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Nonlinear Resonance in Asymmetric Oscillators

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We study the periodic solutions of equations with asymmetric nonlinearities "at resonance" with the Fučík spectrum. We compute the associated topological degree and prove existence, multiplicity, and stability of large-amplitude oscillations for equations with a small friction term. Such equations can be viewed, e.g., as simple models for investigating vertical oscillations of long-span suspension bridges. The results are typically of a nonlinear nature, as some of the situations observed cannot occur with a linear equation. © 1998 Academic Press

1. INTRODUCTION

Consider the periodic problem

$$(P) \qquad \begin{cases} x'' + \mu x^+ - \nu x^- = g(t, x) \\ x(0) - x(2\pi) = 0 = x'(0) - x'(2\pi), \end{cases}$$

where $x^+ = \max\{x, 0\}$ is the positive part of x, and $x^- = \max\{-x, 0\}$ is its negative part; we will assume throughout μ, ν to be positive real numbers such that there is a positive integer n for which

$$\frac{1}{\sqrt{\mu}} + \frac{1}{\sqrt{\nu}} = \frac{2}{n}$$

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(i.e., the pair (μ, ν) belongs to the nth Fučík curve). We are in the situation where the homogeneous problem

$$(H) \qquad \begin{cases} x'' + \mu x^+ - \nu x^- = 0 \\ x(0) - x(2\pi) = 0 = x'(0) - x'(2\pi) \end{cases}$$

has nontrivial solutions. Indeed, consider the $2\pi/n-$periodic function φ which, on the interval $[0, 2\pi/n[$, is defined as follows:

$$\varphi(t) = \begin{cases} \dfrac{1}{\sqrt{\mu}} \sin(\sqrt{\mu}\, t) & \left(t \in \left[0, \dfrac{\pi}{\sqrt{\mu}}\right]\right), \\ -\dfrac{1}{\sqrt{\nu}} \sin\left(\sqrt{\nu}\left(t - \dfrac{\pi}{\sqrt{\mu}}\right)\right) & \left(t \in \left[\dfrac{\pi}{\sqrt{\mu}}, \dfrac{2\pi}{n}\right]\right). \end{cases}$$

Denoting by φ_θ the function

$$\varphi_\theta(t) = \varphi(t + \theta),$$

it can be seen that any nontrivial solution of (H) is of the form $x(t) = A\varphi_\theta(t)$, for some $A > 0$ and $\theta \in [0, 2\pi/n[$ (cf. [4]).

Concerning the function $g : [0, 2\pi] \times \mathbb{R} \to \mathbb{R}$, we assume it satisfies Carathéodory conditions, i.e., $g(\cdot, x)$ is measurable for any $x \in \mathbb{R}$, $g(t, \cdot)$ is continuous for a.e. $t \in [0, 2\pi]$, and we also assume g to be bounded by a function $h \in L^1(0, 2\pi)$:

$$|g(t, x)| \leqslant h(t) \qquad (x \in \mathbb{R}, \text{ a.e. } t \in [0, 2\pi]). \tag{1}$$

Problems like (P) arise in a variety of asymmetric mechanical devices. For example, in [6], Lazer and McKenna proposed these as simple models describing vertical oscillations of long-span suspension bridges.

As a particular case, we can consider a function $g(t, x) = f(t)$, independent of x. There are examples in the literature of such functions for which problem (P) has no solutions: In [1, 2] a function $f(t)$ is defined as the characteristic function of a small interval, while in [5], the case $f(t) = \cos(nt)$ is considered, requiring however that μ, ν be confined in the interval $](n-1)^2, (n+1)^2[$. In these two cases, problem (P) has no solutions; this will be related below to the fact that the $(2\pi/n)-$periodic function

$$\Phi(\theta) = \int_0^{2\pi} f(t)\, \varphi(t + \theta)\, dt$$

has exactly two simple zeros in the interval $[0, 2\pi/n[$. As a matter of fact, we will see that, when Φ has only simple zeros, this is the only case where the problem can have no solution.

On the other hand, when the function $\Phi(\theta)$ is of constant sign, it has been proved in [1, 2] (see also [3]) that problem (P) does have a solution. The sign condition on Φ can be interpreted as a Landesman–Lazer type of condition (see also [8] for results concerning the boundedness of solutions).

Notice that in the linear case $\mu = \nu$, only two possibilities can occur: either the function $\Phi(\theta)$ identically vanishes, or it has exactly two simple zeros in the interval $[0, 2\pi/n[$, in which case the problem has no solution.

In this paper we are mainly interested in the situation when the function $\Phi(\theta)$ has more than two zeros in the interval $[0, 2\pi/n[$. We then prove that, if all zeros are simple, problem (P) always has a solution. More precisely, if the number of such zeros is $2z$, we prove that the topological degree associated to problem (P) is equal to $1-z$.

Further, adding a linear damping term to the differential equation in (P), we show that when the damping coefficient is small and positive, there are z families of asymptotically stable solutions having large amplitude.

In the last section, we consider some concrete examples to which our theory applies.

2. COMPUTING OF THE DEGREE

In the setting described in the introduction, we consider the four L^1-functions $\gamma_\pm$, $\Gamma_\pm$, defined by

$$\liminf_{x \to \pm\infty} g(t, x) = \gamma_\pm(t), \qquad \limsup_{x \to \pm\infty} g(t, x) = \Gamma_\pm(t).$$

Given two functions $\alpha, \beta \in L^1(0, 2\pi)$, satisfying $\gamma_- \leqslant \alpha \leqslant \Gamma_-$, $\gamma_+ \leqslant \beta \leqslant \Gamma_+$, we need to consider the function $\Phi_{\alpha,\beta}$ defined by

$$\Phi_{\alpha,\beta}(\theta) := \int_0^{2\pi} (\beta(t)\, \varphi_\theta^+(t) - \alpha(t)\, \varphi_\theta^-(t))\, dt.$$

Equivalently, we could write

$$\Phi_{\alpha,\beta}(\theta) = \int_{\varphi_\theta < 0} \alpha(t)\, \varphi_\theta(t)\, dt + \int_{\varphi_\theta > 0} \beta(t)\, \varphi_\theta(t)\, dt$$

(where $\varphi_\theta > 0$ indicates the set $\{t \in [0, 2\pi] : \varphi_\theta(t) > 0\}$, and similarly for $\varphi_\theta < 0$). It can be seen that $\Phi_{\alpha,\beta}$ is a $(2\pi/n)-$periodic C^1 function whose derivative is given by

$$\Phi'_{\alpha,\beta}(\theta) = \int_{\varphi_\theta < 0} \alpha(t)\, \varphi'_\theta(t)\, dt + \int_{\varphi_\theta > 0} \beta(t)\, \varphi'_\theta(t)\, dt.$$

For any θ, we introduce the following notations:

$$m(\theta) := \inf\{\Phi_{\alpha,\beta}(\theta) : \gamma_- \leqslant \alpha \leqslant \Gamma_-, \gamma_+ \leqslant \beta \leqslant \Gamma_+\},$$
$$M(\theta) := \sup\{\Phi_{\alpha,\beta}(\theta) : \gamma_- \leqslant \alpha \leqslant \Gamma_-, \gamma_+ \leqslant \beta \leqslant \Gamma_+\},$$
$$d(\theta) := \inf\{\Phi'_{\alpha,\beta}(\theta) : \gamma_- \leqslant \alpha \leqslant \Gamma_-, \gamma_+ \leqslant \beta \leqslant \Gamma_+\},$$
$$D(\theta) := \sup\{\Phi'_{\alpha,\beta}(\theta) : \gamma_- \leqslant \alpha \leqslant \Gamma_-, \gamma_+ \leqslant \beta \leqslant \Gamma_+\}.$$

LEMMA 1. *The following equalities hold*:

$$m(\theta) = \Phi_{\Gamma_-,\gamma_+}(\theta), \qquad M(\theta) = \Phi_{\gamma_-,\Gamma_+}(\theta),$$

$$d(\theta) = \int_{\varphi_\theta<0,\,\varphi'_\theta>0} \gamma_-(t)\,\varphi'_\theta(t)\,dt + \int_{\varphi_\theta<0,\,\varphi'_\theta<0} \Gamma_-(t)\,\varphi'_\theta(t)\,dt$$
$$+ \int_{\varphi_\theta>0,\,\varphi'_\theta>0} \gamma_+(t)\,\varphi'_\theta(t)\,dt + \int_{\varphi_\theta>0,\,\varphi'_\theta<0} \Gamma_+(t)\,\varphi'_\theta(t)\,dt,$$

$$D(\theta) = \int_{\varphi_\theta<0,\,\varphi'_\theta>0} \Gamma_-(t)\,\varphi'_\theta(t)\,dt + \int_{\varphi_\theta<0,\,\varphi'_\theta<0} \gamma_-(t)\,\varphi'_\theta(t)\,dt$$
$$+ \int_{\varphi_\theta>0,\,\varphi'_\theta>0} \Gamma_+(t)\,\varphi'_\theta(t)\,dt + \int_{\varphi_\theta>0,\,\varphi'_\theta<0} \gamma_+(t)\,\varphi'_\theta(t)\,dt.$$

Proof. The first two identities follow directly from the fact that the function $(\alpha, \beta) \mapsto \Phi_{\alpha,\beta}(\theta)$ is decreasing in α and increasing in β. Similar considerations about monotonicity in α and β over the four considered sets yield the third and the fourth identity. ∎

The following assumption concerning the functions $\gamma_\pm, \Gamma_\pm$ is crucial for the sequel.

Assumption A. If, for some $\theta \in [0, 2\pi[$,

$$m(\theta) \leqslant 0 \leqslant M(\theta),$$

then

$$d(\theta)\,D(\theta) > 0.$$

Remark 1. In case the four functions $\gamma_\pm, \Gamma_\pm$ coincide with the same function f, Assumption A just states that the function

$$\Phi(\theta) = \int_0^{2\pi} f(t)\,\varphi(t+\theta)\,dt$$

only has simple zeros.

Remark 2. If it happens that, for every θ, either $m(\theta) > 0$ or $M(\theta) < 0$, we are in a situation of Landesman–Lazer type (see [1, 3]).

Remark 3. Assumption A is equivalent to the following: For any fixed θ, either for every choice of functions $\alpha, \beta \in L^1(0, 2\pi)$, satisfying $\gamma_- \leqslant \alpha \leqslant \Gamma_-$, $\gamma_+ \leqslant \beta \leqslant \Gamma_+$, one has $\Phi_{\alpha,\beta}(\theta) \neq 0$, or for every choice of such functions one has $\Phi'_{\alpha,\beta}(\theta) \neq 0$.

LEMMA 2. *Under Assumption* A, *the functions* $\Phi_{\alpha,\beta}$, *with* $\gamma_- \leqslant \alpha \leqslant \Gamma_-$, $\gamma_+ \leqslant \beta \leqslant \Gamma_+$, *all have the same number of zeros in the interval* $[0, 2\pi/n[$, *all of which are simple. Moreover, there is a constant* $\bar{c} > 0$ *such that, if for some* $\theta \in [0, 2\pi[$,

$$[m(\theta), M(\theta)] \cap [-\bar{c}, \bar{c}] \neq \varnothing,$$

then

$$[d(\theta), D(\theta)] \cap [-\bar{c}, \bar{c}] = \varnothing.$$

Proof. The first part is a direct consequence of the convexity of the set of functions verifying $\gamma_- \leqslant \alpha \leqslant \Gamma_-$, $\gamma_+ \leqslant \beta \leqslant \Gamma_+$, and of the fact that, by Remark 3, the functions $\Phi_{\alpha,\beta}$ considered cannot have multiple zeros. For the second part, by contradiction, assume that there exists a sequence (θ_n) such that

$$m(\theta_n) \leqslant \frac{1}{n}, \qquad M(\theta_n) \geqslant -\frac{1}{n},$$

and

$$d(\theta_n) \leqslant \frac{1}{n}, \qquad D(\theta_n) \geqslant -\frac{1}{n}.$$

Extracting a subsequence, still denoted by (θ_n), converging to some $\bar{\theta}$, and using the continuity with respect to θ, we obtain

$$m(\bar{\theta}) \leqslant 0 \leqslant M(\bar{\theta}), \qquad d(\bar{\theta}) \leqslant 0 \leqslant D(\bar{\theta}),$$

in contradiction with Assumption A. ∎

Notation. We will denote by $2z$ the common number of zeros, in the interval $[0, 2\pi/n[$, of all the functions $\Phi_{\alpha,\beta}$, when α, β satisfy $\gamma_- \leqslant \alpha \leqslant \Gamma_-$, $\gamma_+ \leqslant \beta \leqslant \Gamma_+$ (that number is even, because of the periodicity of $\Phi_{\alpha,\beta}$).

Remark 4. The arguments of the last part of Lemma 2 also imply that Assumption A still holds under a small perturbation of g.

We are now able to state our main existence result of this section. For that purpose, we introduce the operator

$$L : D(L) \subset C^1([0, 2\pi]) \to L^1(0, 2\pi) : x \mapsto x'',$$

where

$$D(L) = \{x \in W^{2,1}(0, 2\pi) \cap C^1([0, 2\pi]) : x(0) - x(2\pi) = 0 = x'(0) - x'(2\pi)\};$$

we also introduce the Nemytskii operator $N : C^1([0, 2\pi]) \to L^1(0, 2\pi)$ defined by $[Nx](t) = -\mu x^+(t) + \nu x^-(t) + g(t, x(t))$. We shall consider the coincidence degree $d_L(L - N, B_R)$, with respect to a ball B_R, for large values of R (for the definition and the properties of that topological degree, see e.g. [7]).

THEOREM 1. *Assume that, for some positive integer* n,

$$\frac{1}{\sqrt{\mu}} + \frac{1}{\sqrt{\nu}} = \frac{2}{n},$$

and that g *satisfies the Carathéodory conditions and* (1). *Let Assumption* A *be satisfied. Then, for* $R > 0$ *sufficiently large, the coincidence degree* $d_L(L - N, B_R)$ *is equal to* $1 - z$. *Consequently, if* $z \neq 1$, *problem* (P) *has at least one solution.*

The proof will be divided in several steps. We first prove that it is possible to find an a priori estimate on the possible solutions of (P).

LEMMA 3. *Under the assumptions of Theorem* 1, *there is a constant* $C > 0$ *such that, if* x *is a solution of* (P), *then*

$$\|x\|_\infty := \max_{t \in [0, 2\pi]} |x(t)| \leqslant C.$$

Proof. By contradiction, assume that there is a sequence (x_k) of solutions of (P) such that $\|x_k\|_\infty \to \infty$ as $k \to \infty$. Set $u_k(t) := x_k(t)/\|x_k\|_\infty$. Then u_k satisfies

$$(\tilde{P}) \qquad \begin{cases} u_k'' + \mu u_k^+ - \nu u_k^- = \|x_k\|_\infty^{-1}\, g(t, x_k(t)) \\ u_k(0) - u_k(2\pi) = 0 = u_k'(0) - u_k'(2\pi). \end{cases}$$

Because of (1), the r.h.s. of the first equation goes to zero in the L^1 – norm. A standard compactness argument (see e.g. [7]) yields a subsequence, still denoted (u_k), which converges in the C^1-norm to a nontrivial solution $\bar{A}\varphi_{\bar{\theta}}(t)$ of (H). Let $\rho_k(t) > 0$, $\theta_k(t)$ be defined, for k sufficiently large, by

$$(u_k(t), u_k'(t)) = \rho_k(t)(\varphi(\theta_k(t)), \varphi'(\theta_k(t))) \tag{2}$$

(notice that, for k large, $u_k(t)$ and $u_k'(t)$ cannot vanish simultaneously, so that $\rho_k(t)$, $\theta_k(t)$ can be defined as continuous and even C^1 functions of t, with $\rho_k(t) > 0$). Then, adding if necessary a multiple of 2π to $\theta_k(t)$, one has

$$\lim_k \rho_k(t) = \bar{A} > 0, \qquad \lim_k \theta_k(t) = t + \bar{\theta}, \tag{3}$$

uniformly in $t \in [0, 2\pi]$. Computing u_k' and u_k'' from (2) and using the equations defining u_k and φ, we get

$$\rho_k'(t)\, \varphi(\theta_k(t)) + \rho_k(t)\, \varphi'(\theta_k(t))(\theta_k'(t) - 1) = 0, \tag{4}$$

$$\rho_k'(t)\, \varphi'(\theta_k(t)) + \rho_k(t)\, \varphi''(\theta_k(t))(\theta_k'(t) - 1) = \|x_k\|_\infty^{-1}\, g(t, x_k(t)). \tag{5}$$

Moreover, one has

$$[\varphi'(\theta_k(t))]^2 - \varphi(\theta_k(t))\, \varphi''(\theta_k(t)) = 1. \tag{6}$$

Multiplying (4) by $\varphi'(\theta_k(t))$, (5) by $\varphi(\theta_k(t))$, and subtracting, we obtain, by (6),

$$\rho_k(t)(\theta_k'(t) - 1) = -\|x_k\|_\infty^{-1}\, g(t, x_k(t))\, \varphi(\theta_k(t)).$$

Dividing by $\rho_k(t)$ and integrating over $[0, 2\pi]$, we have

$$\int_0^{2\pi} g(t, x_k(t))\, \varphi(\theta_k(t))[\rho_k(t)]^{-1}\, dt = 0, \tag{7}$$

since, by (3) and the periodicity of u_k, $\int_0^{2\pi} \theta_k'(t)\, dt = 2\pi$, for k large enough. Using (3) and (1), we see that Fatou's lemma can be applied which gives

$$\int_{\varphi_{\bar{\theta}}<0} \Gamma_- \varphi_{\bar{\theta}} + \int_{\varphi_{\bar{\theta}}>0} \gamma_+ \varphi_{\bar{\theta}} \leqslant 0 \leqslant \int_{\varphi_{\bar{\theta}}<0} \gamma_- \varphi_{\bar{\theta}} + \int_{\varphi_{\bar{\theta}}>0} \Gamma_+ \varphi_{\bar{\theta}}. \tag{8}$$

On the other hand, multiplying the first equation in $(\tilde{P})$ by $u_k'(t)$ and integrating yields

$$\int_0^{2\pi} g(t, x_k(t))\, u_k'(t)\, dt = 0. \tag{9}$$

Since u_k converges to $\varphi_{\bar\theta}$ in the C^1 – norm, by Fatou's lemma we have

$$\int_{\varphi_{\bar\theta}<0,\, \varphi'_{\bar\theta}>0} \gamma_- \varphi'_{\bar\theta} + \int_{\varphi_{\bar\theta}<0,\, \varphi'_{\bar\theta}<0} \Gamma_- \varphi'_{\bar\theta}$$

$$+ \int_{\varphi_{\bar\theta}>0,\, \varphi'_{\bar\theta}>0} \gamma_+ \varphi'_{\bar\theta} + \int_{\varphi_{\bar\theta}>0,\, \varphi'_{\bar\theta}<0} \Gamma_+ \varphi'_{\bar\theta} \leqslant 0$$

$$\leqslant \int_{\varphi_{\bar\theta}<0,\, \varphi'_{\bar\theta}>0} \Gamma_- \varphi'_{\bar\theta} + \int_{\varphi_{\bar\theta}<0,\, \varphi'_{\bar\theta}<0} \gamma_- \varphi'_{\bar\theta}$$

$$+ \int_{\varphi_{\bar\theta}>0,\, \varphi'_{\bar\theta}>0} \Gamma_+ \varphi'_{\bar\theta} + \int_{\varphi_{\bar\theta}>0,\, \varphi'_{\bar\theta}<0} \gamma_+ \varphi'_{\bar\theta}. \tag{10}$$

But (8) and (10) are in contradiction with Assumption A (see Lemma 1), and hence the lemma is proved. ∎

We now consider the Cauchy problems

$$(C_{\theta, A}) \qquad \begin{cases} x'' + \mu x^+ - \nu x^- = g(t, x) \\ x(0) = A\varphi(\theta), \qquad x'(0) = A\varphi'(\theta), \end{cases}$$

where $A > 0$ will be chosen sufficiently large, and $\theta \in [0, 2\pi/n]$. For the next lemma, we will assume that problem $(C_{\theta, A})$ has a unique solution, which will be denoted by $x(t; \theta, A)$. Equivalently, the function $u(t; \theta, A) = A^{-1}x(t; \theta, A)$ verifies

$$(\tilde C_{\theta, A}) \qquad \begin{cases} u'' + \mu u^+ - \nu u^- = A^{-1} g(t, Au(t)) \\ u(0) = \varphi(\theta), \qquad u'(0) = \varphi'(\theta). \end{cases}$$

Setting $w(t) = (u(t; \theta, A) - \varphi_\theta(t), u'(t; \theta, A) - \varphi'_\theta(t))$, one has

$$w' = (u' - \varphi'_\theta,\ -\mu(u^+ - \varphi_\theta^+) + \nu(u^- - \varphi_\theta^-) + A^{-1} g(\cdot, Au)),$$

and hence, by (1),

$$\|w(t)\| \leqslant \int_0^t \|w'(s)\|\, ds$$

$$\leqslant \int_0^t (\max\{\mu, \nu\}\ \|w(s)\| + A^{-1} h(s))\, ds.$$

By Gronwall's inequality, there is a constant $L > 0$ such that

$$|u(t; \theta, A) - \varphi(t+\theta)| \leqslant LA^{-1}, \qquad |u'(t; \theta, A) - \varphi'(t+\theta)| \leqslant LA^{-1}, \tag{11}$$

for all $t \in [0, 2\pi]$, $\theta \in [0, 2\pi/n]$ and $A > 0$.

Let us consider the curve $\mathscr{V}_A : [0, 2\pi/n] \to \mathbb{R}^2$ defined by

$$\mathscr{V}_A(\theta) = (x(2\pi; \theta, A) - A\varphi(\theta), x'(2\pi; \theta, A) - A\varphi'(\theta));$$

that curve is easily seen to be closed. By Lemma 3, if $A > 0$ is large enough, then $\mathscr{V}_A(\theta) \neq 0$ for every $\theta \in [0, 2\pi/n]$. Hence, we may look for the number of rotations around the origin performed by the vector $\mathscr{V}_A(\theta)$, while θ varies from 0 to $2\pi/n$. That number will be denoted by d_A; it is computed in the next lemma, where $2z$ is, as before, the number of zeros of the functions $\Phi_{\alpha,\beta}$ in the interval $[0, 2\pi/n[$, all zeros being assumed to be simple.

LEMMA 4. *Let problem $C_{\theta,A}$ have a unique solution for all $\theta \in [0, 2\pi/n[$ and all $A > 0$. Then, under the assumptions of Theorem* 1, $d_A = 1 - z$, *for A sufficiently large.*

Proof. Multiplying the first equation in $(C_{\theta,A})$ by $\varphi(t+\theta) = \varphi_\theta(t)$ and integrating by parts yields

$$\langle \mathscr{V}_A(\theta), (-\varphi'(\theta), \varphi(\theta)) \rangle + (\nu - \mu) \int_{x\varphi_\theta < 0} x(t; \theta, A) \, |\varphi_\theta(t)| \, dt$$

$$= \int_0^{2\pi} g(t, x(t; \theta, A)) \, \varphi(t+\theta) \, dt,$$

where $\langle \cdot, \cdot \rangle$ denotes the euclidean scalar product in $\mathbb{R}^2$. Using (11) and taking into account the fact that the interval on which $|\varphi(t+\theta)|$ is less than LA^{-1} has a length going to 0 when A goes to infinity, it is readily seen that

$$\lim_{A \to +\infty} A \int_{x\varphi_\theta < 0} x(t; \theta, A) \, |\varphi_\theta(t)| \, dt = 0,$$

uniformly in θ, and hence

$$\left| \int_0^{2\pi} g(t, x(t; \theta, A)) \, \varphi(t+\theta) \, dt - \langle \mathscr{V}_A(\theta), (-\varphi'(\theta), \varphi(\theta)) \rangle \right| \leqslant \frac{1}{A},$$

for every $\theta \in [0, 2\pi/n]$ and $A > 0$ sufficiently large. By Fatou's Lemma, we see that

$$\liminf_{A \to +\infty} \int_0^{2\pi} g(t, x(t; \theta, A)) \, \varphi(t+\theta) \, dt \geqslant \Phi_{\Gamma_-, \gamma_+}(\theta),$$

$$\limsup_{A \to +\infty} \int_0^{2\pi} g(t, x(t; \theta, A)) \, \varphi(t+\theta) \, dt \leqslant \Phi_{\gamma_-, \Gamma_+}(\theta),$$

uniformly in θ. Consequently, taking A sufficiently large, we have

$$m(\theta) - \frac{\bar{c}}{2} \leqslant \langle \mathscr{V}_A(\theta), (-\varphi'(\theta), \varphi(\theta)) \rangle \leqslant M(\theta) + \frac{\bar{c}}{2}, \tag{12}$$

where the number $\bar{c} > 0$ is given by Lemma 2. On the other hand, multiplying the first equation in $(C_{\theta, A})$ by $\varphi'(t+\theta)$ and integrating by parts, we obtain

$$[x'(2\pi; \theta, A) - A\varphi'(\theta)]\, \varphi'(\theta) + (\mu - \nu) \int_{x\varphi_\theta < 0} x'(t; \theta, A)\, |\varphi_\theta(t)|\, dt$$

$$= \int_0^{2\pi} g(t, x(t; \theta, A))\, \varphi'_\theta(t)\, dt.$$

By (11), we have

$$\lim_{A \to \infty} \int_{x\varphi_\theta < 0} x'(t; \theta, A)\, |\varphi_\theta(t)|\, dt = 0,$$

and, again by Fatou's Lemma and Lemma 1,

$$\liminf_{A \to \infty} \int_0^{2\pi} g(t, x(t; \theta, A))\, \varphi'_\theta(t)\, dt \geqslant d(\theta),$$

$$\limsup_{A \to \infty} \int_0^{2\pi} g(t, x(t; \theta, A))\, \varphi'_\theta(t)\, dt \leqslant D(\theta),$$

uniformly in θ. Hence, for A sufficiently large, we obtain

$$d(\theta) - \frac{\bar{c}}{2} \leqslant [x'(2\pi; \theta, A) - A\varphi'(\theta)]\, \varphi'(\theta) \leqslant D(\theta) + \frac{\bar{c}}{2}. \tag{13}$$

Let us consider, for simplicity, the case $z = 1$. Then, each function $\Phi_{\alpha, \beta}$ has exactly two zeros in $[0, 2\pi/n[$, and, by Lemma 2, there exist $\theta_1 < \theta_2 < \theta_3 < \theta_4$ in $\mathbb{R}$ such that $\theta_4 - \theta_1 < 2\pi/n$, and

$$\begin{aligned}
m(\theta) &\geqslant \bar{c} && \text{if} \quad \theta_1 \leqslant \theta \leqslant \theta_2,\\
D(\theta) &\leqslant -\bar{c} && \text{if} \quad \theta_2 \leqslant \theta \leqslant \theta_3,\\
M(\theta) &\leqslant -\bar{c} && \text{if} \quad \theta_3 \leqslant \theta \leqslant \theta_4,\\
d(\theta) &\geqslant \bar{c} && \text{if} \quad \theta_4 \leqslant \theta \leqslant \theta_1 + 2\pi/n.
\end{aligned}$$

Consequently, by (12) and (13), we have

$$\begin{aligned}
\langle \mathscr{V}_A(\theta), (-\varphi'(\theta), \varphi(\theta))\rangle &> 0 && \text{if} \quad \theta_1 \leqslant \theta \leqslant \theta_2, \\
[x'(2\pi; \theta, A) - A\varphi'(\theta)]\, \varphi'(\theta) &< 0 && \text{if} \quad \theta_2 \leqslant \theta \leqslant \theta_3, \\
\langle \mathscr{V}_A(\theta), (-\varphi'(\theta), \varphi(\theta))\rangle &< 0 && \text{if} \quad \theta_3 \leqslant \theta \leqslant \theta_4, \\
[x'(2\pi; \theta, A) - A\varphi'(\theta)]\, \varphi'(\theta) &> 0 && \text{if} \quad \theta_4 \leqslant \theta \leqslant \theta_1 + 2\pi/n.
\end{aligned}$$

Let $\omega_A(\theta)$ be the angle, varying continuously with respect to θ, obtained moving counter-clockwise from the vector $(\varphi(\theta), \varphi'(\theta))$ to the vector $\mathscr{V}_A(\theta)$. Adding if necessary an integer multiple of 2π, the following must hold:

$$\begin{aligned}
0 < \omega_A(\theta) &< \pi && \text{if} \quad \theta_1 \leqslant \theta \leqslant \theta_2, \\
0 < \omega_A(\theta) &< 2\pi && \text{if} \quad \theta_2 \leqslant \theta \leqslant \theta_3, \\
\pi < \omega_A(\theta) &< 2\pi && \text{if} \quad \theta_3 \leqslant \theta \leqslant \theta_4, \\
\pi < \omega_A(\theta) &< 3\pi && \text{if} \quad \theta_4 \leqslant \theta \leqslant \theta_1 + 2\pi/n.
\end{aligned}$$

We then conclude that the vector $\mathscr{V}_A(\theta)$ rotates counter-clockwise exactly once with respect to the vector $(\varphi(\theta), \varphi'(\theta))$ when the value of θ increases of $2\pi/n$ (remember that $\mathscr{V}_A$ is $2\pi/n-$periodic). In the general situation, we would have found $\theta_1 < \theta_2 < \cdots < \theta_{2z}$, with $\theta_{2z} - \theta_1 < 2\pi/n$, and with a similar argument we would have obtained that $\mathscr{V}_A(\theta)$ rotates counter-clockwise exactly z times with respect to $(\varphi(\theta), \varphi'(\theta))$, when θ increases of $2\pi/n$. Taking into account the fact that this latter rotates clockwise exactly once, we obtain the number $1 - z$ we were looking for, and this completes the proof of the lemma. ∎

We are now able to complete the proof of our theorem.

Proof of Theorem 1. Consider a sequence $g_k : [0, 2\pi] \times \mathbb{R} \to \mathbb{R}$ of Carathéodory functions such that $g_k(t, .)$ is locally Lipschitz-continuous for a.e. $t \in [0, 2\pi]$, and

$$|g_k(t, x) - g(t, x)| \leqslant \frac{1}{k} \qquad (x \in \mathbb{R}, \text{ a.e. } t \in [0, 2\pi]).$$

Using Remark 4, we have that Lemmas 3 and 4 hold if $g(t, x)$ is replaced by $g_k(t, x)$. Let the operator $L : x \mapsto x''$ be defined as above; on the other hand,

$$N, N_k : C^1([0, 2\pi]) \to L^1(0, 2\pi)$$

will be the Nemytskii operators defined by $[Nx](t) = -\mu x^+(t) + \nu x^-(t) + g(t, x(t))$, and $[N_k x](t) = -\mu x^+(t) + \nu x^-(t) + g_k(t, x(t))$. The Poincaré map $T_k : \mathbb{R}^2 \to \mathbb{R}^2$ for the equation

$$x' = y, \qquad y' = -\mu x^+ + \nu x^- + g_k(t, x)$$

is well defined. Let $\Omega_A \subset \mathbb{R}^2$ be the open set bounded by the curve $(A\varphi(\theta), A\varphi'(\theta))$. Define the set $\tilde{\Omega}_A$ as follows:

$$\tilde{\Omega}_A = \{x \in C^1([0, 2\pi]) : (x(t), x'(t)) \in \Omega_A \quad (t \in [0, 2\pi])\}.$$

A duality theorem asserts that

$$\deg_B(I - T_k, \Omega_A) = d_L(L - N_k, \tilde{\Omega}_A),$$

where $\deg_B$ denotes the Brouwer degree in $\mathbb{R}^2$ and d_L denotes the coincidence degree (cf. [7]). But, the Brouwer degree with respect to the set Ω_A is precisely the number of rotations around the origin of the vector $\mathscr{V}_A$. Hence, by Lemma 4, for A sufficiently large, $\deg_B(I - T_k, \Omega_A) = 1 - z$. On the other hand, the coincidence degree $d_L(L - N, \tilde{\Omega}_A)$ is well defined by Lemma 3 and, being invariant by small perturbations, it has to be equal to $d_L(L - N_k, \tilde{\Omega}_A)$. Then, $d_L(L - N, \tilde{\Omega}_A) = 1 - z$, and, if $z \neq 1$, equation $Lx = Nx$, which is equivalent to (P), has a solution. Moreover, by the excision property of the degree, using Lemma 3, it follows that $d_L(L - N, B_R)$ must have the same value $1 - z$ for every $R > 0$ large enough. ∎

3. OSCILLATORS WITH DAMPING

We now consider the problem with damping

$$(DP) \qquad \begin{cases} x'' + \delta x' + \mu x^+ - \nu x^- = f(t) \\ x(0) - x(2\pi) = 0 = x'(0) - x'(2\pi), \end{cases}$$

where, for the sake of simplicity, the forcing term $f : [0, 2\pi] \to \mathbb{R}$, independent of x, is assumed to be continuous. Consider the function

$$\Phi(\theta) = \int_0^{2\pi} f(t)\, \varphi(t + \theta)\, dt.$$

We denote by $2z$ its number of zeros, assumed to be simple, in the interval $[0, 2\pi/n[$. Comparing the degree with respect to large balls for $\delta = 0$ (which, by Theorem 1, is $1 - z$) and for $\delta \neq 0$ (which can be seen to be equal to 1), it is easily shown that, when $z \neq 0$, a branch of solutions must exist that goes to infinity when δ tends to 0. We obtain a more precise result below through a different approach, namely by using the implicit function theorem. That approach has the advantage of providing an asymptotic estimate of those solutions and of giving information about their number.

We will need the following result due to Lazer and McKenna [5]; it ensures the differentiability of the solutions with respect to initial conditions, in circumstances where the classical regularity results do not apply, due to the lack of differentiability of the function in the differential equation.

LEMMA 5. *Let $x(t;\xi,\eta)$ be the solution of the Cauchy problem*

$$\begin{cases} x''+\delta x'+\mu x^+ - \nu x = f(t) \\ x(0)=\xi/\delta, \qquad x'(0)=\eta/\delta. \end{cases}$$

Suppose that the zeros of x are isolated in $[0,2\pi]$. Then, for $t\in\,]0,2\pi[$, the partial derivatives of x and x' with respect to ξ,η exist and are continuous.

Let us state the main result of this section.

THEOREM 2. *Assume that, for some positive integer n,*

$$\frac{1}{\sqrt{\mu}}+\frac{1}{\sqrt{\nu}}=\frac{2}{n}.$$

Let θ^* be a simple zero of Φ. There exists a $\delta^*>0$ such that, if $|\delta|\leqslant\delta^*$ and $\delta\Phi'(\theta^*)>0$, problem (DP) has a solution $x(t;\delta)$ such that

$$(x(t;\delta),x'(t;\delta))=\frac{1}{|\delta|}\,\rho(t;\delta)(\varphi(\theta(t;\delta)),\varphi'(\theta(t;\delta))); \tag{14}$$

the functions $\rho(t;\delta)$ and $\theta(t;\delta)$ depend continuously on δ, $\lim_{\delta\to 0}\theta(t;\delta)=t+\theta^*$, and $\lim_{\delta\to 0}\rho(t;\delta)=\rho^*:=\operatorname{sgn}(\delta)\,\Phi'(\theta^*)/\pi$. Moreover, the solution $x(t;\delta)$ is asymptotically stable if $\delta>0$, unstable if $\delta<0$.

Proof. We will write the proof for $\delta>0$ (the case $\delta<0$ can be treated in a similar way). Like in the proof of Lemma 3, provided that $\rho(t;\delta)>0$, computing $x'(t;\delta)$ and $x''(t;\delta)$ from (14), we have a solution of (DP) if

$$\begin{cases} \rho'(t;\delta)\,\varphi(\theta(t;\delta))+\rho(t;\delta)\,\varphi'(\theta(t;\delta))[\theta'(t;\delta)-1]=0 \\ \rho'(t;\delta)\,\varphi'(\theta(t;\delta))+\rho(t;\delta)\,\varphi''(\theta(t;\delta))[\theta'(t;\delta)-1] \\ \qquad =\delta f(t;\delta)-\delta\rho(t;\delta)\,\varphi'(\theta(t;\delta)) \\ \rho(2\pi;\delta)=\rho(0;\delta), \qquad \theta(2\pi;\delta)=\theta(0;\delta)+2\pi. \end{cases}$$

Using (6), we equivalently have

$$(\odot)\qquad \begin{cases} \rho'(t;\delta) = -\delta\rho(t;\delta)[\varphi'(\theta(t;\delta))]^2 + \delta f(t)\,\varphi'(\theta(t;\delta)) \\ [\theta'(t;\delta)-1] = \delta\varphi(\theta(t;\delta))\,\varphi'(\theta(t;\delta)) - \delta f(t)\,\varphi(\theta(t;\delta))[\rho(t;\delta)]^{-1} \\ \rho(2\pi;\delta) = \rho(0;\delta), \qquad \theta(2\pi;\delta) = \theta(0;\delta) + 2\pi. \end{cases}$$

Let us denote by $\rho(t;\delta;\rho_0,\theta_0)$, $\theta(t;\delta;\rho_0,\theta_0)$ the solution of $(\odot)$ corresponding to the initial conditions $\rho(0;\delta)=\rho_0$, $\theta(0;\delta)=\theta_0$. In particular,

$$\rho(t;0;\rho_0,\theta_0) = \rho_0, \qquad \theta(t;0;\rho_0,\theta_0) = t + \theta_0.$$

Integrating the differential equations in $(\odot)$, the periodicity conditions

$$\rho(2\pi;\delta;\rho_0,\theta_0) = \rho(0;\delta;\rho_0,\theta_0), \qquad \theta(2\pi;\delta;\rho_0,\theta_0) = \theta(0;\delta;\rho_0,\theta_0) + 2\pi$$

will be satisfied for $\delta \neq 0$, if and only if $\Psi(\delta,\rho_0,\theta_0) = (0,0)$, with $\Psi = (\Psi_1,\Psi_2)$ defined by

$$\Psi_1(\delta,\rho_0,\theta_0) = -\int_0^{2\pi} \rho(t;\delta;\rho_0,\theta_0)[\varphi'(\theta(t;\delta;\rho_0,\theta_0))]^2\,dt$$

$$+\int_0^{2\pi} f(t)\,\varphi'(\theta(t;\delta;\rho_0,\theta_0))\,dt,$$

$$\Psi_2(\delta,\rho_0,\theta_0) = \int_0^{2\pi} \varphi(\theta(t;\delta;\rho_0,\theta_0))\,\varphi'(\theta(t;\delta;\rho_0,\theta_0))\,dt$$

$$-\int_0^{2\pi} f(t)\,\varphi(\theta(t;\delta;\rho_0,\theta_0))[\rho(t;\delta;\rho_0,\theta_0)]^{-1}\,dt.$$

We will use the implicit function theorem to prove, for δ small, the existence of $(\rho_0(\delta),\theta_0(\delta))$, close to (ρ^*,θ^*), depending continuously on δ, satisfying $\Psi(\delta,\rho_0(\delta),\theta_0(\delta)) = (0,0)$. Although the nonlinear term of the differential equation in (DP) is not differentiable, that theorem can be used here because the functions $\rho(t;\delta;\rho_0,\theta_0)$, $\theta(t;\delta;\rho_0,\theta_0)$ are C^1 functions of ρ_0,θ_0, as results from Lemma 5. Indeed, given ξ,η with $\xi^2+\eta^2\neq 0$, the solution of the Cauchy problem

$$\begin{cases} x'' + \delta x' + \mu x^+ - \nu x^- = f(t) \\ x(0) = \xi/\delta, \qquad x'(0) = \eta/\delta. \end{cases}$$

has only simple zeros in $[0,2\pi[$, at least for δ small, because, in that case, the solution is close, in the C^1 sense, to a (large) multiple of $\varphi(t+\tau)$ for some $\tau\in[0,2\pi[$. Using the fact that $\int_0^{2\pi}[\varphi'(t+\theta_0)]^2\,dt = \int_0^{2\pi}[\varphi'(t)]^2\,dt = \pi$ and $\int_0^{2\pi}\varphi(t+\theta_0)\,\varphi'(t+\theta_0)\,dt = 0$, we have

$$\Psi_1(0, \rho_0, \theta_0) = -\rho_0 \pi + \int_0^{2\pi} f(t)\, \varphi'(t+\theta_0)\, dt,$$

$$\Psi_2(0, \rho_0, \theta_0) = -\frac{1}{\rho_0} \int_0^{2\pi} f(t)\, \varphi(t+\theta_0)\, dt.$$

Hence, $\Psi_1(0, \rho^*, \theta^*) = -\rho^* \pi + \Phi'(\theta^*) = 0$, and $\Psi_2(0, \rho^*, \theta^*) = -\Phi(\theta^*)/\rho^* = 0$. Moreover, the jacobian matrix, given by

$$\frac{\partial \Psi}{\partial(\rho_0, \theta_0)}(0, \rho^*, \theta^*) = \begin{pmatrix} -\pi & \int_0^{2\pi} f(t)\, \varphi''(t+\theta^*)\, dt \\ 0 & -\frac{1}{\rho^*} \int_0^{2\pi} f(t)\, \varphi'(t+\theta^*)\, dt \end{pmatrix},$$

is invertible since $(1/\rho^*) \int_0^{2\pi} f(t)\, \varphi'(t+\theta^*)\, dt = (1/\rho^*)\, \Phi'(\theta^*) = \pi$. The implicit function theorem can therefore be applied, giving, for δ small, the existence of $(\rho_0, \theta_0) = (\rho_0(\delta), \theta_0(\delta))$, solution of $\Psi(\delta, \rho_0, \theta_0) = (0, 0)$ close to (ρ^*, θ^*) depending continuously on δ. Being $\rho^* > 0$, the corresponding solution

$$(\rho(t; \delta), \theta(t; \delta)) = (\rho(t; \delta; \rho_0(\delta), \theta_0(\delta)), \theta(t; \delta; \rho_0(\delta), \theta_0(\delta)))$$

of ($\odot$) is such that $\rho(t; \delta) > 0$ for δ small, validating the transformation made at the beginning of the proof.

For the stability of that solution, consider the vector valued function $\sigma(t; \delta) = (\rho(t; \delta), \theta(t; \delta) - t)$. We have $\sigma(t; 0) = \sigma^* := (\rho^*, \theta^*)$, and problem ($\odot$) is equivalent to

$$\begin{cases} \sigma'(t; \delta) = \delta F(t, \sigma(t; \delta)) \\ \sigma(2\pi; \delta) = \sigma(0; \delta), \end{cases}$$

where, writing $\sigma = (\sigma_1, \sigma_2)$, F is defined by

$$F(t, \sigma) = (f(t)\, \varphi'(t+\sigma_2) - \sigma_1 [\varphi'(t+\sigma_2)]^2,$$
$$\varphi(t+\sigma_2)\, \varphi'(t+\sigma_2) - f(t)\, \varphi(t+\sigma_2)\, \sigma_1^{-1}).$$

The eigenvalues of

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{\partial F}{\partial \sigma}(t, \sigma^*)\, dt = \frac{1}{2\pi} \frac{\partial \Psi}{\partial(\rho_0, \theta_0)}(0, \rho^*, \theta^*)$$

are both equal to $(-1/2)$. A classical result (cf. [9, Theorem 5.3, p. 150]) concerning weakly nonlinear systems tells us that, provided δ^* is small enough, the solution $\sigma(t;\delta)$ is asymptotically stable if $\delta > 0$ is sufficiently small. The asymptotic stability in the variables x, x' then easily follows. ▮

Remark 5. From the eigenvalues of the jacobian matrix found above, we deduce that, for δ "small," the characteristic exponents of the variational equation associated to the solution $x(t;\delta)$ are close to $-\delta/2$. That limiting value, independent of μ, ν, and f, is thus seen to be the same as for the linear case ($\mu = \nu$).

The result of Theorem 2 can be complemented when Φ has only simple zeros. In that case, it can be proved that the solutions given by Theorem 2 are the only solutions which tend to infinity (in the $\|\cdot\|_\infty -$norm), when δ goes to 0. This is based on the following lemma.

LEMMA 6. *Assume that there exist sequences* (x_k), (δ_k) *with* $\delta_k \neq 0$, $\delta_k \to 0$ *and* $\|x_k\|_\infty \to \infty$ *as* $k \to \infty$ *such that*

$$(DP_k) \qquad \begin{cases} x_k'' + \delta_k x_k' + \mu x_k^+ - \nu x_k^- = f(t) \\ x_k(0) - x_k(2\pi) = 0 = x_k'(0) - x_k'(2\pi). \end{cases}$$

If the function Φ *has only simple zeros, there exist positive constants* c_1, c_2 *such that, for* k *sufficiently large, one can write*

$$(x_k(t), x_k'(t)) = \frac{1}{|\delta_k|}\, \rho_k(t)(\varphi(\theta_k(t)), \varphi'(\theta_k(t))), \tag{15}$$

with $c_1 \leqslant \rho_k(t) \leqslant c_2$ *for every* $t \in [0, 2\pi]$. *Moreover,* $\rho(t;\delta_k) = \rho_k(t)$ *and* $\theta(t;\delta_k) = \theta_k(t)$ *satisfy* ($\odot$), *with* $\delta = \delta_k$.

Proof. We assume $\delta_k > 0$. Setting $u_k(t) = x_k(t)/\|x_k\|_\infty$, one has

$$\begin{cases} u_k'' + \delta_k u_k' + \mu u_k^+ - \nu u_k^- = \|x_k\|_\infty^{-1} f(t) \\ u_k(0) - u_k(2\pi) = 0 = u_k'(0) - u_k'(2\pi). \end{cases}$$

Then, for a subsequence, we can assume that u_k converges in the C^1-sense to some function $\bar{A}\varphi(t+\bar{\theta})$. Writing

$$(u_k(t), u_k'(t)) = \tilde{\rho}_k(t)(\varphi(\theta_k(t)), \varphi'(\theta_k(t)));$$

since $\theta_k(t) \to t + \bar{\theta}$, for k large enough one has $\theta_k(2\pi) = \theta_k(0) + 2\pi$. Hence, $\tilde{\rho}_k(t)$ and $\theta_k(t)$ satisfy ($\odot$) with $\delta = \delta_k$, $f(t)$ replaced by $\|x_k\|_\infty^{-1} f(t)$, $\rho(t;\delta_k) = \delta_k \tilde{\rho}_k(t)$ and $\theta(t;\delta_k) = \theta_k(t)$. Consequently, we obtain

$$\int_0^{2\pi} f(t)\, \varphi(\theta_k(t))[\tilde{\rho}_k(t)]^{-1}\, dt = \delta_k \|x_k\|_\infty \int_0^{2\pi} \varphi(\theta_k(t))\, \varphi'(\theta_k(t))\, dt,$$

and

$$\int_0^{2\pi} f(t)\,\varphi'(\theta_k(t))\,dt = \delta_k\,\|x_k\|_\infty \int_0^{2\pi} \tilde{\rho}_k(t)[\varphi'(\theta_k(t))]^2\,dt.$$

Since we are assuming that Φ has only simple zeros, there must be a constant $b_1 > 0$ for which $\delta_k\,\|x_k\|_\infty \geqslant b_1$, for k sufficiently large. Let us see that there also is a constant $b_2 > 0$ such that $\delta_k\,\|x_k\|_\infty \leqslant b_2$. Multiplying the differential equation in (DP_k) by x_k', and integrating over $[0, 2\pi]$, yields

$$\delta_k \int_0^{2\pi} [x_k'(t)]^2\,dt = \int_0^{2\pi} f(t)\,x_k'(t)\,dt,$$

from which follows that

$$\|x_k'\|_2 \leqslant \frac{1}{\delta_k}\,\|f\|_2$$

(where $\|\cdot\|_p$ denotes the norm in $L^p(0, 2\pi)$). If x_k vanishes at some point $\tau \in [0, 2\pi]$, the norm $\|x_k\|_\infty$ is bounded by $\sqrt{2\pi}\,\|x_k'\|_2$ and, consequently,

$$\delta_k\,\|x_k\|_\infty \leqslant b_2 := \sqrt{2\pi}\,\|f\|_2.$$

If x_k did not vanish in $[0, 2\pi]$ (actually, it will result from the present lemma that this cannot happen for k large), its mean value would easily be shown to be bounded, in absolute value, by $\|f\|_1/\min\{\mu, \nu\}$ and the same result would then be obtained, adapting the value of b_2.

Setting $\rho_k(t) = \delta_k\,\|x_k\|_\infty\,\tilde{\rho}_k(t)$, the proof is easily completed, since $\tilde{\rho}_k(t) \to \bar{A} > 0$ and

$$0 < b_1 \leqslant \delta_k\,\|x_k\|_\infty \leqslant b_2. \quad ∎$$

The following corollaries are consequences of the above lemma and of the arguments in the proof of Theorem 2.

COROLLARY 1. *If Φ has exactly $2z$ zeros, all simple, in $[0, 2\pi/n[$, then it is possible to find $\delta^* > 0$ and $R^* > 0$ such that, if $0 < |\delta| \leqslant \delta^*$, there are exactly z solutions of problem (DP) having $\|\cdot\|_\infty$ — norm larger than R^*.*

Proof. Consider two sequences (δ_k), (R_k) with $\delta_k \neq 0$, $\delta_k \to 0$ and $R_k \to \infty$ as $k \to \infty$. Let x_k be a solution of (DP_k) such that $\|x_k\|_\infty \geqslant R_k$. By Lemma 6, we can write $(x_k(t), x_k'(t))$ as in (15), with $0 < c_1 \leqslant \rho_k(t) \leqslant c_2$ for every $t \in [0, 2\pi]$, and the functions $\rho(t; \delta_k) = \rho_k(t)$ and $\theta(t; \delta_k) = \theta_k(t)$

must be solutions of ($\odot$), with $\delta = \delta_k$. Then, from ($\odot$), $\rho_k'(t) \to 0$ and $\theta_k'(t) \to 1$, uniformly in $[0, 2\pi]$, showing that there must be two constants $\rho^* > 0$ and $\theta^* \in [0, 2\pi/n[$ such that, for some subsequences, $\rho_k(t) \to \rho^*$ and $\theta_k(t) \to t + \theta^*$. Moreover, we must have $\Psi(0, \rho^*, \theta^*) = 0$, so that $\Phi(\theta^*) = 0$, and $\delta_k \Phi'(\theta^*) > 0$. As there are exactly z such zeros of Φ, using the local uniqueness of solutions and the continuous dependence on δ_k provided by the implicit function theorem, we can assert that, for k sufficiently large, problem (DP_k) has also exactly z solutions with $\|x_k\|_\infty \geqslant R_k$. The proof can now be completed arguing by contradiction. ∎

Corollary 2. *If Φ has 2 zeros, both simple, in $[0, 2\pi/n[$, and if the problem (DP) has no solution for $\delta = 0$, there exists $\delta^* > 0$, such that, if $0 < |\delta| \leqslant \delta^*$, problem (DP) has a unique solution.*

Proof. By the previous corollary, we already know that there is a unique family of solutions going to infinity when δ goes to 0. There can be no other family of solutions. Indeed, if it was the case, letting δ go to 0, we could find a sequence of solutions of (DP), bounded for the $\|\cdot\|_\infty$ − norm. Extracting a converging subsequence, this would lead to a solution of the equation for $\delta = 0$, in contradiction with the hypotheses. ∎

It can be checked that the conditions of the above corollary are verified by the function $f(t) = \cos(nt)$, provided that $\mu, \nu \in](n-1)^2, (n+1)^2[$ (see [5], where the uniqueness is proved in that particular case).

In the following Corollary, subharmonic solutions can be obtained, i.e., solutions whose minimal period is an integer multiple of the minimal period of the forcing term.

Corollary 3. *Assume that f has minimal period $2\pi/m$, m being an integer, and that Φ has at least two simple zeros, with derivatives of opposite sign. Let $p = l.c.m.(m, n)$ and $q = g.c.d.(m, n)$. Then, the equation*

$$x'' + \delta x' + \mu x^+ - \nu x^- = f(t) \tag{16}$$

has, for $\delta \neq 0$ and sufficiently small, at least p/n periodic solutions having minimal period $2\pi/q$. If $\delta > 0$, those solutions are asymptotically stable. Their amplitude goes to infinity as $\delta \to 0$.

Remark 6. It must be noticed that the p/n solutions provided by Corollary 3 can be deduced from one particular solution in the list. Indeed, if $x(t;\delta)$ is a solution, the same is true for $x(t + 2\pi/m; \delta)$. Starting from a solution of minimal period $2\pi/q$ and taking translates by multiples of $2\pi/m$ provides $q/m = p/n$ distinct solutions. We now give the proof of Corollary 3.

Proof. Since the solution φ of the positively homogeneous equation is of period $2\pi/n$, and f of period $2\pi/m$, the function Φ must be of period $2\pi/n$ and of period $2\pi/m$ and, therefore, of period $2\pi/p$, with $p = \text{l.c.m.}(m, n)$. By periodicity, Φ will then have at least p/n zeros with strictly positive derivative and p/n zeros with strictly negative derivative in the interval $[0, 2\pi/n[$. Adapting Theorem 2 to the search of solutions of period $2\pi/q$, where $q = \text{g.c.d.}(m, n)$, we can assert, for δ sufficiently small, $\delta \neq 0$, the existence of p/n families of $2\pi/q-$periodic solutions $x(t; \delta)$ whose amplitude increase to infinity and such that $|\delta|\, x(t; \delta) \to A\varphi_\theta(t)$ as $\delta \to 0$, for some $A > 0$ and $\theta \in [0, 2\pi/n[$. Hence, the minimal period of those solutions has to be, for δ sufficiently small, an integer multiple of $2\pi/n$, the period of φ. But, it also has to be an integer multiple of $2\pi/m$, the minimal period of the forcing term. The minimal period of those solutions must therefore be $2\pi/q$. ∎

Remark 7. If n is not a multiple of m, the existence of a $2\pi/m$ periodic solution can be proved for δ sufficiently small, including $\delta = 0$, by a fairly simple degree argument, the situation not being one of resonance. Moreover, using Theorem 1, adapted to the search of $2\pi/q-$periodic solutions, we conclude that, for $\delta = 0$, the coincidence degree, with respect to large balls, of the operator associated to the boundary value problem, has value $1 - p/n$. If $p \neq 2n$, the degree is strictly less than -1 ($p = n$ is excluded, since we have assumed that n is not a multiple of m). In regular cases, we will then have at least two solutions of period $2\pi/q$ when $\delta = 0$, one being of period $2\pi/m$. (By regular case we mean that if $2\pi/q-$periodic solutions exist, then the variational equation cannot have a $2\pi/q-$periodic solution.) If the other solution is of minimal period $2\pi/q$, p/n distinct translates can be associated to it, explaining the value $1 - p/n$. Looking at the problem with $\delta \neq 0$, we have to add, to the $p/n + 1$ solutions discussed above, the p/n solutions going to infinity when $\delta \to 0$, as given by Corollary 3, so that we expect a total of $2p/n + 1$ solutions when $p \neq 2n$, $\delta \neq 0$, corresponding to 3 distinct orbits at least.

If n is a multiple of m, examples can be found for which the problem has no solution for $\delta = 0$ (see [1, 2, 5]). In that case, for $\delta \neq 0$, δ small, Corollary 3 establishes only the existence of one solution of period $2\pi/m$, that solution going to infinity when $\delta \to 0$.

4. EXAMPLE OF APPLICATION

Let $p = \text{l.c.m.}(m, n)$ and $q = \text{g.c.d.}(m, n)$, as above. We first consider equation (16) with $\delta \neq 0$ and forcing term of the type

$$f(t) = a\cos(mt) + b\cos(pt).$$

Notice that, if m is a multiple of n, then $p=m$. In this case, we assume for simplicity that $a=0$; otherwise, let $a\neq 0$. If m is not a multiple of n, it is easy to see that

$$\int_0^{2\pi} \cos(mt)\,\varphi(t+\theta)\,dt=0.$$

Consequently, in both cases, we have

$$\begin{aligned}\Phi(\theta)&=b\int_0^{2\pi}\cos(pt)\,\varphi(t+\theta)\,dt\\&=nb\int_0^{2\pi/n}\cos(p(t-\theta))\,\varphi(t)\,dt\\&=nb\cos(p\theta)\int_0^{2\pi/n}\cos(pt)\,\varphi(t)\,dt+nb\sin(p\theta)\int_0^{2\pi/n}\sin\,(pt)\,\varphi(t)\,dt\end{aligned}$$

Assuming $b\neq 0$, two possibilities can occur. Either

$$\int_0^{2\pi/n}\cos(pt)\,\varphi(t)\,dt=\int_0^{2\pi/n}\sin(pt)\,\varphi(t)\,dt=0,$$

in which case Φ vanishes identically (this can happen only when $\mu=\nu=n^2$, and $p\neq n$). Or, Φ has exactly $2p/n$ zeros in $[0, 2\pi/n[$, all of which are simple. Corollary 3 then applies, predicting, for δ small enough, $\delta\neq 0$, the existence of p/n families of solutions of minimal period $2\pi/q$, whose amplitudes go to infinity if $\delta\to 0$. If $\delta>0$, those solutions are asymptotically stable.

Consider now equation (16) with $\delta=0$ and let

$$f(t)=a\cos(mt)+b\cos(kpt),$$

$k\geqslant 1$ being an integer. As above, if $m=kp$, we assume for simplicity that $a=0$. We then see that, if $\mu\neq\nu$ and $b\neq 0$, Φ has exactly $2kp/n$ zeros in $[0, 2\pi/n[$, all of which are simple. Theorem 1 then applies, telling us that the degree is $1-kp/n$. If this degree is not zero (for instance if $k\geqslant 2$), then for any $b\neq 0$ there is a $2\pi/q$ – periodic solution. Moreover, the set of those solutions, as b varies in a neighborhood of 0, has to be unbounded, except perhaps if k is such that the degree corresponding to $b=0$, which cannot be deduced from Theorem 1, happens to be $1-kp/n$, as well.

Remark 8. The conclusions of Corollary 3 remain unchanged, when n is even, if we add to $f(t)$ a function $e(t)$ which is such that

$$e(t+\pi)=-e(t)$$

(the Fourier series of e contains only harmonic terms of odd order). In fact, setting

$$\Phi_1(t) = \int_0^{2\pi} e(t)\,\varphi(t+\theta)\,dt,$$

it is easily seen that $\Phi_1(\theta+\pi) = -\Phi_1(\theta)$. Since Φ_1 is $2\pi/n-$periodic, n being even, this implies that Φ_1 is identically 0. Consequently, we find the same expression for Φ, and we conclude as above.

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