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BOUNDED SOLUTIONS OF NONLINEAR SECOND ORDER ORDINARY DIFFERENTIAL EQUATIONS

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To the memory of Mark Aleksandrovič Krasnosel'skiĭ

Abstract. We prove the existence of bounded solutions to second order differential equations of Liénard type under asymptotic conditions generalizing recent results of Ahmad and Ortega.

1. Introduction. In this paper we are concerned with bounded solutions of the Liénard equation

$$x'' + f(x)x' + g(x) = e(t). \tag{1.1}$$

This is a classical topic in the theory of ODE's. It has been widely investigated, mainly during the period from the fourties to the seventies, by the use of phase-plane methods, for equations which generalize the linear case $f(x) = c$, $g(x) = bx$, with $b, c > 0$ (see, e.g., [14], [12], [3], [2] and the references therein). The main assumptions on the forcing term usually required the boundedness, either of $e(t)$, or of its primitive.

These results, in particular, provide the existence of T-periodic solutions of (1.1), in the case of a T-periodic forcing, by the use of Massera's theorem [6]. In the meanwhile, functional-analytic approaches were also developed to consider different situations in which periodic solutions can arise. With this respect, we recall the pioneering works by Lazer [5] for the equation

$$x'' + cx' + g(x) = e(t), \tag{1.2}$$

and Mawhin [7] for equation (1.1). In this setting, a typical condition on the restoring term $g(x)$ is the following:

$$\limsup_{x\to-\infty} g(x) < \bar{e} < \liminf_{x\to+\infty} g(x)$$

(also referred as a Landesman - Lazer type condition), where $\bar{e}$ is the mean value of the T-periodic forcing $e(t)$:

$$\bar{e} := \frac{1}{T}\int_0^T e(t)\,dt.$$

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Recently, starting with Ahmad [1], there has been a renewed interest in the boundedness of the solutions of equation (1.2), under suitable modifications of the Landesman - Lazer condition. In [1], the forcing term $e(t)$ is assumed to be *bounded* and such that there exists a real number e_0 for which

$$\lim_{\tau\to+\infty} \frac{1}{\tau} \int_a^{a+\tau} e(t)\, dt = e_0$$

holds uniformly with respect to $a \in \mathbb{R}$. The (possibly unbounded) restoring term $g(x)$ is assumed to be such that the limits

$$g(-\infty) := \lim_{x\to-\infty} g(x), \quad g(+\infty) := \lim_{x\to+\infty} g(x)$$

exist and satisfy

$$g(-\infty) < e_0 < g(+\infty). \tag{1.3}$$

Under these hypotheses it is proved that equation (1.2), with $c > 0$, has a bounded solution, i.e., a solution $x(\cdot)$ such that

$$\sup_{t\in\mathbb{R}} \{\, |x(t)| + |x'(t)| \,\} < +\infty.$$

An improvement to condition (1.3) has been obtained by Ortega [10] and Ortega - Tineo [11]. In [11], the following condition is introduced:

$$\limsup_{x\to-\infty} g(x) < \hat{e} \le \check{e} < \liminf_{x\to+\infty} g(x), \tag{1.4}$$

where

$$\hat{e} = \lim_{r\to+\infty} \inf_{v-u\ge r} \frac{1}{v-u} \int_u^v e(t)\, dt\,, \qquad \check{e} = \lim_{r\to+\infty} \sup_{v-u\ge r} \frac{1}{v-u} \int_u^v e(t)\, dt\,.$$

(It is clear that (1.4) generalizes (1.3).) Then, the existence of a bounded solution to equation (1.2) (and even for higher order equations) is obtained by assuming (1.4), $c > 0$ and $g(x)$ to be *bounded*. In [10], only one-sided boundedness of $g(x)$ is required, but a stronger form of (1.4) is needed. The general case of an unbounded $g(x)$ was left by Ortega as an open problem.

Let us focus our attention on condition (1.4). It can be seen (cf.[11]) that, if (1.4) holds, it is possible to make a splitting of the form

$$e(t) = e^*(t) + e^{**}(t)$$

with e^* having a bounded primitive and e^{**} satisfying

$$\limsup_{x\to-\infty} g(x) < \inf_{t\in\mathbb{R}} e^{**}(t) \le \sup_{t\in\mathbb{R}} e^{**}(t) < \liminf_{x\to+\infty} g(x).$$

The special case of $e(t)$ having a bounded primitive (i.e., $e^{**} \equiv 0$), had been already extensively studied in the past for equation (1.1) (see the previously quoted classical papers). In this case, the above results can be deduced from some general theorems on equation (1.1), using quite mild dissipativity conditions on the function f, like, for instance,

$$\lim_{x\to\pm\infty} \int_0^x f(\xi)\, d\xi = \pm\infty \tag{1.5}$$

(see, e.g., [13], [2]).

At this point, one could ask if bounded solutions for equation (1.1) exist, assuming (1.4) alone for g and e, and a condition like (1.5) for f. In particular, it could be of some interest to see whether the classical phase-plane methods can be adapted to generalize the results in [1] and [10].

In this paper, we obtain some results in this direction, for a more general planar system (see Section 2). As an example of applicability of our main theorem, let us consider the following.

Theorem 1. *Let $e, f, g : \mathbb{R} \to \mathbb{R}$ be continuous functions. Assume* (1.4) *and let*

$$\lim_{x\to\pm\infty} |x|^{-1/2} \int_0^x f(\xi)\, d\xi = \pm\infty. \tag{1.6}$$

Then there exists a bounded solution of equation (1.1).

Notice that (1.6) is clearly satisfied in the case of equation (1.2) with $c > 0$. Concerning assumption (1.4), observe that we need no extra boundedness conditions on $g(x)$ or $e(t)$. Thus, Theorem 1 extends the existence results in [1] and [10], providing a positive answer to the problem raised by Ortega [10].

We do not know whether condition (1.6) could be replaced by (1.5). However, we provide variants of Theorem 1 in which we are able to relax (1.6) (and even (1.5)) at the expense of requiring some extra conditions on the function g. The possibility of one-sided growth restrictions is also discussed.

2. Main results.

We consider the planar system

$$x' = y - h_1(x) + e_1(t), \qquad y' = -h_2(x) + e_2(t), \tag{P}$$

where, for $i = 1, 2$, $h_i : \mathbb{R} \to \mathbb{R}$ are continuous functions, and $e_i : \mathbb{R} \to \mathbb{R}$ are continuous and bounded: we set

$$e_i := \sup_{t\in\mathbb{R}} |e_i(t)|, \qquad (i = 1, 2),$$

and define $H_2(x) = \int_0^x h_2(\xi)\, d\xi$. We make the following

Assumption A. There exist positive constants d, δ such that, for $i = 1, 2$,

$$\operatorname{sgn}(x) h_i(x) \geq e_i + 2\delta, \qquad \text{for every} \qquad |x| \geq d.$$

It seems to be an interesting open problem whether Assumption A alone is sufficient for the existence of bounded solutions to system (P).

Definition. *The solutions of (P) are uniformly ultimately bounded if there is a compact set K in $\mathbb{R}^2$ such that, for every solution (x, y) of (P), there is a time T such that $(x(t), y(t)) \in K$ for every $t \geq T$.*

The following result will be proved in Section 3.

Theorem 2. *Besides Assumption A, assume one of the following two conditions: either*

$$\lim_{x\to+\infty} \frac{h_1(x)^2}{H_2(x)} = +\infty, \tag{2.1}$$

or

$$\lim_{x\to\pm\infty} \frac{h_1(x)^2 + \sqrt{H_2(x)}}{|x|} = +\infty. \tag{2.2}$$

Then, the solutions of (P) are uniformly ultimately bounded, and there is at least one solution of (P) which is bounded on $\mathbb{R}$.

We now discuss some applications of Theorem 2 to equation (1.1). Let $e : \mathbb{R} \to \mathbb{R}$ be the sum of a function with bounded primitive and a function which is bounded and continuous, and $f, g : \mathbb{R} \to \mathbb{R}$ be continuous. Define $F(x) = \int_0^x f(\xi)\, d\xi$ and $G(x) = \int_0^x g(\xi)\, d\xi$. We will assume (1.4) and that the following expression holds:

$$\liminf_{x\to+\infty} F(x) - \limsup_{x\to-\infty} F(x) = +\infty. \tag{2.3}$$

Set $\bar{e} = \frac{1}{2}(\hat{e} + \check{e})$, and let $\eta > 0$ be such that

$$\limsup_{x\to-\infty} g(x) < \bar{e} - \eta < \hat{e} \le \check{e} < \bar{e} + \eta < \liminf_{x\to+\infty} g(x).$$

By Lemma 3 in [11], it is possible to write $e(t) = e^*(t) + e^{**}(t)$, with $e^*(t)$ having bounded primitive, and $e^{**}(t)$ continuous such that

$$\bar{e} - \eta < e^{**}(t) < \bar{e} + \eta, \quad \text{for every } t \in I\!R.$$

Then, chosing appropriately a constant $\bar{c} \in I\!R$ and defining

$$\begin{aligned} h_1(x) &= F(x) + \bar{c}, h_2(x) &&= g(x) - \bar{e}, \\ e_1(t) &= \textstyle\int_0^t e^*(s)\,ds, e_2(t) &&= e^{**}(t) - \bar{e}, \end{aligned}$$

(1.1) can be written in the form of problem (P) and Assumption A is readily verified.

As an immediate consequence of Theorem 2, we then have the following.

Theorem 3. *Besides (1.4) and (2.3), assume one of the following two conditions: either*

$$\lim_{x\to+\infty} \frac{F(x)^2}{G(x) - \bar{e}x} = +\infty, \tag{2.4}$$

or

$$\lim_{x\to\pm\infty} \frac{F(x)^2 + \sqrt{|G(x)|}}{|x|} = +\infty. \tag{2.5}$$

Then, the solutions of (1.1) are uniformly ultimately bounded, and there is at least one solution of (1.1) which is bounded on $I\!R$.

Notice that condition (2.5) is in particular satisfied if (1.6) holds, showing that Theorem 1 is a consequence of Theorem 3.

Another situation when condition (2.5) is satisfied is when

$$\lim_{x\to\pm\infty} \frac{G(x)}{x^2} = +\infty. \tag{2.6}$$

One could also assume (1.6) only at $+\infty$ together with (2.6) only at $-\infty$, or viceversa.

On the other hand, condition (2.4) is satisfied, for instance, if (1.6) holds only at $+\infty$ and the function g is bounded on $I\!R^+$. Other variants are omitted for briefness.

3. Proof of Theorem 2. The proof is based on the construction of a family of compact positively invariant sets containing the origin. Using a classical approach, the boundary of each of these sets will be the union of a finite number of simple curves.

Let us start from a point $P_1 = (-d, y_1)$, where $y_1 > 0$ will be fixed later. Define the point $P_2 = (d, y_1 + \delta)$, and let ℓ_1 be the segment joining P_1 and P_2, with outward normal $(-\delta, 2d)$. Along ℓ_1, we have

$$\begin{aligned} (-\delta, 2d)\cdot(x', y') &= -\delta(y - h_1(x) + e_1(t)) + 2d(-h_2(x) + e_2(t)) \\ &\le -\delta\left(y_1 - \max_{|x|\le d}|h_1(x)| - e_1\right) + 2d\left(\max_{|x|\le d}|h_2(x)| + e_2\right) \\ &< 0, \end{aligned}$$

provided y_1 is taken large enough.

Let us now consider the function

$$V(x,y) = \frac{1}{2}(y - \delta)^2 + H_2(x) - e_2(x - d).$$

If $x \geq d$ and $y \geq h_1(x) - e_1 - \delta$, then

$$\begin{aligned} V' &= (y-\delta)y' + (h_2(x) - e_2)x' \\ &= (y-\delta)(-h_2(x) + e_2(t)) + (h_2(x) - e_2)(y - h_1(x) + e_1(t)) \\ &= (y-\delta)(e_2(t) - e_2) + (h_2(x) - e_2)(\delta - h_1(x) + e_1(t)) \\ &< 0. \end{aligned}$$

Hence, V is a Lyapunov function for $x \geq d$ and $y \geq h_1(x) - e_1 - \delta$. We follow the curve $V(x,y) = V(P_2)$ until we reach the curve $y = h_1(x) - e_1 - \delta$. Let us call $P_3 = (x_3, h_1(x_3) - e_1 - \delta)$ the first intersection point, so that

$$\frac{1}{2}(h_1(x_3) - e_1 - 2\delta)^2 + H_2(x_3) - e_2(x_3 - d) = \frac{1}{2}y_1^2 + H_2(d), \tag{3.1}$$

and call ℓ_2 this piece of curve joining P_2 and P_3. Define now the point $P_4 = (x_3, \delta)$, and let ℓ_3 be the segment joining P_3 and P_4, with outward normal $(1,0)$. Notice that along ℓ_3 we have

$$\begin{aligned} (1,0)\cdot(x',y') &= y - h_1(x_3) + e_1(t) \\ &\leq (h_1(x_3) - e_1 - \delta) - h_1(x_3) + e_1 \\ &< 0. \end{aligned}$$

Consider now the function $W(x,y) = \frac{1}{2}(y-\delta)^2 + H_2(x) + e_2(x-d)$. If $x \geq d$ and $y \leq \delta$, then

$$\begin{aligned} W' &= (y-\delta)y' + (h_2(x) + e_2)x' \\ &= (y-\delta)(-h_2(x) + e_2(t)) + (h_2(x) + e_2)(y - h_1(x) + e_1(t)) \\ &= (y-\delta)(e_2(t) + e_2) + (h_2(x) + e_2)(\delta - h_1(x) + e_1(t)) \\ &< 0. \end{aligned}$$

Hence, W is a Lyapunov function for $x \geq d$ and $y \leq \delta$. We follow the curve $W(x,y) = W(P_4)$ until we reach the line $x = d$. Let $P_5 = (d, y_5)$ be the intersection point, so that

$$\frac{1}{2}(y_5 - \delta)^2 + H_2(d) = H_2(x_3) + e_2(x_3 - d), \tag{3.2}$$

and call ℓ_4 this piece of curve joining P_4 and P_5.

We now consider two different situations, which will make us use either (2.1) or (2.2).

We first assume (2.1), and construct the walk $\tilde{P}_1\tilde{P}_2\tilde{P}_3\tilde{P}_4\tilde{P}_5$ by chosing $\tilde{P}_1 = P_5$ and proceeding in the construction analogously as for $P_1P_2P_3P_4P_5$ (see figure 1). In this case, the Lyapunov functions will be

$$\tilde{V}(x,y) = \frac{1}{2}(y+\delta)^2 + H_2(x) + e_2(x+d),$$

and

$$\tilde{W}(x,y) = \frac{1}{2}(y+\delta)^2 + H_2(x) - e_2(x+d).$$

From (3.1) we have

$$y_1 = \sqrt{(h_1(x_3) - e_1 - 2\delta)^2 + 2(H_2(x_3) - H_2(d) - e_2(x_3 - d))}, \tag{3.3}$$

while from (3.2) we get

$$y_5 = \delta - \sqrt{2(H_2(x_3) - H_2(d) + e_2(x_3 - d))}. \tag{3.4}$$

Analogously, we find

$$\tilde{y}_1 = -\sqrt{(h_1(\tilde{x}_3) + e_1 + 2\delta)^2 + 2(H_2(\tilde{x}_3) - H_2(-d) + e_2(\tilde{x}_3 + d))}, \tag{3.5}$$

with $\tilde{y}_1 = y_5$, and

$$\tilde{y}_5 = -\delta + \sqrt{2(H_2(\tilde{x}_3) - H_2(-d) - e_2(\tilde{x}_3 + d))}. \tag{3.6}$$

Notice that, if $y_1 \to +\infty$, then $x_3 \to +\infty$, and consequently $y_5 \to -\infty$, $\tilde{x}_3 \to -\infty$, and $\tilde{y}_5 \to +\infty$. We want to show that, assuming (2.1), if y_1 is sufficiently large, then $\tilde{y}_5 \leq y_1$. Using (3.4) and (3.5), being $\tilde{y}_1 = y_5$, we find

$$H_2(x_3) - H_2(d) + e_2(x_3 - d) \geq H_2(\tilde{x}_3) - H_2(-d) + e_2(\tilde{x}_3 + d). \tag{3.7}$$

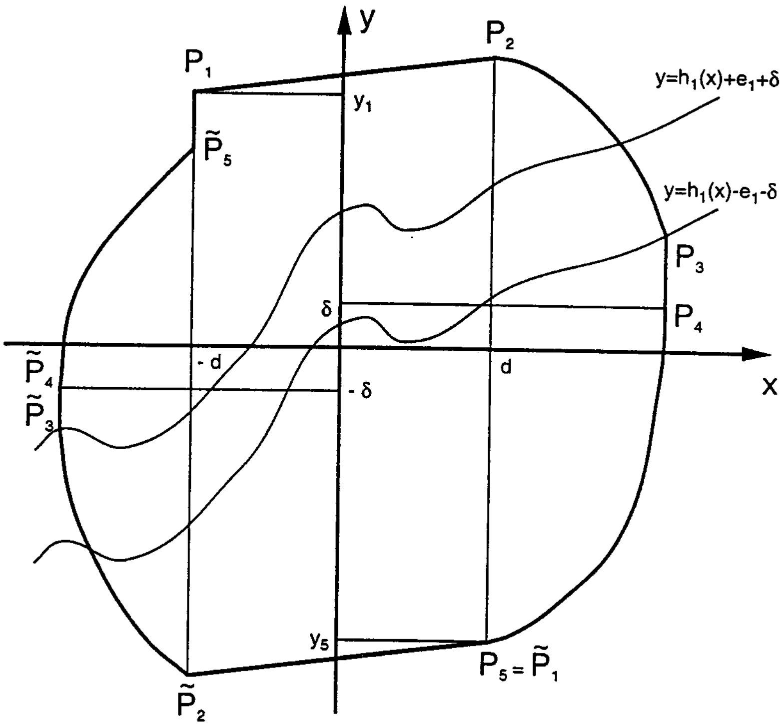

Figure 1

By Assumption A, we have

$$H_2(x_3) - H_2(d) \geq (e_2 + 2\delta)(x_3 - d), \tag{3.8}$$

and, on the other hand,

$$H_2(\tilde{x}_3) - H_2(-d) + e_2(\tilde{x}_3 + d) \geq \frac{\delta}{e_2 + \delta}(H_2(\tilde{x}_3) - H_2(-d) - e_2(\tilde{x}_3 + d)) \tag{3.9}$$

By contradiction, assume $\tilde{y}_5 > y_1$. Then, from (3.3) and (3.6), we have:

$$\begin{aligned} 0 > y_1^2 - (\tilde{y}_5 + \delta)^2 \geq & (h_1(x_3) - e_1 - 2\delta)^2 + \\ & + 2(H_2(x_3) - H_2(d) - e_2(x_3 - d)) - 2(H_2(\tilde{x}_3) - H_2(-d) - e_2(\tilde{x}_3 + d)), \end{aligned}$$

which, together with (3.7), (3.8) and (3.9), gives

$$0 > (h_1(x_3) - e_1 - 2\delta)^2 - \frac{4e_2}{\delta}(H_2(x_3) - H_2(d)).$$

We then reach a contradiction with (2.1) when x_3 is large enough.

Along the segment joining $\tilde{P}_5$ and P_1, with outward normal $(-1,0)$, one has

$$(-1,0)\cdot(x',y') = -y + h_1(-d) - e_1(t) \leq -\tilde{y}_5 + h_1(-d) + e_1 < 0.$$

Hence, the region bounded by the closed curve $P_1P_2P_3P_4P_5\tilde{P}_2\tilde{P}_3\tilde{P}_4\tilde{P}_5P_1$ is positively invariant.

We now assume (2.2) and show that $|y_5| \leq y_1$, provided y_1 is chosen sufficiently large. Assume by contradiction that $|y_5| > y_1$, that is

$$\sqrt{2(H_2(x_3) - H_2(d) + e_2(x_3 - d))} > \\ > \delta + \sqrt{(h_1(x_3) - e_1 - 2\delta)^2 + 2(H_2(x_3) - H_2(d) - e_2(x_3 - d))}.$$

Sqaring both sides, we have

$$(h_1(x_3) - e_1 - 2\delta)^2 - 4e_2(x_3 - d) + \delta^2 + \\ +2\delta\sqrt{(h_1(x_3) - e_1 - 2\delta)^2 + 2(H_2(x_3) - H_2(d) - e_2(x_3 - d))} < 0.$$

From Assumption A, we easily see that, for some positive constants c_1 and c_2,

$$(h_1(x_3) - e_1 - 2\delta)^2 - 4e_2(x_3 - d) + \delta^2 + \\ +2\delta\sqrt{(h_1(x_3) - e_1 - 2\delta)^2 + 2(H_2(x_3) - H_2(d) - e_2(x_3 - d))} \geq \\ \geq c_1\left(h_1(x_3)^2 + \sqrt{H_2(x_3)}\right) - 4e_2x_3 - c_2$$

(remember that, if $y_1 \to +\infty$, then $x_3 \to +\infty$). Using (2.2), we get the contradiction if y_1 is large enough.

We consider the walk $\tilde{P}_1\tilde{P}_2\tilde{P}_3\tilde{P}_4\tilde{P}_5$, analogously as above, with $\tilde{P}_1 = P_5$ (see figure 1). By the symmetry of the hypotheses, one proves that $\tilde{y}_5 \leq |\tilde{y}_1|$. Hence, being $\tilde{y}_1 = y_5$, one has $\tilde{y}_5 \leq y_1$, so that the region bounded by the closed curve $P_1P_2P_3P_4P_5\tilde{P}_2\tilde{P}_3\tilde{P}_4\tilde{P}_5P_1$ is positively invariant.

Now we prove that the solutions are uniformly ultimately bounded. Consider, by chosing y_1 arbitrarily but sufficiently large, all the curves which are constructed as above. Following [2], we call them $C-$curves. We want to show that one of these curves bounds a region which is entered by any solution of (P). By contradiction, let (x,y) be a solution which does not enter a $C-$curve C_1. In this case, it has to stay in the region bounded by two such curves, C_1, and a larger curve C_2, by the invariance proved above. Hence, the solution turns around the origin an infinite number of times, and in doing so, crosses the positive $y-$axis at some points $(0, w_n)$. There exists a $\bar{w} > 0$ which is a cluster point for the w_n's. Consider the $C-$curve C_3, between C_1 and C_2, which passes through the point $\bar{P} = (0, \bar{w})$. At the point $\bar{P}$, if C_1 is sufficiently large, we have $|dy/dx| < \delta/4d$, and the inequality extends to a neighborhood of $\bar{P}$. This implies that, taking n sufficiently large, the solution, after passing through $(0, w_n)$, has to enter in the interior of the region bounded by C_3. Then, the solution crosses another $C-$curve C_4, between C_1 and C_3, and remains inside it, in contradiction with the fact that $\bar{w}$ is a cluster point.

In order to prove that there is a solution of (P) which is bounded on $\mathbb{R}$, we proceed like in [8, Lemma 4]. We give a sketch of the argument, for the reader's convenience. Let $K_0 \subset \mathbb{R}^2$ be a compact positively invariant set containing the origin. Consider, for each integer n, a solution (x_n, y_n) of (P) satisfying the initial condition $x_n(-n) = 0$, $y_n(-n) = 0$. The sequence $(x_n, y_n)_{n\geq 1}$, restricted to the interval $[-1,1]$, is uniformly equicontinuous and bounded. By Ascoli-Arzelà's theorem, there is a subsequence $(x_n^{(1)}, y_n^{(1)})_{n\geq 1}$ converging uniformly on $[-1,1]$. Similarly, the sequence $(x_n^{(1)}, y_n^{(1)})_{n\geq 2}$, restricted to the interval $[-2,2]$, is uniformly equicontinuous and bounded, and there is a subsequence $(x_n^{(2)}, y_n^{(2)})_{n\geq 2}$ converging uniformly on $[-2,2]$. In this way, for every $k \geq 1$, we construct a subsequence $(x_n^{(k)}, y_n^{(k)})_{n\geq k}$ which converges uniformly on $[-k,k]$. The diagonal sequence $(x_n^{(n)}, y_n^{(n)})_{n\geq 1}$ converges uniformly on every compact subset of $\mathbb{R}$, to a solution (x,y) of (P) which is such that $(x(t), y(t)) \in K_0$ for all $t \in \mathbb{R}$. This completes the proof of Theorem 2.

Remark. In the proof of Theorem 2 we have constructed a family of compact positively invariant sets. Each of these sets has a Jordan curve as boundary, and, therefore (by classical results on plane topology), has the fixed point property. This fact implies the existence of at least one T-periodic solution for system (P), when $e_1(t)$ and $e_2(t)$ are T-periodic functions. Alternatively, one could obtain periodic solutions as an application of Massera's theorem [6], using the uniform ultimate boundedness of the solutions.

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