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PERIODIC SOLUTIONS FOR A CONSERVATIVE SYSTEM OF DIFFERENTIAL EQUATIONS WITH A SINGULARITY OF REPULSIVE TYPE

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1. INTRODUCTION

We consider the system

$$\ddot{u}(t) + \nabla G(u(t)) = e(t), \tag{1}$$

where $G: \mathbb{R}^N\backslash\{0\} \to \mathbb{R}$ is a continuously differentiable potential with a singularity of repulsive type at zero, and $e: \mathbb{R} \to \mathbb{R}^N$ is a locally integrable T-periodic forcing ($T > 0$).

As a model for our equation, we have in mind the motion of a charged particle of unitary mass in presence of a fixed charge with the same sign. Once an exterior force $e(t)$ is introduced, varying periodically in time, with the natural choice of coordinates, we have the equation

$$\ddot{u}(t) - \kappa \frac{u(t)}{|u(t)|^3} = e(t), \tag{2}$$

where $\kappa > 0$ is a constant. We will show that, if the mean value of $e(t)$

$$\bar{e} = \frac{1}{T}\int_0^T e(t)\,\mathrm{d}t$$

is nonzero, equation (2) has infinitely many periodic solutions, whose periods are integer multiples of T.

The assumption $\bar{e} \neq 0$ comes out to be also necessary for the existence of periodic solutions of (2) when the amplitude of the forcing term is small enough (see [1]); on the other hand, circular orbits can be easily found, e.g. when $e(t) = (\rho\cos t, \rho\sin t, 0, \dots, 0)$, for $\rho > 0$ sufficiently large.

Our main result is the following theorem.

THEOREM 1. Assume the following conditions:

(i) $$\lim_{x\to 0} G(x) = +\infty;$$

(ii) $$\limsup_{x\to 0}\langle \nabla G(x), x\rangle < 0;$$

(iii) $$\exists\, \hat{w} \in \mathbb{R}^N: \limsup_{|x|\to\infty}|\nabla G(x) - \hat{w}| < |\bar{e} - \hat{w}|.$$

Then, equation (1) as a sequence $(u_k)_{k\geq 1}$ of kT-periodic solutions whose minimal periods tend to infinity.

It is easy to see that the assumptions of theorem 1 are satisfied in the setting of equation (2), taking $\hat{w} = 0$, when $\bar{e} \neq 0$. It seems to be an open problem to know whether, under the assumptions of theorem 1, equation (1) has periodic solutions having minimal period kT, for any sufficiently large integer k. However, if T is the minimal period of $e(t)$, theorem 1 implies the existence of solutions having kT as minimal period, for any sufficiently large prime integer k.

The existence of at least one T-periodic solution for equation (1) has already been proved by Solimini [1], and Habets and Sanchez [2] under assumptions similar to those of theorem 1 (see also [3]). More precise results are available when $N = 1$, where, besides topological degree arguments [4, 5] and variational methods [6, 7], a specific analysis in the phase plane permits the application of powerful Poincaré–Birkhoff-type theorems (see [8, theorem 6.3]).

The proof of theorem 1 uses some variational arguments which have been introduced for a different setting in [9]. However, we are not able to consider equation (1) directly, because of the loss of compactness generated by the singularity. So, we are led to introduce some approximating equations with no singularities. The action functionals corresponding to the approximating equations have a geometry of mountain-pass type and satisfy the Palais–Smale condition. The solutions found for the approximating equations are then shown to be also solutions of equation (1). Finally, the minimal periods are shown to tend to infinity by some careful estimates on the critical levels of the considered functionals.

2. PROOF OF THE MAIN RESULT

First notice that, subtracting the mean value of $e(t)$ from both sides of equation (1) and defining $\tilde{G}(x) = G(x) - \langle \bar{e}, x\rangle$ and $\tilde{e}(t) = e(t) - \bar{e}$, the assumptions of theorem 1 remain verified by $\tilde{G}$ and $\tilde{e}$ (in (iii), replace $\hat{w}$ by $\hat{w} - \bar{e}$). Therefore, from now on, we will assume without loss of generality that

$$\bar{e} = \frac{1}{T}\int_0^T e(t)\,dt = 0. \tag{3}$$

We begin with providing an a priori estimate for the periodic solutions of (1) having a fixed period. The following lemma tells us that these solutions are bounded, and bounded away from the origin, as well. We denote the Euclidean product and norm in $\mathbb{R}^N$ by $\langle\cdot,\cdot\rangle$ and $|\cdot|$, respectively. The open ball centered at 0 with radius ρ is denoted by B_ρ and its closure by $\bar{B}_\rho$. The usual L^p-norm (over an interval which will be clear from the context) is denoted by $\|\cdot\|_p$.

LEMMA 1. Under the assumptions of theorem 1, for every $k \geq 1$ there are two positive constants d_k, D_k such that any kT-periodic solution u of (1) satisfies

$$d_k \leq |u(t)| \leq D_k,$$

for every $t \in \mathbb{R}$.

Proof. We argue by contradiction. Fix $k \geq 1$, and, for every $n \geq 1$, let u_n be a kT-periodic solution of (1) whose orbit is not contained in $\bar{B}_n \backslash B_{1/n}$. Let $\rho_n(t) = |u_n(t)|$, and let $\tau_{n_*}, \tau_n^* \in [0, kT]$ be such that $\rho_n(t)$ has its minimum and maximum values at $t = \tau_{n_*}, \tau_n^*$, respectively.

We first prove that the sequence $(\rho_n(\tau_n^*))_n$ is bounded. Integrating both sides of the equation

$$\ddot{u}_n(t) + \nabla G(u_n(t)) = e(t), \tag{4}$$

and taking into account (3), we have

$$\int_0^{kT} \nabla G(u_n(t))\,\mathrm{d}t = 0.$$

By (iii), there must be a $R_1 > 0$ such that every orbit of the solutions u_n intersects the ball B_{R_1}. Assume that $\rho_n(\tau_n^*) > R_1$. Then there are $t_n^{(1)} < \tau_n^* < t_n^{(2)}$ such that

$$\rho_n(t_n^{(1)}) = R_1 = \rho_n(t_n^{(2)}),$$

and

$$R_1 < \rho_n(t) \leq \rho_n(\tau_n^*), \qquad \text{for } t \in (t_n^{(1)}, t_n^{(2)}).$$

Besides, we have

$$\rho_n'(t) = \frac{1}{\rho_n(t)} \langle \dot{u}_n(t), u_n(t) \rangle,$$

so that

$$\rho_n''(t) = \frac{1}{\rho_n(t)} [\langle \ddot{u}_n(t), u_n(t) \rangle + |\dot{u}_n(t)|^2 - (\rho_n'(t))^2] \geq -|\ddot{u}_n(t)|.$$

Then, for $t \in [t_n^{(1)}, \tau_n]$, we have

$$\begin{aligned}
\rho_n(t) &= \rho_n(t_n^{(1)}) + \int_{t_n^{(1)}}^{t} [\rho_n'(\tau_n) + \int_{\tau_n}^{s} \rho_n''(\xi)\,\mathrm{d}\xi]\,\mathrm{d}s \\
&= R_1 - \int_{t_n^{(1)}}^{t} \int_{s}^{\tau_n} \rho_n''(\xi)\,\mathrm{d}\xi\,\mathrm{d}s \\
&\leq R_1 + \int_{t_n^{(1)}}^{t} \int_{s}^{\tau_n} |\ddot{u}_n(\xi)|\,\mathrm{d}\xi\,\mathrm{d}s \\
&\leq R_1 + kT\|e\|_1 + \int_{t_n^{(1)}}^{t} \int_{s}^{\tau_n} |\nabla G(u_n(\xi))|\,\mathrm{d}\xi.
\end{aligned}$$

By (iii), we can conclude that there is a constant $R_2 > R_1$ such that, for every n, $\rho_n(\tau_n^*) \leq R_2$, i.e.

$$\|u_n\|_\infty \leq R_2. \tag{5}$$

Then, for $n \geq R_2$, it must be

$$\rho_n(\tau_{n_*}) < \frac{1}{n}. \tag{6}$$

Multiplying both sides of (4) by u_n and integrating, we get

$$\int_0^{kT} |\dot{u}_n|^2 = \int_0^{kT} \langle \nabla G(u_n), u_n \rangle - \int_0^{kT} \langle e, u_n \rangle. \tag{7}$$

Using (ii), (iii) and (5), we find a constant $C_1 > 0$ such that, for every n,

$$\|\dot{u}_n\|_2 \leq C_1. \tag{8}$$

Let $\delta_1 > 0$ and $r_1 > 0$ be such that, by (ii),

$$\langle \nabla G(u), u \rangle + \|e\|_1 |u| \leq -\delta_1, \qquad \text{for } |u| \leq r_1.$$

Then, from (7), we see that there exists a $\delta_2 > 0$ such that

$$\text{meas}\{t \in [0, kT]: \rho_n(t) \geq r_1\} \geq \delta_2,$$

for every n. Then, because of (8), there is a constant $C_2 > 0$ and, for every n, there is a $t_n^{(3)}$ such that

$$\rho_n(t_n^{(3)}) \in [r_1, R_2] \qquad \text{and} \qquad |\dot{u}_n(t_n^{(3)})| \leq C_2.$$

Let us now multiply equation (4) by $\dot{u}_n$ and integrate over $[t_n^{(3)}, t]$. We have

$$\tfrac{1}{2}|\dot{u}_n(t)|^2 - \tfrac{1}{2}|\dot{u}_n(t_n^{(3)})|^2 + G(u_n(t)) - G(u_n(t_n^{(3)})) = \int_{t_n^{(3)}}^{t} \langle e(s), \dot{u}_n(s) \rangle \, ds. \tag{9}$$

Then,

$$\tfrac{1}{2}\|\dot{u}_n\|_\infty^2 \leq \tfrac{1}{2}|\dot{u}_n(t_n^{(3)})|^2 + G(u_n(t_n^{(3)})) + \max\{-G(x): x \in \bar{B}_{R_2}\} + \|e\|_1 \|\dot{u}_n\|_\infty,$$

and we can conclude that there is a constant C_3 such that, for every n,

$$\|\dot{u}_n\|_\infty \leq C_3.$$

Taking $t = \tau_{n_*}$ in (9), we get

$$G(u_n(\tau_{n_*})) \leq G(u_n(t_n^{(3)})) + \tfrac{1}{2}C_2^2 + \|e\|_1 C_3,$$

and we have a contradiction with (i) and (6). ∎

Remark that condition (iii) together with (3) imply that

$$\lim_{m \to +\infty} G(m\hat{w}) = +\infty. \tag{10}$$

By (i) and the above, since adding a constant to G does not affect equation (1), we may assume from now on, without loss of generality, that G be positive on the half-line $\{r\hat{w}: r > 0\}$.

In order to avoid some difficulties coming from the fact that G has a singularity, we will modify the equation by a truncation, like in [1]. Consider the family of functions $\eta_m \in C^1(\mathbb{R}, \mathbb{R})$, $m > 0$, such that

$$\eta_m(s) = s \qquad \text{for } s \leq m,$$

$$0 \leq \eta_m'(s) \leq 1 \qquad \text{for every } s \in \mathbb{R},$$

$$\eta_m(s) = m + 1 \qquad \text{for } s \geq m + 2.$$

Set $m^* := \max_{|x|=1} G(x)$, and define, for $m > m^*$, the modified potential

$$G_m(x) = \begin{cases} G(x) & \text{if } |x| \geq 1 \\ \eta_m(G(x)) & \text{if } |x| < 1 \text{ and } x \neq 0 \\ m+1 & \text{if } x = 0. \end{cases}$$

Then, $G_m\colon \mathbb{R}^N \to \mathbb{R}$ is continuously differentiable, and if $|x| < 1$ and $x \neq 0$,

$$\nabla G_m(x) = \eta_m'(G(x))\, \nabla G(x). \tag{11}$$

Notice, moreover, that G_m is positive on the half-line $\{r\hat{w}: r > 0\}$. This fact will be used at the end of the proof, when we will have to estimate the critical levels of some functionals. We consider the modified equations

$$\ddot{u}(t) + \nabla G_m(u(t)) = e(t), \tag{12}$$

and, for every $k \geq 1$, we define the functionals

$$\varphi_{k,m}(u) = \int_0^{kT} [\tfrac{1}{2}|\dot{u}(t)|^2 - G_m(u(t)) + \langle e(t), u(t)\rangle]\, \mathrm{d}t,$$

defined on the Sobolev space H^1_{kT} of kT-periodic functions with square integrable derivative. It is well known that the critical points of $\varphi_{k,m}$ correspond to the kT-periodic solutions of (12).

LEMMA 2. The functionals $\varphi_{k,m}$ satisfy the Palais–Smale condition.

Proof. Let $(u_n)_n$ be a sequence in H^1_{kT} such that $(\varphi_{k,m}(u_n))_n$ is bounded and $\varphi'_{k,m}(u_n) \to 0$ as $n \to \infty$. In particular, for any $\varepsilon > 0$ one has

$$|\varphi'_{k,m}(u_n)v| = \left| \int_0^{kT} [\langle \dot{u}_n(t), \dot{v}(t)\rangle - \langle \nabla G_m(u_n(t)) - e(t), v(t)\rangle]\, \mathrm{d}t \right| \leq \varepsilon \|v\|_{H^1_{kT}},$$

for n large enough. Taking $v = \tilde{u}_n := u_n - (1/kT)\int_0^{kT} u_n(s)\,\mathrm{d}s$, by the use of Wirtinger's inequality we see that $(\tilde{u}_n)$ has to be bounded. Assume by contradiction that $(u_n)_n$ is not bounded. Then, for a subsequence, $\min|u_n(t)| \to \infty$ as $n \to \infty$. Taking $v(t) \equiv \hat{w}$, we have

$$\int_0^{kT} \langle \nabla G_m(u_n(t)), \hat{w}\rangle\, \mathrm{d}t \leq \varepsilon (kT)^{1/2} |\hat{w}|.$$

By (iii), there is a $\delta > 0$ such that, for any $|x|$ large enough,

$$\langle \nabla G_m(x), \hat{w}\rangle \geq \delta |\hat{w}|.$$

Taking $\varepsilon < \delta(kT)^{1/2}$, we get a contradiction. Hence, the sequence $(u_n)_n$ is bounded, and a standard argument then shows that it must have a convergent subsequence (cf. [10]). ■

Define, for $k \geq 1$, the set

$$U_k = \{u \in H^1_{kT}: |u(t)| \geq 1 \text{ for every } t, \text{ and there exists a } t_u \text{ for which } |u(t_u)| = 1\}.$$

The following lemma shows that the functionals $\varphi_{k,m}$ have, for m large enough, a mountain pass geometry.

LEMMA 3. For every $k \geq 1$, there are $a_k \in \mathbb{R}$ and $m_k > 0$ such that, for any $m \geq m_k$,

$$\max\{\varphi_{k,m}(0), \varphi_{k,m}(m\hat{w})\} < a_k < \inf_{U_k} \varphi_{k,m}.$$

Proof. From assumption (iii), it is easy to see that a constant $c > 0$ exists such that, for every $|x| \geq 1$ and $m > m^*$,

$$G_m(x) = G(x) \leq c(1 + |x|).$$

Hence, for $u \in U_k$,

$$\varphi_{k,m}(u) \geq \tfrac{1}{2}\|\dot{u}\|_2^2 - ckT(1 + \|u\|_\infty) - \|e\|_1\|u\|_\infty.$$

Since, for $u \in U_k$, one has

$$\|u\|_\infty \leq 1 + (kT)^{1/2}\|\dot{u}\|_2,$$

one can easily find a $a_k \in \mathbb{R}$ such that

$$\inf_{U_k} \varphi_{k,m} \geq a_k,$$

for every $k \geq 1$ and $m > m^*$.

On the other hand, we have

$$\varphi_{k,m}(0) = -kT(m + 1),$$

$$\varphi_{k,m}(m\hat{w}) = -kTG_m(m\hat{w}).$$

From (10), we have

$$\lim_{m \to +\infty} G_m(m\hat{w}) = +\infty,$$

and it is then easy to find a $m_k > M^*$ such that, for $m \geq m_k$, one has

$$\min\{(m + 1), G_m(m\hat{w})\} > -\frac{a_k}{kT}.$$

The proof follows immediately. ∎

We are now in the position to apply the theory of critical points of min-max type (cf. [10, 11]). For any $k \geq 1$ and $m \geq m_k$, there exists a critical point $u_{k,m}$ of $\varphi_{k,m}$ such that

$$\varphi_{k,m}(u_{k,m}) = \inf_{\gamma \in \Gamma_{k,m}} \max_{\xi \in [0,1]} \varphi_{k,m}(\gamma(\xi)),$$

where $\Gamma_{k,m} = \{\gamma \in C([0, 1], H^1_{kT})\colon \gamma(0) = 0, \gamma(1) = m\hat{w}\}$. Consequently, $u_{k,m}$ is a kT-periodic function such that

$$\ddot{u}_{k,m}(t) + \nabla G_m(u_{k,m}(t)) = e(t), \tag{13}$$

and

$$\varphi_{k,m}(u_{k,m}) \geq a_k. \tag{14}$$

We will now construct an increasing sequence $(m'_k)_{k\ge 1}$, $m'_k \ge \max\{m_k, 2k\}$, such that, if $m \ge m'_k$, then $u_{k,m}$ is a solution of equation (1) and, moreover,

$$G_{m'_k}(x) = G(x) \qquad \text{for every } x \in \bar{B}_{D_k}\backslash B_{d_k}, \tag{15}$$

where d_k and D_k are given by lemma 1.

Let us fix $k \ge 1$. Integrating (13), we have

$$\int_0^{kT} \nabla G_m(u_{k,m}(t))\,\mathrm{d}t = 0.$$

By (iii), there is a $R_1 > 0$ such that the orbit of $u_{k,m}$ intersects B_{R_1}, for every $m > m^*$. Arguing as in lemma 1, we find a constant $R_2 > R_1$ such that

$$\|u_{k,m}\|_\infty \le R_2, \tag{16}$$

for every $m > m^*$. Multiplying (13) by $u_{k,m}$ and integrating, we have

$$\int_0^{kT} |\dot{u}_{k,m}|^2 = \int_0^{kT} \langle \nabla G_m(u_{k,m}), u_{k,m}\rangle - \int_0^{kT} \langle e, u_{k,m}\rangle.$$

By (ii), (iii), (11) and (16), we can find a constant $C_1 > 0$ such that

$$\|\dot{u}_{k,m}\|_2 \le C_1, \tag{17}$$

for every $m > m^*$. Take $r_1 > 0$ and $m_k^0 \ge \max\{m_k, 2k\}$ such that, for $m \ge m_k^0$,

$$\min_{B_{r_1}} G_m > \frac{1}{kT}\left(\frac{1}{2}C_1^2 - a_k + \|e\|_1 r_1\right).$$

By (14), there exists a $\delta > 0$ such that

$$\text{meas}\{t \in [0, kT]: u_{k,m}(t) \notin B_{r_1}\} \ge \delta.$$

Then, by (17), there is a $C_2 > 0$ and, for every $m \ge m_k^0$, there is a t_m such that

$$u_{k,m}(t_m) \in \bar{B}_{R_2}\backslash B_{r_1} \qquad \text{and} \qquad |\dot{u}_{k,m}(t_m)| \le C_2.$$

By an energy estimate, like in the proof of lemma 1, we may see that $(G_m(u_{k,m}(t)))$ is bounded from above, for every $t \in [0, kT]$ and $m \ge m_k^0$. Hence, there is a $r_2 > 0$ such that

$$u_{k,m}(t) \ge r_2,$$

for every $t \in [0, kT]$ and $m \ge m_k^0$. We can now take $m'_k \ge m_k^0$ as an increasing sequence such that $G_{m'_k}(x) = G(x)$ when $x \in (\bar{B}_{R_2}\backslash B_{r_2}) \cup (\bar{B}_{D_k}\backslash B_{d_k})$. Then, $u_{k,m}$ is a solution of (1), for every $m \ge m'_k$.

Set $u_k := u_{k,m'_k}$. By the above, u_k is a kT-periodic solution of (1). We want to prove that the minimal periods of these solutions tend to infinity. To this end, we first prove that

$$\lim_{k\to\infty} \frac{1}{k}\varphi_{k,m'_k}(u_k) = -\infty. \tag{18}$$

We will construct, for each $k \geq 1$, a continuous deformation $\gamma_k: [0, 1] \to H^1_{kT}$ such that, for $t \in [0, kT]$, $\gamma_k(0)(t) \equiv 0$, $\gamma_k(1)(t) \equiv m'_k \hat{w}$ and

$$\lim_{k \to \infty} \frac{1}{k} \max_{\xi \in [0,1]} \varphi_{k,m'_k}(\gamma_k(\xi)) = -\infty.$$

Let us denote by $|k/2|$ the integer part of $k/2$. We define $\gamma_k(\frac{1}{2})$ as follows

$$\gamma_k\left(\frac{1}{2}\right)(t) = \begin{cases} \left[1 + \sin\left(\frac{2\pi}{|k/2|T}t - \frac{\pi}{2}\right)\right]k\hat{w} & \text{if } t \in [0, |k/2|T] \\ 0 & \text{if } t \in (|k/2|T, kT]. \end{cases}$$

For $\xi \in [0, \frac{1}{2}]$, define $\gamma_k(\xi)$ as a convex combination of 0 and $\gamma_k(\frac{1}{2})$, and for $\xi \in [\frac{1}{2}, 1]$, as a convex combination of $\gamma_k(\frac{1}{2})$ and $m'_k \hat{w}$. Precisely,

$$\gamma_k(\xi) = \begin{cases} 2\xi\gamma_k(\frac{1}{2}) & \text{if } \xi \in [0, \frac{1}{2}] \\ 2[(1 - \xi)\gamma_k(\frac{1}{2}) + (\xi - \frac{1}{2})m'_k \hat{w}] & \text{if } \xi \in (\frac{1}{2}, 1]. \end{cases}$$

Let $\bar{\xi}_k \in [0, 1]$ be such that

$$\varphi_{k,m'_k}(\gamma_k(\bar{\xi}_k)) = \max_{\xi \in [0,1]} \varphi_{k,m'_k}(\gamma_k(\xi)).$$

We now estimate the three summands in

$$\frac{1}{k}\varphi_{k,m'_k}(\gamma_k(\bar{\xi}_k)) = \frac{1}{k}\int_0^{kT} \frac{1}{2}\left|\frac{\mathrm{d}}{\mathrm{d}t}\gamma_k(\bar{\xi}_k)(t)\right|^2 \mathrm{d}t - \frac{1}{k}\int_0^{kT} G_{m'_k}(\gamma_k(\bar{\xi}_k)(t))\,\mathrm{d}t$$
$$+ \frac{1}{k}\int_0^{kT} \langle e(t), \gamma_k(\bar{\xi}_k)(t)\rangle\,\mathrm{d}t.$$

We have

$$\frac{1}{k}\int_0^{kT} \frac{1}{2}\left|\frac{\mathrm{d}}{\mathrm{d}t}\gamma_k(\bar{\xi}_k)(t)\right|^2 \mathrm{d}t \leq \frac{1}{k}\int_0^{|k/2|T} \frac{1}{2}\left|\cos\left(\frac{2\pi}{|k/2|T}t - \frac{\pi}{2}\right)\frac{2\pi k}{|k/2|T}\hat{w}\right|^2 \mathrm{d}t$$
$$\leq \frac{2k\pi^2}{|k/2|T}|\hat{w}|^2,$$

and, hence, the first summand is bounded.

Let $E(t)$ be a primitive of $e(t)$. By (3), $E(t)$ is T-periodic, continuous, and

$$\frac{1}{k}\int_0^{kT} \langle e(t), \gamma_k(\bar{\xi}_k)(t)\rangle\,\mathrm{d}t = \frac{1}{k}\int_0^{|k/2|T} \left\langle E(t), \frac{\mathrm{d}}{\mathrm{d}t}\gamma_k(\bar{\xi}_k)(t)\right\rangle \mathrm{d}t$$
$$= \frac{4\pi c(\bar{\xi}_k)}{|k/2|T}\int_0^{|k/2|T} \left\langle E(t), \cos\left(\frac{2\pi}{|k/2|T}t - \frac{\pi}{2}\right)\hat{w}\right\rangle \mathrm{d}t,$$

where $c(\bar{\xi}_k)$ is either $\bar{\xi}_k$ or $(1 - \bar{\xi}_k)$, according to whether $\bar{\xi}_k$ is in $[0, \frac{1}{2}]$ or in $(\frac{1}{2}, 1]$, respectively. Writing the Fourier series of $E(t)$, one realizes that the above is equal to zero as far as $k \geq 4$. Hence,

$$\frac{1}{k}\int_0^{kT} \langle e(t), \gamma_k(\bar{\xi}_k)(t)\rangle \, dt = 0.$$

It remains to show that

$$\lim_{k\to\infty} \frac{1}{k}\int_0^{kT} G_{m'_k}(\gamma_k(\bar{\xi}_k)(t)) \, dt = +\infty.$$

We consider two possibilities. Remember that $G_{m'_k}$ is positive on the half-line $\{r\hat{w}: r > 0\}$.

Either, $\bar{\xi}_k \in [0, \frac{1}{2}]$. Then,

$$\frac{1}{k}\int_0^{kT} G_{m'_k}(\gamma_k(\bar{\xi}_k)(t)) \, dt \geq \frac{1}{k}\int_{|k/2|T}^{kT} G_{m'_k}(\gamma_k(\bar{\xi}_k)(t)) \, dt = \frac{k - |k/2|}{k} T(m'_k + 1),$$

which tends to $+\infty$ as $k \to \infty$, since $m'_k \geq 2k$.

Or, $\bar{\xi}_k \in [\frac{1}{2}, 1]$. Then,

$$\begin{aligned}
&\frac{1}{k}\int_0^{kT} G_{m'_k}(\gamma_k(\bar{\xi}_k)(t)) \, dt \\
&\geq \frac{1}{k}\int_{|k/2|T/4}^{3|k/2|T/4} G_{m'_k}(\gamma_k(\bar{\xi}_k)(t)) \, dt \\
&= \frac{1}{k}\int_{|k/2|T/4}^{3|k/2|T/4} G_{m'_k}\left(2\left[1 + \sin\left(\frac{2\pi}{|k/2|T}t - \frac{\pi}{2}\right)\right](1 - \bar{\xi}_k)k\hat{w} + (2\bar{\xi}_k - 1)m'_k\hat{w}\right) dt \\
&= \frac{|k/2|T}{2\pi k}\int_{\pi/2}^{3\pi/2} G_{m'_k}\left(2\left[1 + \sin\left(s - \frac{\pi}{2}\right)\right](1 - \bar{\xi}_k)k\hat{w} + (2\bar{\xi}_k - 1)m'_k\hat{w}\right) dt,
\end{aligned}$$

which also tends to $+\infty$ as $k \to \infty$, by (10) and the fact that $m'_k \geq 2k$ implies

$$2\left[1 + \sin\left(s - \frac{\pi}{2}\right)\right](1 - \bar{\xi}_k)k + (2\bar{\xi}_k - 1)m'_k \geq k,$$

for $s \in [\pi/2, 3\pi/2]$.

We are now able to finish the proof of theorem 1. Assume by contradiction that the minimal periods of the kT-periodic solutions u_k do not tend to infinity. Then, for a subsequence, there must exist a common period, say $\bar{k}$T. By lemma 1, there are two positive constants $d_{\bar{k}}, D_{\bar{k}}$ such that the orbit of any u_k is contained in $\bar{B}_{D_{\bar{k}}} \backslash B_{d_{\bar{k}}}$. Since $(m'_k)_k$ is increasing, by (15) we have that

$$\varphi_{k,m'_k}(u_k) = \varphi_{k,m'_{\bar{k}}}(u_k),$$

for every $k \geq \bar{k}$. Take $k_n = n\bar{k}$. Then,

$$\frac{1}{k_n}\varphi_{k_n,m'_{k_n}}(u_{k_n}) = \frac{1}{k_n}\varphi_{k_n,m'_{\bar{k}}}(u_{k_n}) = \frac{1}{\bar{k}}\varphi_{\bar{k},m'_{\bar{k}}}(u_{k_n}),$$

which remains bounded from below, in contradiction with (18). The proof is, therefore, completed.

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