin{cases} f_+(t) - x & \text{if } x > 0, \\ f_-(t) - x & \text{if } x < 0. \end{cases}$$

Since $\int_0^{2\pi} [\varphi_{\mu,\nu}]^2 = n\pi(\mu^{-3/2} + \nu^{-3/2})/2$, we have

$$\Psi_{\mu,\nu}(\rho, \theta) = \left(\Phi'_{\mu,\nu}(\theta), \frac{n}{4}(\mu^{-3/2} + \nu^{-3/2}) - \frac{1}{\rho}\Phi_{\mu,\nu}(\theta)\right).$$

Fix $\gamma > 0$. We can choose θ_1, θ_2 such that

$$\Phi'_{\bar{\mu},\bar{\nu}}(\theta_1)\Phi'_{\bar{\mu},\bar{\nu}}(\theta_2) < 0,$$

and

$$|\Phi_{\bar\mu,\bar\nu}(\theta)-\sigma^*|\le \min\left\{\gamma,\frac{\sigma^*}{2},\frac{n\gamma}{8}(\bar\mu^{-3/2}+\bar\nu^{-3/2})\right\},$$

for every $\theta\in[\theta_1,\theta_2]$. Then we choose ρ_1,ρ_2 in $\mathbb{R}^+$ such that, for every $\theta\in[\theta_1,\theta_2]$,

$$\frac{4\sigma^*}{n(\bar\mu^{-3/2}+\bar\nu^{-3/2})}-\gamma\le\rho_1<\frac{4\Phi_{\bar\mu,\bar\nu}(\theta)}{n(\bar\mu^{-3/2}+\bar\nu^{-3/2})}<\rho_2\le\frac{4\sigma^*}{n(\bar\mu^{-3/2}+\bar\nu^{-3/2})}+\gamma.$$

In this way, we have

$$\frac{1}{\rho_2}\Phi_{\bar\mu,\bar\nu}(\theta)<\frac{n}{4}(\bar\mu^{-3/2}+\bar\nu^{-3/2})<\frac{1}{\rho_1}\Phi_{\bar\mu,\bar\nu}(\theta),$$

for every $\theta\in[\theta_1,\theta_2]$. By Miranda's Theorem, setting $U=]\rho_1,\rho_2[\times]\theta_1,\theta_2[$, we have that the degree $d_B(\Psi_{\bar\mu,\bar\nu},U,0)$ is nonzero. By Theorem 1, we have a solution with $\rho(t)\in]\rho_1,\rho_2[$ and $\theta(t)\in]\theta_1,\theta_2[$, and the proof is complete. □

Remark 3. If $\Phi_{\bar\mu,\bar\nu}$ takes only strictly positive values and is nonconstant, Corollary 2 provides at least two solutions going to infinity when $\varepsilon\to 0$, since in this case there are at least a local minimum and a local maximum in $[0,2\pi[$ which do not coincide. The positiveness of $\Phi_{\bar\mu,\bar\nu}$ can be interpreted as a Landesman-Lazer condition (see [2, 3, 4]). In that case, it is known that, for $\varepsilon=0$, a solution exists corresponding to a nonzero degree with respect to large balls. That degree remains different from zero for ε small. Hence, there exists a solution which remains in a bounded set for $\varepsilon\to 0$. On the whole, we see that at least three solutions are obtained for ε "small", $\varepsilon>0$. This result should be compared with [9], where the problem

$$\begin{cases} x''+\varepsilon x=f(t,x)\\ x(t+2\pi)=x(t)\end{cases}$$

(with $\mu=\nu=0$) is considered assuming Landesman-Lazer conditions on f. However, while in [9] one has a change in the topological degree when ε attains 0, it is remarkable that, in our situation, the topological degree remains unchanged.

Remark 4. Assume that f does not depend on x. Since $\Phi_{\mu,\nu}$ is a correlation product of f and $\varphi_{\mu,\nu}$, it is constant if and only the Fourier series of f and $\varphi_{\mu,\nu}$ do not contain harmonics of the same order. Thus, if f has nonzero harmonics of order kn for some integer k and if $\bar\mu,\bar\nu$ verify (1), the function $\Phi_{\bar\mu,\bar\nu}$ will not be constant. Moreover, since $\int_0^{2\pi}\varphi_{\bar\mu,\bar\nu}=2(\frac{1}{\bar\mu}-\frac{1}{\bar\nu})$, the mean value of $\Phi_{\bar\mu,\bar\nu}$ (which is equal to the product of the mean values of f and $\varphi_{\bar\mu,\bar\nu}$) will have the same sign as ε provided that

$$\varepsilon(\bar\nu-\bar\mu)\int_0^{2\pi}f(t)\,dt>0\,.$$

In this case, $\Phi_{\bar{\mu},\bar{\nu}}$ taking values of the same sign as ε, by Corollary 2, the problem $(P^{(2)}_{\bar{\mu},\bar{\nu},\varepsilon})$ has at least one 2π-periodic solution whose amplitude goes to infinity when ε goes to 0. As an example, we have

$$\begin{cases} x'' + (\mu+\varepsilon)x^+ - (\nu+\varepsilon)x^- = c + \cos(jt) \\ x(t+2\pi) = x(t), \end{cases}$$

where j is a multiple of n, $\bar{\mu} < \bar{\nu}$ and εc is positive.

4 Subharmonic solutions

In this section we assume that $\bar{\mu}, \bar{\nu}$ satisfy (1) and let (μ, ν) lie in a neighborhood of $(\bar{\mu}, \bar{\nu})$, with μ, ν not necessarily satisfying (1). We consider the case when $\Phi_{\bar{\mu},\bar{\nu}}$ *only takes strictly positive values* (a Landesman-Lazer type of situation).

For simplicity, we assume $f : \mathbb{R} \to \mathbb{R}$ to be a 2π-periodic, continuous function of t (independent of x). Consequently, for $\mu > 0, \nu > 0$, we have

$$\Phi_{\mu,\nu}(\theta) = \frac{1}{2\pi}\int_0^{2\pi} f(t)\varphi_{\mu,\nu}(t+\theta)\,dt.$$

We first prove two preliminary results.

Lemma 1. *Let $\bar{\mu}, \bar{\nu}$ satisfy* (1)*, and assume that*

$$\Phi_{\bar{\mu},\bar{\nu}}(\theta) > 0,$$

for every $\theta \in [0, 2\pi]$. Then, there is a neighborhood U of $(\bar{\mu}, \bar{\nu})$ such that, if μ, ν satisfy

$$\mu > 0, \quad \nu > 0, \quad \frac{1}{\sqrt{\mu}} + \frac{1}{\sqrt{\nu}} = \frac{2p}{q}, \tag{3}$$

for some positive integers p, q, and $(\mu, \nu) \in U$, then

$$\Phi^{(p)}_{\mu,\nu}(\theta) := \frac{1}{2\pi p}\int_0^{2\pi p} f(t)\varphi_{\mu,\nu}(t+\theta)\,dt > 0,$$

for every $\theta \in [0, 2\pi]$.

Proof. Let us denote by m the minimum (positive) value assumed by $\Phi_{\bar{\mu},\bar{\nu}}$. We have

$$\begin{aligned} \Phi^{(p)}_{\mu,\nu}(\theta) &= \frac{1}{2\pi p}\sum_{j=0}^{p-1}\int_{2\pi j}^{2\pi(j+1)} f(t)\varphi_{\mu,\nu}(t+\theta)\,dt \\ &= \frac{1}{2\pi p}\sum_{j=0}^{p-1}\int_0^{2\pi} f(t)\varphi_{\mu,\nu}(t+\theta+2\pi j)\,dt, \end{aligned}$$

f being of period 2π. Since $\varphi_{\mu,\nu}$ is of period $2\pi p/q$, we have

$$\varphi_{\mu,\nu}(t+\theta+2\pi j) = \varphi_{\mu,\nu}(t+\theta+2\pi j_{\mu,\nu}),$$

for some $j_{\mu,\nu} \in [0, p/q[$. If (μ,ν) tends to $(\bar\mu,\bar\nu)$, then $\varphi_{\mu,\nu}$ converges to $\varphi_{\bar\mu,\bar\nu}$ on any fixed compact interval. Given any $\gamma > 0$, there exists a bounded neighborhood U of $(\bar\mu,\bar\nu)$ such that, if μ,ν satisfy (3) and $(\mu,\nu) \in U$, then $j_{\mu,\nu} \in [0, 2/n]$ (since p/q can be assumed close to $1/n$), and for any $t \in [0, 2\pi]$ and $\theta \in [0, 2\pi]$, we have

$$|\varphi_{\mu,\nu}(t+\theta+2\pi j_{\mu,\nu}) - \varphi_{\bar\mu,\bar\nu}(t+\theta+2\pi j_{\mu,\nu})| \le \gamma.$$

Assume $0 < \gamma < 2m\pi/\|f\|_1$ (here $\|\cdot\|_1$ is the usual norm in L^1); then,

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi} f(t)\varphi_{\mu,\nu}(t+\theta+2\pi j_{\mu,\nu})\,dt &= \Phi_{\bar\mu,\bar\nu}(\theta+2\pi j_{\mu,\nu})\\ &+ \frac{1}{2\pi}\int_0^{2\pi} f(t)[\varphi_{\mu,\nu}(t+\theta+2\pi j_{\mu,\nu}) - \varphi_{\bar\mu,\bar\nu}(t+\theta+2\pi j_{\mu,\nu})]\,dt\\ &\ge m - \frac{\gamma}{2\pi}\|f\|_1 > 0.\end{aligned}$$

Consequently, each term in the sum appearing above is strictly positive. □

Lemma 2. *Assume $f(t)$ is nonconstant, and let $T = 2\pi/N$ be its minimal period. Let p, q be positive integers such that Np and q are relatively prime, and fix $R > 0$. Then, there are $\delta_{p,q} > 0$ and $\Gamma_{p,q} > 0$ such that, if μ,ν satisfy* (3) *and $|\mu|+|\nu| \le R$, then for every (α,β) such that $|\alpha-\mu| + |\beta-\nu| \le \delta_{p,q}$, for any periodic solution x of the equation*

$$x'' + \alpha x^+ - \beta x^- = f(t) \tag{4}$$

one has that, either $x(t)$ has minimal period greater than or equal to $2\pi p$, or $\|x\|_\infty \le \Gamma_{p,q}$.

Proof. Consider all the Fučík curves relative to the periods $T, 2T, \ldots, (Np-1)T$. All these curves are different from the one defined by (3), since Np and q are relatively prime (indeed, otherwise there would be two integers $m \ge 1$ and $k \in \{1, 2, \ldots, Np-1\}$ for which

$$\frac{kT}{m\pi} = \frac{2p}{q},$$

i.e. $k/m = Np/q$, which is impossible). Then, we can choose $\delta_{p,q}$ such that the set

$$\{(\alpha,\beta) : |\alpha-\mu| \le \delta_{p,q}, |\beta-\nu| \le \delta_{p,q}\}$$

does not intersect any of them. Let x be a periodic solution of (4). Then, the minimal period of x is an integer multiple of T, say $\bar kT$. If $\bar kT < 2\pi p$, i.e. $\bar k < pN$, the above arguments about the Fučík curves indicates that we are in a "nonresonant" situation and it is well known that the $\bar{k}T$-periodic solutions of (4) are a priori bounded. Hence, there is a constant Γ, depending on μ, ν and on $\delta_{p,q}$ such that, if $x(t)$ is a solution of (4) having minimal period less than $2\pi p$, then $\|x\|_\infty \leq \Gamma$. Being μ, ν in a compact set, this bound can be chosen to depend only on p, q, and the proof is complete. □

Lemma 3. *Assume $f(t)$ is nonconstant, and let $T = 2\pi/N$ be its minimal period. Let μ, ν satisfy (3), $\mu \neq \nu$ and such that the square roots $\sqrt{\mu}, \sqrt{\nu}$ are irrational numbers. Then, the function $\Phi^{(p)}_{\mu,\nu}$ is nonconstant if and only if the Fourier development of f for the period $2\pi p$ contains harmonics of order q or a multiple of q.*

Proof. Assume μ, ν satisfy (3), and consider the following Fourier coefficients of $\varphi_{\mu,\nu}$:

$$a_m = \frac{1}{\pi p} \int_0^{2\pi p} \varphi_{\mu,\nu}(t) \cos\left(\frac{mt}{p}\right) dt,$$

$$b_m = \frac{1}{\pi p} \int_0^{2\pi p} \varphi_{\mu,\nu}(t) \sin\left(\frac{mt}{p}\right) dt.$$

Since $\varphi_{\mu,\nu}$ has minimal period $2\pi p/q$, the above Fourier coefficients can be different from zero only if m is a multiple of q. In that case, taking into account the fact that $\sqrt{\mu} \neq kq/p$ and $\sqrt{\nu} \neq kq/p$, $\sqrt{\mu}$ and $\sqrt{\nu}$ being irrational, we compute

$$a_{kq} = \frac{q}{\pi p}\left(\cos\left(\frac{kq\pi}{p\sqrt{\mu}}\right) + 1\right) \frac{\nu - \mu}{(\mu - (\frac{kq}{p})^2)(\nu - (\frac{kq}{p})^2)},$$

and

$$b_{kq} = \frac{q}{\pi p} \sin\left(\frac{kq\pi}{p\sqrt{\mu}}\right) \frac{\nu - \mu}{(\mu - (\frac{kq}{p})^2)(\nu - (\frac{kq}{p})^2)}.$$

Since $\mu \neq \nu$, we have that

$$a_{kq} \neq 0 \quad \text{and} \quad b_{kq} \neq 0,$$

for every $k \geq 1$. Being $\Phi^{(p)}_{\mu,\nu}$ a correlation product of f and $\varphi_{\mu,\nu}$, it is constant if and only the Fourier series of f and $\varphi_{\mu,\nu}$ do not contain harmonics of the same order, i.e., in the case considered, if and only if the Fourier development of f for the period $2\pi p$ does not contain harmonics of order q or a multiple of q. □

Lemma 4. *Assume $f(t)$ is nonconstant, and let $T = 2\pi/N$ be its minimal period. Let $\bar{\mu}, \bar{\nu}$ verify (1), and assume that $\Phi_{\bar{\mu},\bar{\nu}}$ is strictly positive. Then, there is a neighborhood V of $(\bar{\mu}, \bar{\nu})$ with the following property. For every $\Gamma > 0$, and any positive integers p, q such that Np and q are relatively prime, there is a $\varepsilon_{p,q} > 0$ such that, if μ, ν satisfy (3), $\mu \neq \nu$, $(\mu, \nu) \in V$ and $\sqrt{\mu}, \sqrt{\nu}$ are irrational, and* *if the Fourier development of f for the period $2\pi p$ contains harmonics of orders multiple of q, then, for $0 < \varepsilon \leq \varepsilon_{p,q}$, the equation*

$$x'' + (\mu + \varepsilon)x^+ - (\nu + \varepsilon)x^- = f(t) \tag{5}$$

has a periodic solution $x(t)$ with minimal period $2\pi p$ which is such that

$$|x(t)| + |x'(t)| \geq \Gamma,$$

for every $t \in \mathbb{R}$.

Proof. Let U be a bounded set like in Lemma 1. Fix $\Gamma > 0$ and the integers p, q such that the corresponding Fučík curve defined by (3) intersects U. Let μ, ν be such that $(\mu, \nu) \in U$ and satisfy (3). By Lemma 1, the function $\Phi_{\mu,\nu}$ is strictly positive. Moreover, if $\mu \neq \nu$, and $\sqrt{\mu}, \sqrt{\nu}$ are irrational, and if the Fourier development of f for the period $2\pi p$ contains harmonics of orders multiple of q, by Lemma 3, $\Phi^{(p)}_{\mu,\nu}$ is nonconstant. Corollary 2 can be applied to provide, for $\varepsilon > 0$ small enough (depending on p, q), a solution x of (5) which is near a large multiple of $\varphi_{\mu,\nu}$. The minimality of the period follows from Lemma 2, if ε is taken small enough. □

Remark 5. Since f is assumed to be 2π-periodic, the Fourier development of f for the period $2\pi p$ has nonzero coefficients only for those indices corresponding to a multiple of p. In the following, when speaking of the Fourier coefficients of f, we mean those of the Fourier development of f for the period 2π.

Corollary 3. *Let $\bar{\mu}, \bar{\nu}$ verify (1), and assume that $\Phi_{\bar{\mu},\bar{\nu}}$ is strictly positive, and that f has minimal period 2π and nonzero Fourier coefficients of arbitrarily high order. Then, there is a neighborhood V of $(\bar{\mu}, \bar{\nu})$ with the following property. For every $\Gamma > 0$ and any positive integer $\bar{p}$, there is a set D, which is dense in V, such that, for every $(\alpha, \beta) \in D$, the equation*

$$x'' + \alpha x^+ - \beta x^- = f(t) \tag{6}$$

has a periodic solution x whose minimal period is greater than $2\pi\bar{p}$ and which is such that

$$|x(t)| + |x'(t)| \geq \Gamma,$$

for every $t \in \mathbb{R}$.

Proof. Let V be the neighborhood determined in Lemma 4. Fix $\Gamma > 0$ and $\bar{p} \geq 1$. The set

$$\mathcal{Q} = \left\{ \frac{p}{q} : p \geq \bar{p}, f \text{ has a harmonic of order } q, p \text{ and } q \text{ are relatively prime} \right\}$$

is dense in $\mathbb{R}^+$. Let D_1 be the set of those (μ, ν) in V such that μ, ν satisfy (3), for some p, q for which $p/q \in \mathcal{Q}$. Clearly, D_1 is dense in V; it is made of a countable union of pieces of Fučík curves. Set now

$$D_2 := \{(\mu, \nu) \in D_1 : \mu \neq \nu \text{ and } \sqrt{\mu}, \sqrt{\nu} \text{ are irrational}\}.$$

The set D_2 is obtained from D_1 by taking away a countable number of points from each piece of a Fučík curve, and hence is still dense in V. For any $(\mu, \nu) \in D_2$, consider p, q such that μ, ν satisfy (3) and $p/q \in \mathcal{Q}$. Then, the Fourier development of f for the period $2\pi p$ contains a harmonic of order pq. By Lemma 4, there is a $\varepsilon_{p,q} > 0$ such that, if $0 < \varepsilon \leq \varepsilon_{p,q}$, equation (5) has a solution x with minimal period $2\pi p$ and such that

$$|x(t)| + |x'(t)| \geq \Gamma,$$

for every $t \in \mathbb{R}$. Setting

$$D := \{(\mu + \varepsilon, \nu + \varepsilon) : (\mu, \nu) \in D_2, 0 < \varepsilon \leq \varepsilon_{p,q}\},$$

we have that D is dense in V and has the required properties. □

As an example of application, we have $f(t) = c - |t|$ on the interval $[-\pi, \pi]$, extended by 2π-periodicity on $\mathbb{R}$. It is easily seen that the assumptions of Corollary 3 are verified provided that $\bar{\mu} < \bar{\nu}$ and c is sufficiently large and positive.

Remark 6. We emphasize the fact that the set V in Corollary 3 is not chosen in dependence of the amplitude of the solutions. Remark also that subharmonic solutions for equation (6) have been already observed in [5, 6], taking as (α, β) a point below the first Fučík curve (corresponding to (1) with $n = 1$).

5 Bifurcations from infinity for almost periodic solutions

In this section, for some $\varepsilon_0 > 0$, the function $g : \mathbb{R}^3 \times [0, \varepsilon_0] \to \mathbb{R} : (t, x, y, \varepsilon) \mapsto g(t, x, y, \varepsilon)$ is continuous with continuous partial derivatives with respect to x, y, and $g(t, x, y, \varepsilon)$ is almost periodic in t, uniformly with respect to (x, y) in compact sets, for each fixed $\varepsilon \in]0, \varepsilon_0]$ (cf. [8]). Moreover, we make the following

Assumption $\tilde{H}$. *A continuous function* $G : \mathbb{R} \times \mathbb{R}_0 \times \mathbb{R}_0 \to \mathbb{R}$ *can be well defined by*

$$G(t, x, y) = \lim_{\varepsilon \to 0^+} g\left(t, \frac{x}{\varepsilon}, \frac{y}{\varepsilon}, \varepsilon\right),$$

the above limit being uniform for $t \in \mathbb{R}$ *and* (x, y) *in any compact subset of* $\mathbb{R}_0 \times \mathbb{R}_0$. *The function* G *satisfies the following conditions.*

- *G is almost periodic in t, uniformly with respect to (x, y) in compact sets of $\mathbb{R}_0 \times \mathbb{R}_0$.*

- *G has continuous partial derivatives with respect to x, y in $\mathbb{R}_0 \times \mathbb{R}_0$, and moreover*
$$\lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \frac{\partial g}{\partial x} \left(t, \frac{x}{\varepsilon}, \frac{y}{\varepsilon}, \varepsilon\right) = \frac{\partial G}{\partial x}(t, x, y),$$
and
$$\lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \frac{\partial g}{\partial y} \left(t, \frac{x}{\varepsilon}, \frac{y}{\varepsilon}, \varepsilon\right) = \frac{\partial G}{\partial y}(t, x, y),$$
uniformly for $t \in \mathbb{R}$ and (x, y) in any compact subset of $\mathbb{R}_0 \times \mathbb{R}_0$.

We fix $\mu > 0$, $\nu > 0$ (which do not need to satisfy (1)), and denote by φ the solution of $(\mathrm{I}_{\mu,\nu})$ (recall that φ is periodic with minimal period $\pi(1/\sqrt{\mu} + 1/\sqrt{\nu})$). We define the function $\Psi : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^2$, $\Psi = (\Psi_1, \Psi_2)$, by

$$\Psi_1(\rho, \theta) = \lim_{T \to \infty} \frac{1}{T} \int_0^T G(t, \rho\varphi(t + \theta), \rho\varphi'(t + \theta))\, \varphi'(t + \theta)\, dt,$$
$$\Psi_2(\rho, \theta) = -\lim_{T \to \infty} \frac{1}{\rho T} \int_0^T G(t, \rho\varphi(t + \theta), \rho\varphi'(t + \theta))\, \varphi(t + \theta)\, dt.$$

It can be seen that Ψ is well defined and continuous on $\mathbb{R}^+ \times \mathbb{R}$, even if $G(t, x, y)$ is not defined for $x = 0$ or $y = 0$.

Theorem 2. *Assume that there are $\rho^* \in \mathbb{R}^+$, $\theta^* \in \mathbb{R}$ for which $\Psi(\rho^*, \theta^*) = (0, 0)$ and the jacobian matrix $J_\Psi(\rho^*, \theta^*)$ has eigenvalues with non zero real parts. There is a $\varepsilon^* \in]0, \varepsilon_0]$ such that, if $0 < \varepsilon \le \varepsilon^*$, then the equation*
$$x'' + \mu x^+ - \nu x^- = g(t, x, x', \varepsilon)$$
has an almost periodic solution $x(t; \varepsilon)$ of the form
$$(x(t; \varepsilon), x'(t; \varepsilon)) = \frac{1}{\varepsilon}\rho(t; \varepsilon)(\varphi(t + \theta(t; \varepsilon)), \varphi'(t + \theta(t; \varepsilon))),$$
the functions $\rho(t; \varepsilon), \theta(t; \varepsilon)$ being almost periodic in t and such that
$$\lim_{\varepsilon \to 0} \theta(t; \varepsilon) = \theta^*, \quad \lim_{\varepsilon \to 0} \rho(t; \varepsilon) = \rho^*.$$
If both eigenvalues of $J_\Psi(\rho^*, \theta^*)$ have negative real parts, then the solution $x(t; \varepsilon)$ is asymptotically stable. On the contrary, if one or both eigenvalues have positive real parts, then the solution is unstable.

Proof. As in the proof of Theorem 1, the change of variables

$$x = \frac{1}{\varepsilon}\,\rho\varphi(t+\theta), \quad x' = \frac{1}{\varepsilon}\,\rho\varphi'(t+\theta)$$

transforms the differential equation in $(P_{\mu,\nu,\varepsilon})$ into

$$\begin{cases} \rho' = \varepsilon g(t, \frac{\rho}{\varepsilon}\,\varphi(t+\theta), \frac{\rho}{\varepsilon}\,\varphi'(t+\theta), \varepsilon)\varphi'(t+\theta) \\ \theta' = -\frac{\varepsilon}{\rho}\,g(t, \frac{\rho}{\varepsilon}\,\varphi(t+\theta), \frac{\rho}{\varepsilon}\,\varphi'(t+\theta), \varepsilon)\varphi(t+\theta), \end{cases} \qquad (\odot)$$

as soon as $\rho > 0$. By Assumption $\tilde{H}$, we are in the hypotheses of a theorem on almost periodic solutions due to Hale ([8] Theorem 3.1, p.194), which gives us the conclusion (since G is continuous only on $\mathbb{R} \times \mathbb{R}_0 \times \mathbb{R}_0$, some caution is needed; we use the fact that $\rho(t) \to \rho^*$ when $\varepsilon \to 0$, uniformly in $\mathbb{R}$). □

As in Section 3, we now consider two corollaries of the above general theorem. Let $f : \mathbb{R} \to \mathbb{R}$ be an almost periodic and continuous function, and define

$$\Phi(\theta) = \lim_{T\to\infty} \frac{1}{T} \int_0^T f(t)\varphi(t+\theta)\,dt.$$

Corollary 4. *Let θ^* be a simple zero of Φ. There is a $\varepsilon^* > 0$ such that, if $|\varepsilon| \le \varepsilon^*$ and $\varepsilon\Phi'(\theta^*) > 0$, then the equation*

$$x'' + \varepsilon x' + \mu x^+ - \nu x^- = f(t)$$

has an almost periodic solution $x(t;\varepsilon)$ of the form

$$(x(t;\varepsilon), x'(t;\varepsilon)) = \frac{1}{|\varepsilon|}\rho(t;\varepsilon)(\varphi(t+\theta(t;\varepsilon)), \varphi'(t+\theta(t;\varepsilon))),$$

the functions $\rho(t;\varepsilon), \theta(t;\varepsilon)$ being almost periodic in t and such that

$$\lim_{\varepsilon\to 0} \theta(t;\varepsilon) = \theta^*, \quad \lim_{\varepsilon\to 0} \rho(t;\varepsilon) = \rho^* := 2|\Phi'(\theta^*)|.$$

If $\Phi'(\theta^*) > 0$ (and hence $\varepsilon > 0$), this solution is asymptotically stable; otherwise, it is unstable.

Proof. Consider the case $\varepsilon > 0$. We apply Theorem 2 with $g(t,x,y,\varepsilon) = f(t) - \varepsilon y$, which corresponds to $G(t,x,y) = f(t) - y$. Since

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T [\varphi'(t+\theta)]^2\,dt = \left(\frac{\pi}{\sqrt{\mu}} + \frac{\pi}{\sqrt{\nu}}\right)^{-1} \int_0^{\frac{\pi}{\sqrt{\mu}}+\frac{\pi}{\sqrt{\nu}}} [\varphi']^2 = \frac{1}{2},$$

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \varphi'(t+\theta)\varphi(t+\theta)\,dt = \left(\frac{\pi}{\sqrt{\mu}} + \frac{\pi}{\sqrt{\nu}}\right)^{-1} \int_0^{\frac{\pi}{\sqrt{\mu}}+\frac{\pi}{\sqrt{\nu}}} \varphi'\varphi = 0,$$

we have

$$\Psi(\rho,\theta) = \left(\Phi'(\theta) - \frac{\rho}{2}, -\frac{1}{\rho}\Phi(\theta)\right).$$

Notice that $\Psi(\rho^*,\theta^*) = (0,0)$; moreover, the jacobian matrix at (ρ^*,θ^*),

$$J_\Psi(\rho^*,\theta^*) = \begin{pmatrix} -\frac{1}{2} & \Phi''(\theta^*) \\ 0 & -\frac{\Phi'(\theta^*)}{\rho^*} \end{pmatrix},$$

has two negative eigenvalues. Theorem 2 completes the proof. □

As an example, consider the case $f(t) = \cos(jt) + \cos(\sqrt{2}t)$. If μ,ν verify (1) with $\mu \neq \nu$, and j is a multiple of n, then we are in the situation where Corollary 4 applies. Notice that, when $\mu = \nu$ and $j \neq 1$, similar solutions of large amplitude do not exist.

Corollary 5. *Let θ^* be such that*

$$\Phi'(\theta^*) = 0 \quad \text{and} \quad \Phi(\theta^*)\Phi''(\theta^*) > 0.$$

There is a $\varepsilon^* \in]0,\varepsilon_0]$ such that, if $|\varepsilon| \leq \varepsilon^*$ and $\varepsilon\Phi(\theta^*) > 0$, then the equation

$$x'' + (\mu+\varepsilon)x^+ - (\nu+\varepsilon)x^- = f(t)$$

has an unstable almost periodic solution $x(t;\varepsilon)$ of the form

$$(x(t;\varepsilon), x'(t;\varepsilon)) = \frac{1}{|\varepsilon|}\rho(t;\varepsilon)(\varphi(t+\theta(t;\varepsilon)), \varphi'(t+\theta(t;\varepsilon))),$$

the functions $\rho(t;\varepsilon),\theta(t;\varepsilon)$ being almost periodic in t and such that

$$\lim_{\varepsilon\to 0}\theta(t;\varepsilon) = \theta^*, \quad \lim_{\varepsilon\to 0}\rho(t;\varepsilon) = \rho^* := \frac{2\mu\nu}{|\Phi(\theta^*)|}\frac{\sqrt{\mu}+\sqrt{\nu}}{\sqrt{\mu^3}+\sqrt{\nu^3}}.$$

Proof. We consider the case $\varepsilon > 0$ and apply Theorem 2 with $g(t,x,y,\varepsilon) = f(t) - \varepsilon x$, which corresponds to $G(t,x,y) = f(t) - x$. Since

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T [\varphi(t+\theta)]^2\,dt = \left(\frac{\pi}{\sqrt{\mu}} + \frac{\pi}{\sqrt{\nu}}\right)^{-1}\int_0^{\frac{\pi}{\sqrt{\mu}}+\frac{\pi}{\sqrt{\nu}}}[\varphi]^2$$
$$= \frac{1}{2\mu\nu}\frac{\sqrt{\mu^3}+\sqrt{\nu^3}}{\sqrt{\mu}+\sqrt{\nu}},$$

we have

$$\Psi(\rho,\theta) = \left(\Phi'(\theta), \frac{1}{2\mu\nu}\frac{\sqrt{\mu^3}+\sqrt{\nu^3}}{\sqrt{\mu}+\sqrt{\nu}} - \frac{1}{\rho}\Phi(\theta)\right).$$

Then, $\Psi(\rho^*, \theta^*) = (0,0)$; moreover, since $\Phi'(\theta^*) = 0$, the jacobian matrix at (ρ^*, θ^*),

$$J_\Psi(\rho^*, \theta^*) = \begin{pmatrix} 0 & \Phi''(\theta^*) \\ \frac{\Phi(\theta^*)}{(\rho^*)^2} & 0 \end{pmatrix},$$

has a positive and a negative eigenvalue. Theorem 2 completes the proof. □

Notice that the conditions of Corollary 5 cannot be satisfied when $\mu = \nu$, since $\Phi(\theta)$ is then of the form $A\cos(\sqrt{\mu}\theta) + B\sin(\sqrt{\mu}\theta)$. Notice also the differences between Corollaries 2 and 5 : for $\varepsilon > 0$, Corollary 5 provides only solutions corresponding to local positive minima of Φ, whereas Corollary 2, which deals with periodic solutions, provides solutions corresponding to maxima as well as to minima.

As an example of application, consider the case

$$f(t) = c + \cos(jt) + \cos(\sqrt{2}t).$$

If μ, ν verify (1) with $\mu < \nu$, and j is a multiple of n, we are in the situation where Corollary 5 applies, provided that εc is positive and $|c|$ is sufficiently large.

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