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Large-Amplitude Subharmonic Oscillations for Scalar Second-Order Differential Equations with Asymmetric Nonlinearities

A. FONDA* AND M. RAMOS†

Dip. Scienze Matematiche, Università di Trieste, P.le Europa 1, 34127 Trieste, Italy; and
†*C.M.A.F., Universidade de Lisboa, Av. Prof. Gama Pinto 2, 1699 Lisboa Codex, Portugal*

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We provide sufficient conditions for the existence of subharmonic solutions for equations whose nonlinearity may have a different qualitative behaviour to the right and to the left of the origin. The proofs are based on variational methods.

1. INTRODUCTION

Consider the second-order differential equation

$$\ddot{u}(t) + g(t, u(t)) = 0, \tag{1.1}$$

where $g \in \mathscr{C}(\mathbb{R}^2; \mathbb{R})$ is T-periodic in its first variable ($T > 0$). Our purpose is to study the existence of subharmonic solutions of Eq. (1.1), i.e., kT-periodic solutions ($k \geq 1$ is an integer) which are not T-periodic.

There is a vast literature on this problem, even for more general Hamiltonian systems; we refer the reader to [Ek, MW] for a bibliography and recent results. In most of these some convexity conditions on the potential are assumed. Some "generic" type results were proved in [CZ, FW] and, for the one-dimensional case, phase-plane methods were used in [Ja, DZ_1, Ya]. Here we study Eq. (1.1) by means of some careful estimates on the critical levels of mountain pass and saddle point type of the associated functional.

In Section 2 we study the case of a subquadratic potential. Namely, denoting $G(t, x) := \int_0^x g(t, s)\, ds$, we assume that

$$\lim_{|x| \to \infty} \frac{G(t, x)}{x^2} = 0 \tag{1.2}$$

holds, uniformly in t, together with a Landesman–Lazer condition. By developing some ideas in [Gi, FL] we prove the existence of kT-periodic

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solutions of (1.1) whose amplitudes and minimal periods tend to infinity. In particular, when $g(t, x) = g(x) + e(t)$ and $e(\cdot)$ has minimal period T, one has subharmonic solutions with minimal period kT for every sufficiently large prime integer k (see [MT] for a more general condition). We point out that no convexity hypotheses are made upon the potential $G(t, x)$ and that we deal with situations where *a priori* bounds for the periodic solutions cannot be proved. On the other hand, provided $g(t, x)$ depends in a suitable manner on the variables (t, x), we are able to weaken (1.2) to a one-sided growth condition. In this way we extend some results in [FL, FRW].

In Section 3 we show that the equation

$$\ddot{u} + \mu u^+ - \nu u^- + g(t, u) = 0, \tag{1.3}$$

where μ and ν are positive constants and $g(t, x)$ is sublinear, still has subharmonic solutions, provided the couple (μ, ν) lies below the first Fučik curve. For this purpose we obtain a new variational characterization of the first Fučik curve associated to Eq. (1.3). This variational result is much in the spirit of the one in O. Kavian [Ka]; a similar characterization has been found independently by M. Cuesta and J.-P. Gossez [CG].

In contrast with some results in [$\text{HRS}_{1,2}$] where uniqueness theorems for T-periodic solutions were proved, this gives in particular a multiplicity result for kT-periodic solutions of (1.3). We are also able to apply our methods to a suspension bridge model considered by A. C. Lazer and P. J. McKenna [LM_2]. We prove the existence of large amplitude subharmonic oscillations for such a model.

We have learnt that T. Ding and F. Zanolin [DZ_2] have obtained results related to those of Section 2, by means of the Poincaré–Birkhoff theorem.

2. One-Sided Sublinear Nonlinearities

We consider the scalar equation

$$\ddot{u} + g(t, u) = e(t), \tag{2.1}$$

where the following periodic and Carathéodory type assumptions upon the functions $g(t, x)$ and $e(t)$ are assumed throughout the paper:

- for every $x \in \mathbb{R}$, $g(\cdot, x)$ is measurable and periodic with period $T > 0$;
- for a.e. $t \in \mathbb{R}$, $g(t, \cdot)$ is continuous;
- for every $R > 0$ there exists $h_R(t) \in L^1(0, T)$ such that, for a.e. $t \in [0, T]$ and all $|x| \leqslant R$, one has $|g(t, x)| \leqslant h_R(t)$;
- $e(\cdot)$ is a T-periodic, locally integrable function.

We denote by $G(t, \cdot)$ a primitive of $g(t, \cdot)$ and by $\bar{e}$ the mean value of $e(\cdot)$ along a period

$$\bar{e} = \frac{1}{T}\int_0^T e(t)\, dt.$$

For any $k \in \mathbb{N}$ we denote by H^1_{kT} the Sobolev space of kT-periodic absolutely continuous functions whose derivatives are square integrable over a period, equipped with the usual norm $\|u\|_{kT} := (\int_0^{kT}(u^2(t) + \dot{u}^2(t))\, dt)^{1/2}$. We also denote by $\|\cdot\|_p$, $1 \leqslant p \leqslant +\infty$, the usual L^p-norm. Recall that, after identification of the set of constant functions with $\mathbb{R}$, we can write $H^1_{kT} = \mathbb{R} \oplus \tilde{H}^1_{kT}$, where $\tilde{H}^1_{kT}$ is the set of functions with mean value zero over $[0, kT]$. Accordingly, every $u \in H^1_{kT}$ can be written as $u(t) = \bar{u} + \tilde{u}(t)$, $\bar{u} \in \mathbb{R}$, $\tilde{u} \in \tilde{H}^1_{kT}$. We consider the functional $\varphi_k \in \mathscr{C}^1(H^1_{kT}; \mathbb{R})$,

$$\varphi_k(u) := \int_0^{kT} [\tfrac{1}{2}\dot{u}^2(t) - G(t, u(t)) + e(t)\, u(t)]\, dt,$$

whose critical points correspond to the kT-periodic solutions of (2.1).

We begin by considering a symmetric situation at $\pm\infty$.

THEOREM 2.1. *Assume that the following conditions hold.*

(i) $\lim_{|x| \to \infty}(G(t, x)/x^2) = 0$ *uniformly for a.e.* $t \in [0, T]$;

(ii) *there exists* $h(t) \in L^1(0, T)$ *such that*

$$\operatorname{sgn}(x)\, g(t, x) \geqslant h(t)$$

for a.e. $t \in [0, T]$ *and all* $x \in \mathbb{R}$;

(iii) $1/T \int_0^T \limsup_{x \to -\infty} g(t, x)\, dt < \bar{e} < 1/T \int_0^T \liminf_{x \to +\infty} g(t, x)\, dt$.

Then Eq. (2.1) *has a sequence* $(u_k)_{k \geqslant 1}$ *of kT-periodic solutions whose amplitudes and minimal periods tend to infinity.*

Remark 2.2. The assumptions (ii) and (iii) are the well-known Landesman-Lazer conditions. In particular they imply

(iii)′ $\lim_{|x| \to \infty}(1/T \int_0^T G(t, x)\, dt - x\bar{e}) = +\infty$.

(See [Ma, RS] for a proof.) Whenever $g(t, \cdot)$ happens to be increasing for a.e. $t \in [0, T]$, condition (iii)′ is proved to be equivalent to (ii) and (iii); cf. [Ma]. Theorem 2.1 generalizes Theorem 4 in [FRW], where the existence of subharmonics which are not T-periodic was proved assuming $g(t, \cdot)$ to be increasing and a supplementary growth condition on $g(t, x)/x$ was considered.

Remark 2.3. It follows from the proof that condition (i) can be weakened to the following

(i)′ for every $\varepsilon > 0$ there exist functions $\alpha_\varepsilon(t)$, $\beta_\varepsilon(t) \in L^1(0, T)$ such that, for a.e. $t \in [0, T]$ and every $x \in \mathbb{R}$,

$$G(t, x) \leqslant \varepsilon x^2 + \alpha_\varepsilon(t)\,|x| + \beta_\varepsilon(t).$$

Proof of Theorem 2.1. Without loss of generality we assume $\bar{e} = 0$ and $\int_0^T G(t, x)\,dt \geqslant 0$ for every $x \in \mathbb{R}$. Let $k \in \mathbb{N}$ be fixed. We apply the saddle point theorem (see [Ra]) to the functional φ_k defined above. It follows easily from assumptions (i)′ and (iii)′ that $(-\varphi_k)$ is coercive on $\mathbb{R}$ and that φ_k is coercive on $\tilde{H}^1_{kT}$.

We check the Palais–Smale condition for φ_k, namely, that every sequence $(u_n)_{n \geqslant 1}$ in H^1_{kT} such that $(\varphi_k(u_n))$ is bounded and $\nabla\varphi_k(u_n) \to 0$ as $n \to \infty$ admits a convergent subsequence. By standard arguments (cf. [Ra]) it is sufficient to prove that (u_n) is bounded. Assume by contradiction that, for a subsequence, $\|u_n\|_{kT} \to \infty$. From (i)′ and the fact that $(\varphi_k(u_n))$ is bounded from above we get that

$$|\bar{u}_n| \to \infty \qquad \text{and} \qquad \frac{\|\tilde{u}_n\|_{kT}}{|\bar{u}_n|} \to 0$$

as $n \to \infty$ (cf. [FL, RS]). From the identity $u_n(t) = \bar{u}_n(1 + \tilde{u}_n(t)/\bar{u}_n)$ we conclude that $\min_{[0,kT]} |u_n| \to \infty$. Assume that $m_n := \min_{[0,kT]} u_n \to +\infty$, the other case being symmetrical. From (ii) we have, for a.e. $t \in [0, T]$ and every n large, $g(t, u_n(t)) \geqslant h(t)$. On the other hand, by the assumption, there exists a positive constant C such that $\|(\nabla\varphi_k(u_n), v)\|_{kT} \leqslant C\,\|v\|_{kT}$ for every $n \geqslant 1$ and every $v \in H^1_{kT}$. Choosing $v \equiv 1$ and $v \equiv \tilde{u}_n$ yields

$$\left|\int_0^{kT} g(t, u_n(t))\,dt\right| \leqslant CkT,$$

$$\left|\int_0^{kT} [\dot{\tilde{u}}_n^2(t) - (g(t, u_n(t)) - h(t))\,\tilde{u}_n(t) + (e(t) - h(t))\,\tilde{u}_n(t)]\,dt\right| \leqslant C\,\|\tilde{u}_n\|_{kT}$$

so that

$$\int_0^{kT} \dot{\tilde{u}}_n^2(t)\,dt \leqslant C\,\|\tilde{u}_n\|_{kT} + \|\tilde{u}_n\|_\infty (CkT + 2\,\|h\|_1 + \|e\|_1)$$

and we can conclude that $(\tilde{u}_n)$ is bounded. Since $(\varphi_k(u_n))$ is bounded, this implies that $(\int_0^{kT} G(t, u_n(t))\,dt)$ is also bounded. But

$$\begin{aligned}\int_0^{kT} G(t, u_n(t))\,dt &= \int_0^{kT}\left[G(t, m_n) + \int_{m_n}^{u_n(t)} g(t, s)\,ds\right]dt\\ &\geqslant \int_0^{kT} [G(t, m_n) + (u_n(t) - m_n)\,h(t)]\,dt\\ &\geqslant \int_0^{kT} G(t, m_n)\,dt - C'\end{aligned}$$

for a certain constant C', and this contradicts (iii)′. Thus the Palais–Smale condition holds.

We can thus conclude by the saddle point theorem that φ_k admits a critical point u_k, for every $k \geqslant 1$. The corresponding critical level is given, for r_k large enough, by

$$\varphi_k(u_k) = \inf_{\gamma \in \Gamma_k} \max_{\xi \in [-r_k, r_k]} \varphi_k(\gamma(\xi)),$$

where

$$\Gamma_k := \{\gamma \in \mathscr{C}([-r_k, r_k]; H^1_{kT}) : \gamma(\pm r_k) = \pm r_k\}.$$

We show that

$$\lim_{k \to \infty} \frac{1}{k}\,\varphi_k(u_k) = -\infty. \tag{2.2}$$

Indeed, choose $r_k \geqslant k$, consider the path $\gamma_k \in \Gamma_k$ given by

$$\gamma_k(\xi)(t) := \xi + 2k\left(1 - \frac{|\xi|}{r_k}\right)\sin\left(\frac{2\pi t}{kT}\right),$$

and let $\xi_k \in [-r_k, r_k]$ be such that

$$\varphi_k(\gamma_k(\xi_k)) = \max_{[-r_k, r_k]} \varphi_k(\gamma_k(\cdot)).$$

By the assumption, the function $E(t) = \int_0^t e(s)\,ds$ is continuous and T-periodic. Thus, by a Fourier Series' argument,

$$\int_0^{kT} e(t)\,\gamma_k(\xi_k)(t)\,dt = -\int_0^{kT} E(t)\,\frac{d}{dt}\gamma_k(\xi_k)(t)\,dt = 0$$

for any $k \geqslant 2$. Then we have

$$\begin{aligned}\frac{1}{k}\varphi_k(u_k) &\leqslant \frac{1}{k}\varphi_k(\gamma_k(\xi_k)) \\ &\leqslant \frac{4\pi^2}{T} - \frac{1}{k}\int_0^{kT} G(t, \gamma_k(\xi_k)(t))\,dt \\ &= \frac{4\pi^2}{T} - \frac{1}{k}\sum_{i=0}^{k-1}\int_0^T G(t, w_{k,i}(t))\,dt,\end{aligned}$$

where we define, for $t \in [0, T]$,

$$w_{k,i}(t) = \xi_k + 2k\left(1 - \frac{|\xi_k|}{r_k}\right)\sin\left(\frac{2\pi(t+iT)}{kT}\right).$$

It is not difficult to see that, defining

$$m_{k,i} = \begin{cases}\min w_{k,i} & \text{if } \ w_{k,i}(t) \geqslant 0 \text{ for all } t\\ \max w_{k,i} & \text{if } \ w_{k,i}(t) \leqslant 0 \text{ for all } t\\ 0 & \text{otherwise,}\end{cases}$$

one has, for every $t \in [0, T]$,

$$|w_{k,i}(t) - m_{k,i}| \leqslant 4\pi. \tag{2.3}$$

Denote $\mathscr{A}_k := \{i \in \mathbb{N} \colon k/8 \leqslant i \leqslant k/4 - 1\}$ or $\mathscr{A}_k := \{i \in \mathbb{N} \colon 1 - k/4 \leqslant i \leqslant -k/8\}$ according to whether $\xi_k \geqslant 0$ or $\xi_k < 0$, respectively. It is easily seen that

$$\liminf_{k \to \infty} \frac{1}{k}(\#\mathscr{A}_k) > 0 \tag{2.4}$$

and that, for every $i \in \mathscr{A}_k$,

$$|m_{k,i}| \geqslant k. \tag{2.5}$$

Using (2.3) we obtain

$$\begin{aligned}\frac{1}{k}\varphi_k(u_k) &\leqslant \frac{4\pi^2}{T} - \frac{1}{k}\sum_{i=0}^{k-1}\int_0^T [G(t, m_{k,i}) - |w_{k,i}(t) - m_{k,i}|\,|h(t)|]\,dt \\ &\leqslant \frac{4\pi^2}{T} + 4\pi\,\|h\|_1 - \frac{1}{k}\sum_{i \in \mathscr{A}_k}\int_0^T G(t, m_{k,i})\,dt.\end{aligned}$$

By (2.4), (2.5), and (iii)′ the right-hand side tends to $-\infty$ as $k \to \infty$ and (2.2) is proved.

It is clear that $\|u_k\|_\infty \to \infty$ as $k \to \infty$. Let us prove that the amplitudes of u_k tend to infinity, i.e., $(\max_{[0,kT]} u_k - \min_{[0,kT]} u_k) \to +\infty$. For this purpose, we show that $\|\tilde{u}_k\|_\infty \to \infty$ as $k \to \infty$. If not, for a subsequence, we would have, say, $\min_{[0,kT]} u_k \to +\infty$ as $k \to \infty$. Then, from (2.1) and Fatou's lemma,

$$0 = \frac{1}{k}\int_0^{kT} g(t, u_k(t))\,dt = \int_0^T \left[\frac{1}{k}\sum_{i=0}^{k-1} g(t, u_k(t+iT))\right] dt$$
$$\geqslant \int_0^T \liminf_{k\to\infty} \left[\frac{1}{k}\sum_{i=0}^{k-1} g(t, u_k(t+iT))\right] dt$$
$$\geqslant \int_0^T \liminf_{x\to+\infty} g(t, x)\,dt,$$

a contradiction with (iii). An analogous contradiction is obtained if $\max_{[0,kT]} u_k \to -\infty$ as $k \to \infty$.

We now show that the minimal periods of the subharmonic solutions u_k tend to infinity. If not, we could find a subsequence of such solutions whose minimal periods are bounded and for which we would be able to find a common period, say $\bar{k}T$. The sequence $(u_{n\bar{k}})_{n\geqslant 1}$ of critical points of $\varphi_{\bar{k}}$ is such that, by (2.2),

$$\varphi_{\bar{k}}(u_{n\bar{k}}) = \frac{1}{n}\,\varphi_{n\bar{k}}(u_{n\bar{k}}) \to -\infty$$

as $n \to \infty$. In particular $(\varphi_{\bar{k}}(u_{n\bar{k}}))$ is bounded from above and as we saw in the proof of the Palais–Smale condition, this implies that $\min_{[0,\bar{k}T]} |u_{n\bar{k}}(t)| \to \infty$. Assume $\min_{[0,\bar{k}T]} u_{n\bar{k}}(t) \to +\infty$, the other case being treated similarly. From (2.1) and Fatou's lemma we get

$$0 = \int_0^{\bar{k}T} g(t, u_{n\bar{k}}(t))\,dt$$
$$\geqslant \int_0^{\bar{k}T} \liminf_{n\to\infty} g(t, u_{n\bar{k}}(t))\,dt$$
$$\geqslant \int_0^{\bar{k}T} \liminf_{x\to+\infty} g(t, x)\,dt,$$

a contradiction with (iii). This concludes the proof. ∎

Remark 2.4. If (iii)′ is assumed instead of (iii), it follows from the above proof that Eq. (2.4) has a sequence (u_k) of subharmonic solutions which is unbounded in the L^∞-norm. Under stronger conditions, a similar result has been proved for systems in [Gi]. We do not know if the conclusion of Theorem 2.1 still holds in this case.

In our next result, condition (i) of Theorem 2.1 will be replaced by a one-sided growth restriction.

THEOREM 2.5. *Assume that there exists a differentiable function* $G_1\colon]-\infty, 0] \to \mathbb{R}$ *such that*

$$g(t, x) \geqslant G_1'(x) \tag{2.6}$$

for a.e. $t \in [0, T]$ *and all* $x \leqslant 0$, *and*

$$\lim_{x \to -\infty} \frac{G_1(x)}{x^2} = 0. \tag{2.7}$$

If moreover conditions (ii) *and* (iii) *hold, then Eq.* (2.1) *has a sequence* $(u_k)_{k \geqslant 1}$ *of kT-periodic solutions whose amplitudes and minimal periods tend to infinity.*

Proof. Without loss of generality, we assume $\bar{e} = 0$ and $G_1'(x) \leqslant 0$ for every $x \leqslant 0$. We consider a modified problem; let, for $r > 0$,

$$g_r(t, x) = \begin{cases} g(t, x) & \text{if } x \leqslant r \\ g(t, r) & \text{if } x \geqslant r. \end{cases}$$

For any $k \geqslant 1$, choose $r_k > 0$ sufficiently large so that, by (iii), $\int_0^T g(t, r_k)\, dt > 0$, and in such a way that $r_k \to +\infty$ as $k \to \infty$. Define

$$\psi_k(u) := \int_0^{kT} \left[\tfrac{1}{2}\dot{u}^2(t) - G_{r_k}(t, u(t)) + e(t)\, u(t)\right] dt,$$

where $G_{r_k}(t, x)$ denotes any primitive of $g_{r_k}(t, x)$. Then, assumptions (i)', (ii), and (iii) of Theorem 2.1 are satisfied, and, like in the proof of that theorem, we can construct, for each of the functionals ψ_k, a critical point u_k. As shown in [RS], by Fatou's Lemma and (iii),

$$\liminf_{\substack{|x| \to \infty \\ k \to \infty}} \operatorname{sgn}(x) \int_0^T \int_0^1 g_{r_k}(t, xs)\, ds\, dt > 0,$$

and since $\int_0^T G_{r_k}(t, x)\, dt = x \int_0^T \int_0^1 g_{r_k}(t, xs)\, ds\, dt$, we have

$$\lim_{\substack{|x| \to \infty \\ k \to \infty}} \int_0^T G_{r_k}(t, x)\, dt = +\infty. \tag{2.8}$$

Estimating the critical levels as we did in the proof of Theorem 2.1, we have

$$\lim_{k \to \infty} \frac{1}{k} \psi_k(u_k) = -\infty.$$

In fact, because of the Landesman–Lazer assumption (iii), which forces the nonlinearity to stay, on average, a positive distance apart from 0, the estimate can be seen to be uniform with respect to the truncated functions g_{r_k}.

To conclude the proof, it is sufficient to prove that for every $k \geqslant 1$ there exists a sufficiently large $r_k > 0$ with the property that every kT-periodic solution $u(t)$ of

$$\ddot{u} + g_r(t, u) = e(t), \tag{2.9}$$

with $r \geqslant r_k$, satisfies $\|u\|_\infty < r_k$. We use an energy estimate through (2.6), similarly to [OVZ].

Fix $k \geqslant 1$. Assume by contradiction that there exist sequences (r_n), (u_n) with $r_n \to +\infty$, and $\|u_n\|_\infty \to \infty$ as $n \to \infty$, where each $u_n(t)$ is a kT-periodic function satisfying

$$\ddot{u}_n + g_{r_n}(t, u_n) = e(t). \tag{2.10}$$

We claim that, up to a subsequence, we have

$$\min_{[0,kT]} u_n \to -\infty \qquad \text{and} \qquad \max_{[0,kT]} u_n \to +\infty \tag{2.11}$$

as $n \to +\infty$. Indeed, if for instance there is an $m > 0$ such that $\min_{[0,kT]} u_n \geqslant -m$ for all n, then there exists $\eta(t) \in L^1(0, kT; \mathbb{R})$ such that $g_{r_n}(t, u_n(t)) \geqslant \eta(t)$ for a.e. $t \in [0, kT]$. Since $\int_0^{kT} g_{r_n}(t, u_n(t))\, dt = 0$, multiplying (2.10) by $\tilde{u}_n(t)$ and integrating yields

$$\begin{aligned} \int_0^{kT} \dot{\tilde{u}}_n^2(t)\, dt &= \int_0^{kT} (g_{r_n}(t, u_n(t)) - \eta(t))\, \tilde{u}_n(t)\, dt \\ &\quad + \int_0^{kT} (\eta(t) + e(t))\, \tilde{u}_n(t)\, dt \\ &\leqslant \|\tilde{u}_n\|_\infty\, [2\, \|\eta\|_1 + \|e\|_1] \end{aligned}$$

so that $(\tilde{u}_n)$ is bounded and $\min_{[0,kT]} u_n \to +\infty$. A contradiction follows then from (iii) as in the end of the proof of Theorem 2.1. Similarly one proves that $\max_{[0,kT]} u_n \to +\infty$.

Extending the functions by kT-periodicity over $\mathbb{R}$, it follows from the claim that we can find an interval $[\alpha_n, \beta_n]$ containing a point t_n at which $u_n(t_n) = \min_{\mathbb{R}} u_n$ such that $(\beta_n - \alpha_n) \leqslant kT$ and

$$u_n(\alpha_n) = 0 = u_n(\beta_n),$$

$$u_n(t_n) \leqslant u_n(t) \leqslant 0 \qquad \text{for all} \quad t \in [\alpha_n, \beta_n].$$

For $t \in [\alpha_n, \beta_n]$ we can write Eq. (2.10) as

$$\begin{aligned} \dot{u}_n(t) &= v_n(t) + \int_{\alpha_n}^{t} e(s)\,ds \\ \dot{v}_n(t) &= -g(t, u_n(t)). \end{aligned} \tag{2.12}$$

By (ii) we have that the function $v_n(\cdot) - \int_{\alpha_n}^{\cdot} h(s)\,ds$ is increasing, so that

$$\begin{aligned} u_n(t_n) &= \int_{\alpha_n}^{t_n} \left[\left(v_n(t) - \int_{\alpha_n}^{t} h(s)\,ds \right) + \int_{\alpha_n}^{t} (h(s) - e(s))\,ds \right] dt \\ &\geqslant kTv_n(\alpha_n) - (\|h\|_1 + \|e\|_1). \end{aligned} \tag{2.13}$$

In particular, for n large enough, $v_n(\alpha_n) < -\|e\|_1$. On the other hand, by (2.12), $v_n(t_n) \geqslant -\|e\|_1$. So there exists a smallest value $t'_n \in]\alpha_n, t_n]$ such that $v_n(t'_n) = -\|e\|_1$. We now restrict our analysis to the interval $[\alpha_n, t'_n]$. One has

$$\begin{aligned} &\frac{d}{dt}\left[G_1(u_n(t)) + \tfrac{1}{2}(v_n(t) + \|e\|_1)^2\right] \\ &\quad = G'_1(u_n(t)) \left(v_n(t) + \int_{\alpha_n}^{t} e(s)\,ds \right) - g(t, u_n(t))(v_n(t) + \|e\|_1) \\ &\quad \geqslant (v_n(t) + \|e\|_1)(G'_1(u_n(t)) - g(t, u_n(t))) \\ &\quad \geqslant 0 \end{aligned}$$

and it follows from (2.7) that for every $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$\begin{aligned} G_1(0) + \tfrac{1}{2}(v_n(\alpha_n) + \|e\|_1)^2 &\leqslant G_1(u_n(t'_n)) \\ &\leqslant \varepsilon\, |u_n(t'_n)|^2 + C_\varepsilon \\ &\leqslant \varepsilon\, |u_n(t_n)|^2 + C_\varepsilon. \end{aligned} \tag{2.14}$$

From (2.13) and (2.14) we conclude that $(u_n(t_n))$ is bounded and this contradicts (2.11). The proof is thus complete. ∎

COROLLARY 2.6. *Assume that* $g \in \mathscr{C}(\mathbb{R}^2; \mathbb{R})$, *conditions* (ii)–(iii) *hold, and moreover*

(i)″ $\lim_{x \to -\infty} g(t, x)/x = 0$ *uniformly for* $t \in [0, T]$.

Then one can conclude as in Theorem 2.5.

Proof. Letting $g_1(x) := \min_{[0,T]} g(\cdot, x)$, we have that $g_1 \in \mathscr{C}(\mathbb{R}; \mathbb{R})$ and $\lim_{x \to -\infty} g_1(x)/x = 0$. Thus Theorem 2.5 applies with $G_1(x) := \int_0^x g_1(s)\, ds$. ∎

Remark 2.7. Corollary 2.6 is a generalization of Theorem 6 in [FRW], where the existence of subharmonic solutions which are not T-periodic was proved under the additional assumptions that $g(t, x)$ be bounded from below and increasing in x. We do not know whether condition (i)″ can be replaced by the weaker one-sided assumption $\lim_{x \to -\infty} G(t, x)/x^2 = 0$. It is easy to find examples where this condition is satisfied while the hypotheses of Theorem 2.5 are not.

As noted in Remark 2.4, assumption (iii) was used in the above results in order to prove some kind of *a priori* bound of solutions of certain equations and to have (2.8). It is easy to see, however, that one can replace (ii)–(iii) by (iii)′ and a sign condition which is more convenient when $g(t, x)$ is independent of t. As a conclusion, we have that conditions (ii) and (iii) can be systematically replaced in the above results by the corresponding following ones

(ii)′ there exists $d > 0$ such that $\operatorname{sgn}(x)[g(t, x) - \bar{e}] > 0$, for a.e. $t \in [0, T]$ and all $|x| \geq d$;

(iii)′ $\lim_{|x| \to \infty} [1/T \int_0^T G(t, x)\, dt - x\bar{e}] = +\infty$.

As a consequence of this remark and Theorem 2.5, we have

COROLLARY 2.8. *Assume $e(t) \in \mathscr{C}(\mathbb{R}; \mathbb{R})$, $g(t, x) = g(x)$ and let $T > 0$ be the minimal period of $e(t)$. If*

(a) $\lim_{x \to -\infty} G(x)/x^2 = 0$;

(b) *there exists $d > 0$ such that for all $|x| \geq d$,* $\operatorname{sgn}(x)[g(x) - \bar{e}] > 0$;

(c) $\lim_{|x| \to \infty} [G(x) - x\bar{e}] = +\infty$;

then Eq. (2.1), *besides having T-periodic solutions, also has periodic solutions with minimal period kT, for any sufficiently large prime integer k, and the corresponding amplitudes tend to infinity.*

Remark 2.9. The above theorem generalizes Theorem 2 in [FL], where a two-sided condition was assumed instead of (a), and a further growth condition was considered on $g(x)/x$. We also extend the results in [Ya] where existence theorems for T-periodic solutions were proved for nonlinearities such as $g(t, x) = (x + h(t))/(x^2 + 1)$.

3. ASYMMETRIC NONLINEARITIES

In this section, we consider

$$\ddot{u} + \mu u^{+} - \nu u^{-} + g(t, u) = e(t), \tag{3.1}$$

where μ, ν are positive constants and $u^{+} := \max(u, 0)$, $u^{-} := \max(-u, 0)$. We assume the same regularity conditions on the functions $g(t, x)$ and $e(t)$ as in Section 2. We also assume

(i)‴ $\lim_{|x| \to \infty} g(t, x)/x = 0$ uniformly for a.e. $t \in [0, T]$.

It is known by degree theory that Eq. (3.1) admits a T-periodic solution for a.e. $(\mu, \nu) \in]0, +\infty[\times]0, +\infty[$, precisely, provided that the couple (μ, ν) lies between two consecutive Fučik curves (see, e.g., [Fu, LM_1]). Those curves are given by the formulas

$$\omega\left(\frac{1}{2\sqrt{\mu}} + \frac{1}{2\sqrt{\nu}}\right) = \frac{1}{n}, \qquad n = 1, 2, ..., \tag{3.2}$$

where $\omega := 2\pi/T$, and correspond to the couples (μ, ν) for which the problem

$$\begin{aligned} \ddot{v} + \mu v^{+} - \nu v^{-} &= 0 \\ v(t+T) &= v(t) \end{aligned} \tag{3.3}$$

admits a nontrivial solution. Here we are concerned with the first Fučik curve. We prove the following

THEOREM 3.1. *Assume that conditions* (i)‴ *and* (ii)–(iii) *of Theorem* 2.1 *hold. Then there exists an integer $k_0 \geqslant 2$ having the following property: for any $k \geqslant k_0$ and any $\mu > 0$, $\nu > 0$ such that*

$$\frac{1}{2\sqrt{\mu}} + \frac{1}{2\sqrt{\nu}} > \frac{kT}{2\pi}. \tag{3.4}$$

Equation (3.1), *besides having a T-periodic solution, also has a kT-periodic solution which is not T-periodic.*

The proof of the theorem is based on the following result. Recall that whenever $\mu = \nu$, the first positive eigenvalue of the eigenvalue problem (3.3) is ω^2, given by

$$\omega^2 = \min_{\substack{u \in H^1_T \setminus \{0\} \\ \int_0^T u = 0}} \frac{\int_0^T \dot{u}^2(t)\, dt}{\int_0^T u^2(t)\, dt}.$$

An analogous statement can be made for the general case (3.3).

PROPOSITION 3.2. *For any $\mu > 0$, $\nu > 0$ we have*

$$\min_{\substack{u \in H_T^1 \setminus \{0\} \\ \mu \int_0^T u^+ = \nu \int_0^T u^-}} \frac{\int_0^T \dot{u}^2(t)\, dt}{\int_0^T (\mu u^{+2}(t) + \nu u^{-2}(t))\, dt} = \omega^2 \left(\frac{1}{2\sqrt{\mu}} + \frac{1}{2\sqrt{\nu}} \right)^2.$$

Proof. Denote by m the left-hand side above. Assume first $\mu \geqslant \nu$ and set

$$\psi(u) = \int_0^T [\mu u^+(t) - \nu u^-(t)]\, dt,$$

$$\varphi(u) = \frac{\int_0^T \dot{u}^2(t)\, dt}{\int_0^T [\mu u^{+2}(t) + \nu u^{-2}(t)]\, dt}.$$

Then $\varphi \in \mathscr{C}^1(H_T^1 \setminus \{0\}; \mathbb{R})$ and ψ is convex. We denote by $\partial\psi(u)$ the subdifferential of ψ at the point u, i.e.,

$$u^* \in \partial\psi(u) \quad \text{iff} \quad \psi(v) \geqslant \psi(u) + \langle u^*, v - u \rangle \quad \forall v \in H_T^1, \tag{3.5}$$

where $\langle \cdot, \cdot \rangle$ stands for the scalar product in H_T^1.

By taking a minimizing sequence of φ and using the compact imbedding of H_T^1 into L^2 we can find a point $u \in \psi^{-1}(0) \setminus \{0\}$ such that

$$\begin{aligned} 0 < m = \int_0^T \dot{u}^2(t)\, dt = \min_{\psi^{-1}(0) \setminus \{0\}} \varphi, \\ \int_0^T (\mu u^{+2}(t) + \nu u^{-2}(t))\, dt = 1. \end{aligned} \tag{3.6}$$

By a Lagrange multiplier rule due to F. Clarke [$\mathrm{Cl}_{1,2}$] there exist $u^* \in \partial\psi(u)$ and two scalars $\lambda_1, \lambda_2 \in \mathbb{R}$ not both equal to zero such that

$$\lambda_1 \langle \nabla\varphi(u), \cdot \rangle + \lambda_2 \langle u^*, \cdot \rangle = 0 \quad \text{in } H_T^1.$$

Since

$$\langle \nabla\varphi(u), v \rangle = 2 \int_0^T \dot{u}(t)\, \dot{v}(t)\, dt - 2m \int_0^T (\mu u^+(t) - \nu u^-(t))\, v(t)\, dt$$

in H_T^1, we have in particular that $\langle \nabla\varphi(u), \cdot \rangle|_{\mathbb{R}} \equiv 0$ and thus

$$\lambda_2 \langle u^*, \cdot \rangle|_{\mathbb{R}} \equiv 0. \tag{3.7}$$

On the other hand it follows from (3.5) and (3.6) that for any $v \in]-\infty, 0]$,

$$\langle u^*, u \rangle \geqslant \langle u^*, v \rangle - \psi(v) = \langle u^*, v \rangle + T\nu\, |v|. \tag{3.8}$$

From (3.7), it follows that $\lambda_2 = 0$ and thus $\nabla\varphi(u) = 0$; that is, $u(t)$ is a nontrivial solution of the periodic problem

$$\ddot{u} + m\mu u^+ - m\nu u^- = 0$$
$$u(t+T) = u(t).$$

From (3.2) we deduce that

$$m \geqslant \omega^2 \left(\frac{1}{2\sqrt{\mu}} + \frac{1}{2\sqrt{\nu}}\right)^2. \tag{3.9}$$

In a similar way one can prove that (3.9) in case $\mu < \nu$ holds: simply replace ψ by $-\psi$ and use positive constants in (3.8). To prove the reversed inequality, denote $\theta := \omega^2(1/(2\sqrt{\mu}) + 1/(2\sqrt{\nu}))^2$ and let $v(t)$ be the unique solution of the problem

$$\ddot{v} + \theta\mu v^+ - \theta\nu v^- = 0$$
$$v(t+T) = v(t)$$
$$\int_0^T (\mu v^{+2}(t) + \nu v^{-2}(t))\, dt = 1.$$

Then we have $\psi(v) = 0$ and $m \leqslant \varphi(v) = \theta$, which concludes the proof. ∎

To prove Theorem 3.1 we also need the following elementary

LEMMA 3.3. *Assume that conditions* (i)‴ *and* (ii)–(iii) *of Theorem* 2.1 *hold. Then, for each* $\varepsilon > 0$, *there exists* $R \equiv R_\varepsilon > 0$ *such that any solution* $u(t)$ *of* (3.1) *with*

$$\omega\left(\frac{1}{2\sqrt{\mu}} + \frac{1}{2\sqrt{\nu}}\right) \geqslant 1 + \varepsilon \tag{3.10}$$

satisfies $\|u\|_T \leqslant R$.

Proof. Suppose by contradiction that some sequence $(u_n)_{n \geqslant 1}$ of solutions of (3.1) with (μ_n, ν_n) satisfying (3.10) verifies $\|u_n\|_T \to \infty$ as $n \to \infty$. Following the arguments in the proof of Theorem 2.5 (see (2.11)) we see that (ii)–(iii) imply

$$\max_{[0,T]} u_n \to +\infty \quad \text{and} \quad \min_{[0,T]} u_n \to -\infty. \tag{3.11}$$

We claim that for any $\delta > 0$ we have

$$\min(\mu_n, \nu_n) \geqslant \left(\frac{\pi}{T}\right)^2 - \delta \tag{3.12}$$

for every n large. Otherwise, say, $\mu_n \leqslant (\pi/T)^2 - \delta$ and from (3.11) we can choose an interval $[\alpha_n, \beta_n] \subset [0, T]$ such that $u_n(\alpha_n) = 0 = u_n(\beta_n)$ and $u_n(t) \geqslant 0$ on $[\alpha_n, \beta_n]$. Using (i)‴ and Poincaré's inequality we deduce from (3.1) that, for some constant $C > 0$,

$$\left(\frac{\pi}{\beta_n - \alpha_n}\right)^2 \int_{\alpha_n}^{\beta_n} u_n^2 \leqslant \int_{\alpha_n}^{\beta_n} \dot{u}_n^2 \leqslant \left(\frac{\delta}{2} + \mu\right) \int_{\alpha_n}^{\beta_n} u_n^2 + \int_{\alpha_n}^{\beta_n} e u_n + C,$$

which shows that $(\|\dot{u}_n\|_2)$ is bounded, contradicting (3.11).

Now, from (3.10) and (3.12) with δ small enough, it follows that both (μ_n) and (ν_n) are bounded. Letting $v_n(t) := u_n(t)/\|u_n\|_T$, it follows from (3.1) and standard arguments that, up to subsequences, $\mu_n \to \mu$, $\nu_n \to \nu$, and $v_n \to v$, where v is a nontrivial solution of problem (3.3). From (3.10), either $\mu = 0$ or $\nu = 0$, which is impossible by (3.12). ∎

Proof of Theorem 3.1. Let μ, ν, k satisfy (3.4). We apply the mountain pass theorem (cf. [AR]) to the associated $\mathscr{C}^1$ functional

$$\varphi_k(u) = \int_0^{kT} \left[\tfrac{1}{2}(\dot{u}^2(t) - \mu u^{+2}(t) - \nu u^{-2}(t)) - G(t, u(t)) + e(t)\, u(t)\right] dt.$$

Define

$$S_k := \left\{ u \in H^1_{kT} : \int_0^{kT} (\mu u^+(t) - \nu u^-(t))\, dt = 0 \right\}.$$

It follows from (i)‴ and Proposition 3.2 that $\inf_{S_k} \varphi_k$ is finite. From (ii)–(iii) it follows that $-\varphi_k$ is coercive on $\mathbb{R}$ (see Remark 2.2) and we can choose r_k large enough so that

$$\varphi_k(\pm r_k) < \inf_{S_k} \varphi_k.$$

Define now

$$\Gamma_k := \{\alpha \in \mathscr{C}([0, 1]; H^1_{kT}) : \alpha(0) = -r_k, \alpha(1) = r_k\},$$

$$c_k := \inf_{\alpha \in \Gamma_k} \sup_{t \in [0,1]} \varphi_k(\alpha(t)).$$

From the intermediate value theorem we see that every path in Γ_k meets S_k. Thus we have $c_k \geqslant \inf_{S_k} \varphi_k > \varphi_k(\pm r_k)$. Moreover, φ_k satisfies the Palais–Smale condition over H^1_{kT}, for if $\nabla\varphi_k(u_n) \to 0$ along a sequence $(u_n)_{n \geqslant 1}$ with $\|u_n\|_{kT} \to +\infty$, it follows from (i)‴ that $v_n(t) := u_n(t)/\|u_n\|_{kT}$ converges (up to a subsequence) to some nonzero solution of problem (3.3), which is impossible by (3.2) and (3.4).

We conclude by the mountain pass theorem that c_k is a critical value for φ_k. Since moreover

$$\varphi_k(u) \leqslant \int_0^{kT} [\tfrac{1}{2}\dot{u}^2(t) - G(t, u(t)) + e(t)\, u(t)]\, dt$$

on H^1_{kT}, the argument in the proof of Theorem 2.1 shows that $c_k/k \to -\infty$ as $k \to \infty$, uniformly in (μ, ν). By Lemma (3.3) we may then conclude. ∎

We point out that how large the integer k_0 in the above theorem should be is determined by the *a priori* bound in Lemma 3.3. In some situations this bound can be estimated; in that case a different argument, based on Morse theory, can be used, provided the function g satisfies some "twist" assumptions. Suppose for instance that $g \in \mathscr{C}(\mathbb{R}; \mathbb{R})$, $e \in \mathscr{C}(\mathbb{R}; \mathbb{R})$ and consider

$$\ddot{u} + \mu u^+ - \nu u^- + g(u) = e(t) \tag{3.13}$$

with $\mu > 0$, $\nu > 0$. Assume

(a) $\displaystyle\lim_{|x| \to \infty} \frac{g(x)}{x} = 0;$

(b) $\displaystyle\limsup_{x \to -\infty} g(x) < \bar{e} < \liminf_{x \to +\infty} g(x)$

and, according to Lemma 3.1, let R_0 be such that $\|u\|_\infty < R_0$ for any T-periodic solution of (3.13) with $\omega(1/(2\sqrt{\mu}) + 1/(2\sqrt{\nu})) \geqslant 2$. Denote $h(x) \equiv \mu x^+ - \nu x^- + g(x)$. Then we have

THEOREM 3.4. *Assume that conditions* (a) *and* (b) *hold,* $h \in \mathscr{C}^1(\mathbb{R}; \mathbb{R})$, *and that there exists* $\varepsilon > 0$ *such that*

$$h'(x) \geqslant \varepsilon \qquad \textit{for every} \quad x \in [-R_0, R_0]. \tag{3.14}$$

Then the conclusion of Theorem 3.1 *is true with any integer* $k_0 \geqslant 2$ *such that*

$$k_0 > \frac{2\pi}{T\sqrt{\varepsilon}}.$$

Proof. It follows from our assumptions and the preceding arguments that the $\mathscr{C}^2$ functional φ_k admits a critical value c_k found by the mountain pass theorem. Moreover, according to the results in [LS, So], some critical point v at level c_k has Morse index less or equal to one. Recall that, for any kT-periodic solution u of (3.13), we define the Morse index $m_k(u)$ of u as the number of negative eigenvalues λ (counted with multiplicity) of the linear problem

$$\ddot{w} + (h'(u(t)) + \lambda) w = 0$$
$$w(t + kT) = w(t).$$

Now, for any T-periodic solution $u(t)$ of (3.13) (extended by T-periodicity as a kT-periodic solution), denote by $(\lambda_n)_{n \geqslant 1}$ the sequence of corresponding eigenvalues. From (3.14) and our assumption on k_0 we have $\lambda_2 \leqslant -\varepsilon + (2\pi/kT)^2 < 0$ for $k \geqslant k_0$, so that $m_k(u) \geqslant 2$. Thus the kT-periodic solution v found above cannot be T-periodic and this ends the proof. ∎

We conclude with the following example. In [LM$_2$], the authors considered a nonlinear model of a suspension bridge. When considering no-node type oscillations, they were led to the equation

$$\ddot{y} + EI(\pi/L)^4 y + ly^+ = W_0 + f(t).$$

Here I is the moment of the inertia; E is Young's modulus; L is the length of the bridge; l takes into account the rigidity of the cables of the bridge; W_0 represents the weight; $f(t)$ is a T-periodic forcing term.

Consider the more general equation

$$\ddot{u} + \lambda u + g(u) = e(t), \tag{3.15}$$

where $g \in \mathscr{C}(\mathbb{R}; \mathbb{R})$, $e \in L^1(0, \pi)$ is T-periodic, and

$$\lim_{x \to -\infty} \frac{g(x)}{x} = 0 < l = \lim_{x \to +\infty} \frac{g(x)}{x}. \tag{3.16}$$

By using the preceding arguments, one can prove

THEOREM 3.5. *Assume* (3.16) *holds and moreover*

$$\limsup_{x \to -\infty} g(x) < \bar{e}.$$

Then there exists $\lambda_0 > 0$ *such that, for any* $\lambda \in]0, \lambda_0[$, *Eq.* (3.15) *has a subharmonic solution* u_λ *with amplitude* A_λ *and minimal period* T_λ, *such that*

$$\lim_{\lambda \to 0} A_\lambda = \lim_{\lambda \to 0} T_\lambda = \infty.$$

The fact that λ must be small, i.e., EI/L^4 must be small, means that the bridge is "long and flexible." In this case, if the bridge is subjected to a periodic forcing, one can expect to have large amplitude subharmonic oscillations.

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