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Periodic oscillations of forced pendulums with very small length*

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We prove the existence of an arbitrarily large number of periodic solutions for a class of nonlinear differential equations generalising the dynamics of a forced pendulum with small length.

1. Introduction

Consider a simple pendulum with unitary mass and length l, and assume that there is a periodic time-dependent external force $e(t)$ acting on it. The angular displacement then follows the well-known differential equation

$$x'' + \frac{g}{l}\sin x = e(t), \tag{1.1}$$

g being the acceleration of gravity. Let us vary the length l and see what happens when l becomes smaller and smaller.

A simple change of variable transforms (1.1) into the equation

$$x'' + \sin x = \frac{l}{g}\, e\left(t\sqrt{\frac{l}{g}}\right). \tag{1.2}$$

At first sight, l being small, (1.2) can be considered to be a perturbation of the autonomous equation $x'' + \sin x = 0$, which has infinitely many periodic solutions. On the other hand, a closer look at equation (1.2) shows that the period of the right-hand side becomes larger and larger as l becomes small: if $T > 0$ is the period of $e(t)$, the period of the right-hand side of (1.2) is $T_l = T\sqrt{g/l}$.

Two questions then naturally arise: Can we say that the number of periodic

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solutions of (1.1) becomes larger and larger as l becomes small? Are there subharmonic solutions, i.e. solutions whose minimal period is a multiple of T?

We have found a partial answer to our first question in two papers published by Hammerstein [5] and Iglish [7] in the 1930s. There, a shooting method was used to prove that there are many solutions for equation (1.1), with Dirichlet boundary conditions, when l is small. As an easy consequence, one has many T-periodic solutions for (1.1) if $e(t)$ is assumed to be odd.

Since then, a large number of contributions to the study of equation (1.1) have appeared in the literature. However, to our knowledge, a complete answer to our first question, for a general forcing, seems not yet to be available. In this regard, we refer to the interesting survey by Mawhin [11] on the pendulum equation, considered as an inexhaustible source of dynamical situations and as a testing model for different techniques in nonlinear analysis.

Concerning our second question, the existence of subharmonic solutions has been proved as a by-product of some theorems on 'chaos', but again at the expense of additional assumptions on $e(t)$ (see [**1**, **13**]). Otherwise, one can apply some results from [**4**, **10**], but only for a 'generic' forcing term $e(t)$ with mean value zero.

In this paper, we answer in the affirmative both the above questions, for a more general equation of the type

$$x'' + Af(x) = e(t), \tag{1.3}$$

where $e(t)$ is an arbitrary continuous and T-periodic forcing term. The assumptions we choose on the restoring force $f(x)$ naturally cover the case of equation (1.1). For instance, we can prove the following theorem:

THEOREM 1.1. *Assume $f: \mathbb{R} \to \mathbb{R}$ to be continuous, globally bounded, differentiable at a point x_0 with $f(x_0) = 0$ and $f'(x_0) > 0$. Then, given any two positive integers, M, N, there exists a constant $\bar{A} > 0$ such that, for any $A \geqq \bar{A}$, equation* (1.3) *has, for each $k = 1, 2, \ldots, M$, at least N periodic solutions with minimal period kT. Concerning the T-periodic solutions, they can be chosen to have exactly $2j$ simple crossings with x_0 in the interval $[0, T[$, with $j = 1, 2, \ldots, N$.*

With the aim of obtaining a similar result by assuming only local conditions on the restoring force $f(x)$, we also consider a situation in which the related autonomous equation has a heteroclinic (or homoclinic) orbit. Unfortunately, we are not able to prove in this case that all the above periodic solutions survive to the perturbation. Roughly speaking, we lose control on those having a low number of zeros, maybe because they come too near to the heteroclinic orbit. Nevertheless, we can prove that the number of harmonic and subharmonic periodic solutions still increases to infinity as A grows.

THEOREM 1.2. *Assume $f: [a, b] \to \mathbb{R}$ to be differentiable and let $x_0 \in]a, b[$ be such that $f(x_0) = 0$ and $f'(x_0) > 0$. Assume moreover $f(a) = f(b) = 0$ and*

$$f(x) < 0 \quad \text{if } x \in]a, x_0[, \qquad f(x) > 0 \quad \text{if } x \in]x_0, b[.$$

Then, given any two positive integers M, N, there exists a constant $\bar{A} > 0$ such that, for any $A \geqq \bar{A}$, equation (1.3) *has, for each $k = 1, 2, \ldots, M$, at least N periodic solutions with minimal period kT.*

Theorem 1.2 should be compared with the result in [1], where Battelli and Palmer prove, for large A, the existence of a transversal heteroclinic point of the Poincaré map associated to equation (1.3), with the accompanying chaotic behaviour. However, they need the functions $f(x)$ and $e(t)$ to be at least eight times continuously differentiable and, moreover, the existence of a simple zero for $e(t)$.

Theorems 1.1 and 1.2 are actually obtained as consequences of our main results which will be stated for an equation of the type

$$x'' + f(x) = \varepsilon e(t, x, \varepsilon), \tag{1.4}$$

with a continuous restoring term $f(x)$ and a continuous forcing term $e(t, x, \varepsilon)$ which is periodic in its first variable, with a period which depends on ε. (The case of constant period has been already studied by many authors, cf. [**8, 9, 15**].)

In Section 2 we start with a setting of general assumptions and we prove some auxiliary lemmas which will be used in the proofs of our main theorems. These generalise Theorems 1.1 and 1.2 above and are stated and proved in Section 3. The proofs use a variant of the Poincaré–Birkhoff Fixed-Point Theorem due to W. Ding [3].

Without loss of generality, we assume, from now on, that $x_0 = 0$.

2. Some preliminary results

We consider equation (1.4), which we write in the form

$$x' = y, \quad y' = -f(x) + \varepsilon e(t, x, \varepsilon). \tag{2.1}$$

Here, $f: \mathbb{R} \to \mathbb{R}$ is a continuous function such that there are $a < 0 < b$, for which

(H_1) $\quad xf(x) > 0 \quad$ for all $x \in \,]a, 0[\, \cup \,]0, b[$,

and $e: \mathbb{R} \times \mathbb{R} \times \,]0, 1] \to \mathbb{R}$ is continuous, periodic in its first variable, with a period T_ε which depends on ε in such a way that

(H_2) $\quad \lim_{\varepsilon \to 0^+} T_\varepsilon = +\infty \quad$ and $\quad \lim_{\varepsilon \to 0^+} \varepsilon T_\varepsilon = 0.$

We also assume there is a constant $K > 0$ such that

(H_3) $\quad |e(t, x, \varepsilon)| \leqq K,$

for every $t \in \mathbb{R}$, $x \in \mathbb{R}$ and $\varepsilon \in \,]0, 1]$. We set $F(x) = \int_0^x f(s)\, ds$, and, to fix the ideas, we assume that $F(b) \leqq F(a)$, i.e.

$$\int_a^b f(s)\, ds \leqq 0.$$

Denote, for any $r \in \,]0, b]$, by Γ_r the curve obtained as the connected component of the set

$$\{(x, y) \in \mathbb{R}^2 : \tfrac{1}{2}y^2 + F(x) = F(r)\}$$

such that $\Gamma_r \cap (\mathbb{R} \times \{0\}) \subset [a, b]$. By assumption (H_1), Γ_r is a closed simple curve which delimits an open set Ω_r, contained in $[a, b] \times [-\sqrt{2F(b)}, \sqrt{2F(b)}]$ and star-shaped with respect to the origin. Moreover, $\Gamma_{r_1} \cap \Gamma_{r_2} = \varnothing$ for $r_1 \neq r_2$ and, for any

$r \in]0, b[$, $\lim_{s \to r} d_H(\Gamma_s, \Gamma_r) = 0$, where d_H denotes the Hausdorff distance. (If, instead of the above, we had $F(a) \leqq F(b)$, then we could repeat the same construction, with obvious modifications, taking $r \in [a, 0[$.)

We introduce the set

$$X(\varepsilon, r) = \{(x(t), y(t)) \in \mathbb{R}^2 : (x, y) \text{ is a solution of (2.1)} \\ \text{with } (x(0), y(0)) \in \Gamma_r \text{ and } |t| \leqq T_\varepsilon\},$$

made up of all those points of the plane which are reached at some time t with $|t| \leqq T_\varepsilon$, by the solutions starting at the time $t = 0$ from a point of Γ_r. In particular, $X(0, r) = \Gamma_r$. (We point out that at this stage of our discussion, we assume neither the uniqueness of the solutions for the initial value problems associated to (2.1), nor their continuability on $[-T_\varepsilon, T_\varepsilon]$.)

As a consequence of the following lemma, we have that the set $X(\varepsilon, r)$ approaches Γ_r as ε goes to zero, i.e.

$$\lim_{\varepsilon \to 0^+} \left(\max_{x \in X(\varepsilon, r)} \min_{y \in \Gamma_r} \|x - y\| \right) = 0.$$

LEMMA 2.1. *Assume* $(H_{1\text{-}3})$ *and choose* $r \in]0, b[$. *Then, given any* $\delta \in]0, \min\{r, b - r\}[$, *we have that, for sufficiently small* ε, *any solution starting at the time* $t = 0$ *from a point of* Γ_r *is defined and remains in* $\Omega_{r+\delta} \backslash \bar{\Omega}_{r-\delta}$ *for all* $|t| \leqq T_\varepsilon$.

Proof. By contradiction, suppose that, for some r and δ as above, there is a sequence $(\varepsilon_n)_n$ which tends to zero and, corresponding to each n, there is a solution (x_n, y_n), starting at time $t = 0$ from a point of Γ_r, which stays in $\Omega_{r+\delta} \backslash \bar{\Omega}_{r-\delta}$ until a certain time T_n, with $|T_n| < T_{\varepsilon_n}$, at which time it reaches $\Gamma_{r-\delta}$ or $\Gamma_{r+\delta}$. Writing

$$E_n(t) = \tfrac{1}{2}(y_n(t))^2 + F(x_n(t)),$$

for every $|t| \leqq T_n$ we have

$$\left| \frac{d}{dt} E_n(t) \right| = \varepsilon_n |e(t, x_n(t), \varepsilon_n| \, |y_n(t)| \leqq \varepsilon_n K \sqrt{2F(b)},$$

so that, setting $v = \min\{F(r) - F(r - \delta), F(r + \delta) - F(r)\}$, we have

$$0 < v \leqq |E_n(T_n) - E_n(0)| \leqq \varepsilon_n K \sqrt{2F(b)} T_n < K \sqrt{2F(b)} \varepsilon_n T_{\varepsilon_n}.$$

For n large, this contradicts assumption (H_2). □

We say that a solution (x, y) of (2.1) makes *at least* N rotations around the origin in the time τ if $(x(t), y(t)) \neq (0, 0)$, for all $t \in [0, \tau]$, and, considering polar coordinates

$$x(t) = \rho(t) \cos \theta(t), \quad y(t) = \rho(t) \sin \theta(t),$$

we have that $\theta(0) - \theta(\tau) \geqq 2N\pi$. We say it makes *more than* N rotations if $\theta(0) - \theta(\tau) > 2N\pi$. In the same spirit, we speak of solutions making *at most*, *less than*, or *exactly* N rotations around the origin.

LEMMA 2.2. *Assume* $(H_{1\text{-}3})$ *and*

$$(H_4) \quad \liminf_{s \to 0} \frac{f(s)}{s} > 0.$$

Then we can construct three sequences $(r_n)_n$, $(\varepsilon_n)_n$ *and* $(\tau_n)_n$ *with the following properties:*

(a) *For every* n, $r_n \in]0, b[$, $\varepsilon_n > 0$ *and* $0 < \tau_n < T_{\varepsilon_n}$.

(b) *As* $n \to \infty$, *we have* $r_n \to 0$, $\varepsilon_n \to 0$ *and* $(\tau_n)_n$ *remains bounded.*

(c) *Any solution starting at the time* $t = 0$ *from a point of* Γ_{r_n} *is defined for* $|t| \leqq T_{\varepsilon_n}$ *and makes more than one rotation around the origin in the time* τ_n.

Proof. By Lemma 2.1, we can choose two sequences $(r_n)_n$ and $(\varepsilon_n)_n$, tending to zero, in such a way that

$$X(\varepsilon_n, r_n) \subset \Omega_{2r_n} \backslash \bar{\Omega}_{2^{-1}r_n}$$

and, denoting by d_n the distance of $\Gamma_{2^{-1}r_n}$ from the origin,

$$\lim_{n \to \infty} \frac{\varepsilon_n}{d_n^2} = 0.$$

Let (x_n, y_n) be a solution of (2.1) starting from Γ_{r_n} at time $t = 0$. By the above inequality and the definition of $X(\varepsilon_n, r_n)$, this solution is defined at least on $[-T_{\varepsilon_n}, T_{\varepsilon_n}]$ and, moreover, $(x_n(t), y_n(t)) \neq (0, 0)$ for every t with $|t| \leqq T_{\varepsilon_n}$. Therefore, passing to polar coordinates, we can write

$$x_n(t) = \rho_n(t) \cos \theta_n(t), \quad y_n(t) = \rho_n(t) \sin \theta_n(t).$$

Standard computations lead to

$$-\theta_n'(t) = \frac{y_n^2(t) + [f(x_n(t)) - \varepsilon_n e(t, x_n(t), \varepsilon_n)] x_n(t)}{y_n^2(t) + x_n^2(t)}.$$

By assumption (H_4), there is an $\alpha > 0$ and a $\delta > 0$ such that, for $|s| \leqq \delta$, $f(s)s \geqq \alpha s^2$. Since $r_n \to 0$, for n large enough we have

$$-\theta_n'(t) \geqq \frac{y_n^2(t) + \alpha x_n^2(t)}{y_n^2(t) + x_n^2(t)} - \frac{\varepsilon_n K(b-a)}{d_n^2}.$$

Then, there is a $\gamma > 0$ such that, for sufficiently large n, one has $\theta_n'(t) > \gamma$, for every $|t| \leqq T_{\varepsilon_n}$. For n large, since $T_{\varepsilon_n} \to +\infty$, the solution is defined on $[0, 2\pi/\gamma]$, and makes more than one rotation around the origin in a certain time $\tau_n \leqq 2\pi/\gamma$. Hence, the result easily follows (taking subsequences, if needed). □

REMARK 2.3. Notice that assumption (H_4) itself guarantees the existence of $a < 0 < b$ for which (H_1) holds. This fact will permit us to state Theorem 3.1 below without the need to assume (H_1) explicitly.

LEMMA 2.4. *Assume* $(H_{1\text{-}3})$ *and*

$$(H_5) \quad \limsup_{s \to b^-} \frac{f(s)}{b - s} < +\infty.$$

Then we can construct three sequences $(s_n)_n$, $(\varepsilon_n)_n$ *and* $(\sigma_n)_n$ *with the following properties;*

(a) *For every* n, $s_n \in]0, b[$, $\varepsilon_n > 0$ *and* $0 < \sigma_n \leqq T_{\varepsilon_n}$.

(b) *As* $n \to \infty$, *we have* $s_n \to b$, $\varepsilon_n \to 0$ *and* $\sigma_n \to +\infty$.

(c) *Any solution starting at the time* $t=0$ *from a point of* Γ_{s_n} *is defined for* $|t| \leqq T_{\varepsilon_n}$ *and makes less than one rotation around the origin in the time* σ_n.

Proof. By Lemma 2.1, we can choose two sequences $(s_n)_n$ and $(\varepsilon_n)_n$ in such a way that

$$X(\varepsilon_n, s_n) \subset \Omega_b \backslash \bar{\Omega}_{2s_n - b}.$$

Let (x_n, y_n) be a solution of (2.1) starting from Γ_{s_n} at time $t=0$. This solution is defined at least on $[-T_{\varepsilon_n}, T_{\varepsilon_n}]$ and, for n large, $(x_n(t), y_n(t)) \neq (0,0)$ for every t with $|t| \leqq T_{\varepsilon_n}$. For simplicity of notation, we omit the subscript n in what concerns the solution (x_n, y_n), when no confusion can arise.

Assume that the solution makes at least one rotation around the origin in the time T_{ε_n}. Then one of the following two possibilities occurs. Either there are $t_1 < t_2$ such that

$$x(t_1)=0, \quad x'(t_2)=0 \quad \text{and} \quad x'(t)>0 \quad \text{for every } t_1<t<t_2;$$

or there are $t_1 < t_2$ such that

$$x'(t_1)=0, \quad x(t_2)=0 \quad \text{and} \quad x'(t)<0 \quad \text{for every } t_1<t<t_2.$$

Assume the first possibility occurs. First of all, we observe that $2s_n - b < x(t_2) < b$. Moreover, for every $t_1 < t < t_2$, we have

$$\frac{d}{dt}\left[\frac{1}{2}(x'(t))^2 + F(x(t)) + \varepsilon_n K x(t)\right] = (x''(t) + f(x(t)) + \varepsilon_n K)x'(t) \geqq 0.$$

Hence,

$$\tfrac{1}{2}(x'(t))^2 \leqq F(x(t_2)) - F(x(t)) + \varepsilon_n K(x(t_2) - x(t)).$$

By assumption (H_s), there is a $\beta > 0$ and a $\eta \in]0, b[$ such that, for $b - \eta \leqq s \leqq b$, $f(s) \leqq \beta(b-s)$. Assuming n sufficiently large, let $t_\eta \in]t_1, t_2[$ be such that $x(t_\eta) = b - \eta$. Then, for every $t \in]t_\eta, t_2[$, we have

$$(x'(t))^2 \leqq 2\int_{x(t)}^{x(t_2)} f(s)\,ds + 2\varepsilon_n K\eta \leqq \beta(b - x(t))^2 + 2\varepsilon_n K\eta.$$

Then,

$$\begin{aligned} t_2 - t_\eta &\geqq \int_{t_\eta}^{t_2} \frac{x'(t)}{\sqrt{\beta(b-x(t))^2 + 2\varepsilon_n K\eta}}\,dt \\ &= \int_{x(t_\eta)}^{x(t_2)} \frac{dx}{\sqrt{\beta(b-x)^2 + 2\varepsilon_n K\eta}} \\ &\geqq \int_{b-\eta}^{2s_n - b} \frac{dx}{\sqrt{\beta(b-x)^2 + 2\varepsilon_n K\eta}} \\ &= \frac{1}{\sqrt{\beta}}\left[\sinh^{-1}\left(\frac{\sqrt{\beta}\eta}{\sqrt{2\varepsilon_n K\eta}}\right) - \sinh^{-1}\left(\frac{\sqrt{\beta}(2b - 2s_n)}{\sqrt{2\varepsilon_n K\eta}}\right)\right] := \sigma_n. \end{aligned}$$

Thus the time needed to make at least one rotation around the origin is strictly greater than the above-defined σ_n. The same conclusion is reached if the second possibility occurs. Therefore, in any case, the solution makes less than one rotation around the origin in the time σ_n. The result easily follows (taking subsequences, if needed), since σ_n goes to infinity with an order of growth like $[-\ln(b-s_n)]$. □

In the proof of our main results, we will also need the following lemma concerning the distribution of prime numbers on the real line.

LEMMA 2.5. *Let $(p_n)_n$ and $(q_n)_n$ be two real sequences such that, for every n, $1 \leqq p_n \leqq q_n$ and*

$$\lim_{n\to\infty} \frac{q_n}{p_n} = +\infty.$$

Then the number of primes between p_n and q_n becomes larger and larger as n goes to infinity.

Proof. Denote, for any real $x \geqq 1$, by $\Pi(x)$ the number of primes smaller than x, and set $\Sigma(x) = x/\ln x$. The Prime Number Theorem asserts that

$$\lim_{x\to+\infty} \frac{\Pi(x)}{\Sigma(x)} = 1.$$

The function $\Sigma(x)$ is positive, increasing and concave for $x \geqq e^2$. Without loss of generality, we can assume that $p_n \to \infty$ as $n \to \infty$, and $p_n \geqq e^2$ for every n. Then, setting $R(x) = \Pi(x) - \Sigma(x)$, we have

$$\begin{aligned}\Pi(q_n) - \Pi(p_n) &\geqq \Sigma'(q_n)(q_n - p_n) + R(q_n) - R(p_n)\\ &\geqq \frac{\ln q_n - 1}{(\ln q_n)^2}(q_n - p_n) - \left(\frac{|R(q_n)|}{\Sigma(q_n)} + \frac{|R(p_n)|}{\Sigma(p_n)}\right)\Sigma(q_n)\\ &= \Sigma(q_n)\left[\left(1 - \frac{1}{\ln q_n}\right)\left(1 - \frac{p_n}{q_n}\right) - \left(\frac{|R(q_n)|}{\Sigma(q_n)} + \frac{|R(p_n)|}{\Sigma(p_n)}\right)\right],\end{aligned}$$

which implies that $\Pi(q_n) - \Pi(p_n) \to \infty$ as $n \to \infty$. □

3. The main results

THEOREM 3.1. *Assume (H_{2-4}) and*

$$(H_6) \quad \lim_{|x|\to\infty} \frac{F(x)}{x^2} = 0.$$

Then, given any two positive integers, M, N, there exists a constant $\bar{\varepsilon} > 0$, such that, for any $0 \leqq \varepsilon \leqq \bar{\varepsilon}$, equation (1.3) has, for each $k = 1, 2, \ldots, M$, at least N periodic solutions having period kT_ε. If $k \geqq 2$, these solutions can be chosen not to have a smaller period in the set $\{T_\varepsilon, 2T_\varepsilon, \ldots, (k-1)T_\varepsilon\}$. Concerning the T_ε-periodic solutions, they can be chosen to have exactly 2j simple zeros in the interval $[0, T[$, with $j = 1, 2, \ldots, N$.

Proof. Assumptions (H_3) and (H_6) guarantee that the solutions to the Cauchy problems associated to equation (2.1) are globally defined (cf. [14]). Take two sequences $(r_n)_n$ and $(\varepsilon_n)_n$ as in Lemma 2.2. For n large enough, the solutions starting from any point of Γ_{r_n} will make, in the time T_{ε_n}, at least $N+1$ rotations. On the other hand, condition (H_6) implies that the solutions starting from a point of a sufficiently large circle $S_R=\{(u,v)\in\mathbb{R}^2: u^2+v^2=R^2\}$ will not be able to rotate around the origin even once in time T_{ε_n} (cf. [12]). Let us fix ε and an annulus with inner boundary Γ_r and outer boundary S_R with the above properties.

If we assume that the solutions to the Cauchy problems associated to equation (2.1) are unique, so that the Poincaré map can be defined, then the generalised version of the Poincaré–Birkhoff Fixed-Point Theorem due to W. Ding [3] yields the existence of T_{ε_n}-periodic solutions which make exactly $1, 2, \ldots, N+1$ rotations around the origin, respectively (cf. [2, 14]).

If we do not have uniqueness for the Cauchy problems associated to (2.1), we proceed as follows. First of all, we take $R_1>R$ such that any solution (x, y) of (2.1) with $x(0)^2+y(0)^2\leqq R^2$ is such that $x(t)^2+y(t)^2\leqq R_1^2$, for every $|t|\leqq T_\varepsilon$. Next, we find sequences of functions f_j, e_j, which converge uniformly on compact sets to f, e, respectively, and such that the Cauchy problems associated to the equations

$$x'=y,\quad y'=-f_j(x)+\varepsilon e_j(t,x,\varepsilon) \tag{3.1$_j$}$$

with initial value as above, have a unique solution defined on $[-T_\varepsilon, T_\varepsilon]$ such that $x(t)^2+y(t)^2\leqq R_1^2+1$, for every $|t|\leqq T_\varepsilon$. We want to prove that, for j large enough, any solution from Γ_r will make more than N rotations in time T_ε, while any solution starting from S_R will not be able to rotate even once. Assume by contradiction that, for a subsequence, there are initial points, $(x_{0,j}, y_{0,j})$ in Γ_r from which a solution (x_j, y_j) of $(3.1)_j$ departs which makes at most N rotations around the origin in the time T_ε. For a further subsequence, we can assume there is a $(\bar{x},\bar{y})\in\Gamma_r$ such that $(x_{0,j}, y_{0,j})\to(\bar{x},\bar{y})$. By [6, Theorem 3.2, p. 14], there is a solution (x, y) of (2.1) with $(x(0), y(0))=(\bar{x},\bar{y})$ such that a further subsequence (x_j, y_j) converges to it. This contradicts the fact that (x, y) makes at least $N+1$ rotations around the origin. A similar argument works for the solutions starting from S_R.

Ding's theorem can then be applied to get the desired periodic solutions of equations $(3.1)_j$. A standard argument makes these solutions converge, as j goes to infinity, to solutions of (2.1), which preserve the number of rotations.

It is clear that the analogues of Lemmas 2.1–2.4 hold, replacing T_ε with a new period $\bar{k}T_{\varepsilon_n}$, for $\bar{k}\geqq 2$. The solutions starting from any point of Γ_{r_n} will make more than a certain number of q_n of rotations around the origin in the time $\bar{k}T_{\varepsilon_n}$, with $q_n\to\infty$ as $n\to\infty$. We make n so large that it is possible to find N prime numbers smaller than q_n. On the other hand, condition (H_6) implies that the solutions starting from a point of a sufficiently large circle S_R will not be able to rotate around the origin even once in the time $\bar{k}T_{\varepsilon_n}$.

Ding's theorem then yields the existence of N periodic solutions having period $\bar{k}T_{\varepsilon_n}$ and making a prime number of rotations around the origin in the time $\bar{k}T_{\varepsilon_n}$. It is now easy to check that these solutions cannot have a smaller period in the set $\{T_\varepsilon, 2T_\varepsilon, \ldots, (\bar{k}-1)T_\varepsilon\}$ (see [2, 14]). The proof of the theorem can now be easily completed. □

REMARK 3.2. Notice that Theorem 1.1 follows immediately from Theorem 3.1.

THEOREM 3.3. *Assume $(H_{1\text{-}5})$. Then, given any two positive integers M, N, there exists a $\bar{\varepsilon}>0$ such that, for any $0\leqq\varepsilon\leqq\bar{\varepsilon}$, equation* (1.3) *has, for each $k=1, 2, \ldots, M$, at* *least N periodic solutions having period kT. If $k \geqq 2$, these solutions can be chosen not to have a smaller period in the set $\{T_\varepsilon, 2T_\varepsilon, \dots, (k-1)T_\varepsilon\}$.*

Proof. It can be seen from the proofs that it is possible to choose sequences $(r_n)_n$, $(s_n)_n$, and a sequence $(\varepsilon_n)_n$ to satisfy both Lemmas 2.2 and 2.4. Moreover, the lemmas in Section 2 can be stated for a general period $\bar{k}T_{\varepsilon_n}$, for $\bar{k} \geqq 1$. By Lemma 2.1, the solutions starting either from Γ_{r_n} or from Γ_{s_n} are defined at least on $[-\bar{k}T_{\varepsilon_n}, \bar{k}T_{\varepsilon_n}]$. Those starting from Γ_{r_n} make more than a certain number q_n of rotations around the origin in the time $\bar{k}T_{\varepsilon_n}$, with

$$q_n > \frac{\bar{k}T_{\varepsilon_n}}{\tau_n} - 1,$$

(see Lemma 2.2). On the other hand, the solutions starting from Γ_{s_n} rotate less than p_n times, with

$$p_n \leqq \frac{\bar{k}T_{\varepsilon_n}}{\sigma_n} + 1,$$

(see Lemma 2.4). Since $(\tau_n)_n$ is bounded and $\sigma_n \to \infty$ as $n \to \infty$, we have

$$\lim_{n\to\infty} \frac{q_n}{p_n} = +\infty.$$

The arguments used in the proof of Theorem 3.1 together with Lemma 2.5 can now be used to complete the proof. □

Final remarks

Notice that Theorem 3.3 is of a local nature, and we could have assumed $f(\cdot)$ and $e(t, \cdot, \varepsilon)$ to be defined only on $[a, b]$. In particular, Theorem 3.3 applies, under (H_{1-3}), when $f: [a, b] \to \mathbb{R}$ is a differentiable function with $f'(0) > 0$, with the further assumption that

$$f(b) = 0 \quad \text{and} \quad \int_a^b f(s)\, ds \leqq 0.$$

Symmetrically, as remarked in Section 2, the above can be substituted by

$$f(a) = 0 \quad \text{and} \quad \int_a^b f(s)\, ds \geqq 0.$$

From these facts, Theorem 1.2 can be obtained as a corollary of Theorem 3.3.

We could consider, for equation (1.4), a wide choice of functions $f(x)$ satisfying the above assumptions. For instance, besides the sine function appearing in the pendulum equation, we can deal with any polynomial function having at least two zeros, with a positive derivative in at least one of them. Typical examples of such functions are

$$f(x) = \alpha + \beta x^2 \quad \text{or} \quad f(x) = \alpha x + \beta x^3,$$

with $\alpha\beta < 0$.

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