Alessandro Fonda

Professeur à l'Université de Trieste

Periodic solutions of scalar second order differential equations with a singularity

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1. Introduction

The aim of this paper is to illustrate the use of some topological and variational techniques which provide the existence of periodic solutions to the equation

$$u'' + g(u) = e(t), \qquad (1.1)$$

where $g : (0, +\infty) \to \mathbb{R}$ is a continuous function, 0 being a singularity, and $e : \mathbb{R} \to \mathbb{R}$ is a locally integrable periodic function with period $T > 0$. As a model for equation (1.1), we have in mind an equation like

$$u'' - \frac{1}{u^a} = e(t),$$

where $a \geq 1$ and u is positive.

Lazer and Solimini [LS] gave the start to a series of papers on the existence of periodic solutions to (1.1). Defining $G(x) = \int_1^x g(\xi)\, d\xi$, a primitive of $g(x)$, and $\bar{e} = \frac{1}{T}\int_0^T e(s)\, ds$, the mean value of $e(t)$, they proved the following.

Theorem 1.1. — ([LS]) *Assume that* $g < 0$ *and*

(*i*) $$\lim_{x \to +\infty} g(x) = 0\,;$$

(*ii*) $$\lim_{x \to 0^+} g(x) = -\infty\,;$$

(*iii*) $$\lim_{x \to 0^+} G(x) = +\infty\,;$$

Then equation (1.1) *has a* T*-periodic solution if and only if* $\bar{e} < 0$.

Integrating the differential equation, it is easy to see that a necessary condition for the existence of periodic solutions to (1.1) is that $\bar{e}$ belongs to the image of $g(x)$. This explains why, under Lazer-Solimini's assumptions, $\bar{e}$ has to be negative. Concerning the sufficient conditions, we will show that they can be considerably weakened.

Due to the structure of equation (1.1), which can be considered as a pertubation of an autonomous equation by a periodic forcing, it is natural to investigate whether conditions relying on the potential $G(x)$ rather than on the field $g(x)$ itself still guarantee the existence of periodic solutions. Moreover, it is useful knowing whether in the above setting one can find subharmonic solutions, i.e. periodic solutions whose period is a multiple of T.

We will answer in the positive to the above questions. In section 2 we provide an almost complete generalization of Theorem 1.1, assuming conditions relying only on the potential function $G(x)$, without requiring explicit conditions on the field $g(x)$. The price we have to pay for this is a boundedness assumption on $e(t)$. The proof of this result is a combination of different theoretical approaches. We use the theory of upper and lower solutions, topological degree arguments, and a variational method. This result can be considered as an extension of the one in [Fo], where the author considered an analogous situation for second order differential equations without singularities. See also [MW], [FZ], and [FoZ] for results along these lines.

In section 3 we will prove the existence of infinitely many subharmonic solutions under conditions generalizing the ones in Theorem 1.1. This problem has been treated in a joint paper with Manasevich and Zanolin [FMZ]. The proof is variational, and requires careful estimates for the critical levels of the associated functional. The main idea of the proof was introduced in [FL] and developed in [FR], for equations without singularities. For related results, see also [FW], [FRW] and the references therein.

Section 4 is an appendix, where we briefly explain the methods used in the proofs of our existence theorems. In subsection 4.1 we recall the definition of upper and lower solution, and we state the existence theorem used in the proof of Theorem 2.1. In subsection 4.2 we explain how Mawhin's coincidence degree is defined, generalizing the Leray-Schauder theory (see [Ma_3]), and we recall its main properties and a result of Capietto, Mawhin and Zanolin

[CMZ] which will be needed in the proof of Theorem 2.1. In subsection 4.3 we state the Mountain Pass Theorem of Ambrosetti and Rabinowitz [AR] and the Saddle Point Theorem of Rabinowitz (cf. [Ra]).

There are many directions towards which one can extend the study of an equation like (1.1). Equations with a friction term have been considered by Habets and Sanchez [HS], and Mawhin [Ma_3], providing conditions for the existence of periodic solutions with the same period of the forcing term.

A different behaviour at $+\infty$ can also be considered. The asymptotically linear case has been studied by del Pino, Manasevich and Montero in [DMM]. Here again, the existence of T-periodic solutions was proved, while the existence of subharmonic solutions seems to be an open problem.

On the other hand, the superlinear case has been treated in [FMZ], where the existence of infinitely many subharmonic solutions of any order was proved by the use of the Poincaré-Birkhoff fixed point theorem. In the same paper, the qualitatively similar case where the nonlinearity has two singularities was considered, as well.

2. Periodic solutions having the same period as the forcing term

Consider the periodic problem

$$\text{(P)} \qquad \begin{cases} u'' + g(u) = e(t) \\ u(0) - u(\mathrm{T}) = u'(0) - u'(\mathrm{T}) = 0, \end{cases}$$

where $g: (0, +\infty) \to \mathbb{R}$ is a continuous function, and $e: [0, \mathrm{T}] \to \mathbb{R}$ is an integrable function.

Theorem 2.1. — Assume $e(t)$ to be a bounded measurable function, and the following conditions to hold.

$$(j) \qquad \liminf_{x \to +\infty} \frac{2\mathrm{G}(x)}{x^2} < \left(\frac{\pi}{\mathrm{T}}\right)^2;$$

$$(jj) \qquad \lim_{x \to +\infty} [\mathrm{G}(x) - \bar{e}x] = +\infty;$$

$$(jjj) \qquad \lim_{x \to 0^+} \mathrm{G}(x) = +\infty.$$

Then problem (P) *has at least one solution.*

Remark. — Comparing our result with Theorem 1.1, notice that g is neither assumed to be negative nor to have a limit at the origin (see condition (ii)). Moreover, the assumptions (i) and $\bar{e} < 0$ have been weakened in (j) and (jj). The rather weak hypothesis (j) concerning the inferior limit of $2\,\mathrm{G}(x)/x^2$ was first introduced by Fernandes and Zanolin [FZ], while a condition like (jj) was introduced by Ahmad, Lazer and Paul in [ALP]. Hypotheses (jjj) is necessary for the conclusion, as was shown in [LS].

PROOF. — Without loss of generality, we may assume that $\bar{e} = 0$. This can be seen by substracting $\bar{e}$ to both sides of the equation, and noticing that the assumptions (j)-(jjj) remain unchanged when $G(x)$ is replaced by $(G(x) - \bar{e}x)$, and $\bar{e}$ by 0.

We will be led to distinguish qualitatively different situations. Accordingly, we will use the theory of upper and lower solutions, topological degree arguments, and a variational approach.
We need to consider two different cases.

Case 1 : g is unbounded from above.

In this case, since $e(t)$ is bounded, it is possible to find two positive constants A, B such that

$$g(A) < e(t) < g(B), \tag{2.1}$$

for a.e. $t \in [0, T]$. In particular, the constant function A is an upper solution and B is a lower solution of (P). If $B < A$, the result follows from the classical theory of upper and lower solutions.

Assume now that $A < B$. Choose M such that

$$A < M < B. \tag{2.2}$$

If g is unbounded from below in $(M, +\infty)$, it is possible to find a constant upper solution $A' > B$, and the conclusion follows again. Analogously one can conclude if g is unbounded from above on $(0, M)$. Hence we can from now on assume that, for a constant $c_1 > 0$,

$$\operatorname{sgn}(x - M)g(x) \geq -c_1. \tag{2.3}$$

Consider, for $\lambda \in (0, 1)$, the problems

$$(P_\lambda) \qquad \begin{cases} u'' + g_\lambda(u) = (1 - \lambda)e(t) \\ u(0) - u(T) = u'(0) - u'(T) = 0, \end{cases}$$

where

$$g_\lambda(x) = \lambda \frac{x - M}{x} + (1 - \lambda)g(x).$$

We will find two positive constants R, S satisfying $R < A < B < S$, and such that no solution of (P_λ) belongs to the boundary of the open bounded set

$$\mathcal{A} = \{u \in C([0,T],\mathbb{R}) : \min(u) \in (R,B), \max(u) \in (A,S)\},$$

i.e. to

$$\partial\mathcal{A} = \{u \in C([0,T],\mathbb{R}) : \min(u) \in [R,B], \max(u) \in [A,S] \text{ and}$$
$$\text{either } \min(u) \in \{R,B\} \text{ or } \max(u) \in \{A,S\}\}.$$

It is easy to see that if u is a solution of (P_λ), then $min(u) \neq B$ and $max(u) \neq A$. Just argue by contradiction, write the differential equation at the point where u attains its minimum (resp. maximum), and use (2.1) and (2.2). Remark also that, for any $u \in \partial\mathcal{A}$,

$$\exists t_u \in [0,T] : u(t_u) \in [A,B]. \tag{2.4}$$

We define now $\hat{g}(x) = g(x) + c_1 + 1$, $\hat{G}(x) = G(x) + (c_1 + 1)x$. If μ is taken so that, by (j),

$$\liminf_{x \to +\infty} \frac{2G(x)}{x^2} < \mu < \left(\frac{\pi}{T}\right)^2,$$

one has

$$\limsup_{x \to +\infty} [\mu x^2 - 2\hat{G}(x)] = +\infty ;$$

so, there is a sequence (S_n) such that $S_n \to +\infty$ as $n \to \infty$, and

$$\forall s \in (M,S_n),\ \mu s^2 - 2\hat{G}(s) < \mu S_n^2 - 2\hat{G}(S_n). \tag{2.5}$$

Using an argument like in [FZ], we will show that, if n is large enough, any solution u of (P_λ) satisfying (2.4) is such that $\max(u) \neq S_n$.

Let $E(t)$ be a primitive of $e(t)$. Having supposed $\bar{e} = 0$, $E(t)$ is a periodic function, and problem (P_λ) is equivalent to the system

$$(P'_\lambda) \qquad \begin{cases} u' = v + (1-\lambda)E(t) \\ v' = -g_\lambda(u) \\ u(0) = u(T) \\ v(0) = v(T) \end{cases}$$

Assume by contradiction that there is a subsequence, still denoted (S_n), such that $\max(u_n) = S_n$, where (u_n, v_n) is a solution of (P'_λ), for some $\lambda = \lambda_n \in (0, 1)$, such that $\{u_n(t) : t \in [0,T]\} \cap [A,B] \neq 0$. For n large enough, $S_n > B$, and, extending u_n by T-periodicity, there are $\alpha_n^{(1)} < \beta_n^{(1)} \leq \beta_n^{(2)} < \alpha_n^{(2)}$ such that $\alpha_n^{(2)} - \alpha_n^{(1)} < T$ and

$$u_n(\alpha_n^{(1)}) = B = u_n(\alpha_n^{(2)}),$$

$$u_n(\beta_n^{(1)}) = S_n = u_n(\beta_n^{(2)}),$$

and

$$B < u_n(t) < S_n \ \text{ for } t \in (\alpha_n^{(1)}, \beta_n^{(1)}) \cup (\beta_n^{(2)}, \alpha_n^{(2)}).$$

Let us concentrate on the interval $(\alpha_n^{(1)}, \beta_n^{(1)})$. By the second equation in (P'_λ) and (2.3), the function $(v(t) - c_1 t)$ is decreasing. Integrating the first equation, we get

$$S_n - B = \int_{\alpha_n^{(1)}}^{\beta_n^{(1)}} (v_n(s) + (1 - \lambda_n)E(s))\, ds$$

$$= \int_{\alpha_n^{(1)}}^{\beta_n^{(1)}} (v_n(s) - c_1 s + c_1 s + (1 - \lambda_n)E(s))\, ds$$

$$\leq T v_n(\alpha_n^{(1)}) + 2c_1 T^2 + T \,\| E \|_\infty.$$

So, for n large enough, one has $v_n(\alpha_n^{(1)}) > \| E \|_\infty$. On the other hand, it is easy to see, from the first equation in (P'_{λ_n}) that $| v_n(\beta_n^{(1)}) | < \| E \|_\infty$. Hence, for n large enough, there is a $\gamma_n^{(1)} \in (\alpha_n^{(1)}, \beta_n^{(1)})$ such that $v_n(\gamma_n^{(1)}) = \| E \|_\infty$. Integrating the first equation in (P'_{λ_n}) over $[\gamma_n^{(1)}, \beta_n^{(1)}]$, we get

$$S_n - u_n(\gamma_n^{(1)}) =$$

$$\int_{\gamma_n^{(1)}}^{\beta_n^{(1)}} (v_n(s) - c_1 s + c_1 s + (1 - \lambda_n)E(s))\, ds \leq 2T \,\| E \|_\infty + 2c_1 T^2 := L.$$

On the other hand, for $t \in [\alpha_n^{(1)}, \gamma_n^{(1)}]$,

$$\frac{d}{dt}[\hat{G}(u_n(t)) + \frac{1}{2}(v_n(t) - \| E \|_\infty)^2] =$$

$$= \hat{g}(u_n(t))(v_n(t) + (1 - \lambda_n)E(t)) - g_{\lambda_n}(u_n(t))(v_n(t) - \| E \|_\infty)$$

$$\geq (\hat{g}(u_n(t)) - g_{\lambda_n}(u_n(t)))(v_n(t) - \| E \|_\infty) \geq 0.$$

So, the function $(\hat{G}(u_n(\cdot)) + \frac{1}{2}(v_n(\cdot) - \| E \|_\infty)^2)$ is nondecreasing on $[\alpha_n^{(1)}, \gamma_n^{(1)}]$, and we have

$$\hat{G}(u_n(t)) + \frac{1}{2}(v_n(t) - \| E \|_\infty)^2 \leq \hat{G}(u_n(\gamma_n^{(1)})) \leq \hat{G}(S_n).$$

Consequently, by (2.5),

$$v_n(t) - \| E \|_\infty \leq (2[\hat{G}(S_n) - \hat{G}(u_n(t))])^{1/2} \leq \mu(S_n^2 - u_n^2)^{1/2},$$

and, using the first equation in (P'_λ),

$$\frac{u'_n(t)}{2 \| E \|_\infty + \sqrt{\mu(S_n^2 - u_n^2(t))}} \leq 1,$$

for every $t \in [\alpha_n^{(1)}, \gamma_n^{(1)}]$. Integrating, we have

$$\gamma_n^{(1)} - \alpha_n^{(1)} \geq \frac{1}{\sqrt{\mu}} \int_B^{u_n(\gamma_n^{(1)})} \frac{ds}{2 \| E \|_\infty \mu^{-1/2} + \sqrt{S_n^2 - s^2}}$$

$$\geq \frac{1}{\sqrt{\mu}} \int_B^{S_n - L} \frac{ds}{2 \| E \|_\infty \mu^{-1/2} + \sqrt{S_n^2 - s^2}} := \omega_n^{(1)}.$$

The above integration can be performed explicitly. Setting $\eta = 2 \| E \|_\infty \mu^{-1/2}$, for n large enough we have $S_n > \eta$, and a primitive of $1/(\eta + \sqrt{S_n^2 - s^2})$ in $[0,S_n]$ is given by

$$\arcsin\left(\frac{s}{S_n}\right) -$$

$$\frac{\eta}{\sqrt{S_n + \eta}\sqrt{S_n - \eta}} \log \left| \frac{\sqrt{S_n + \eta} + \sqrt{S_n - \eta} \tan\left(\frac{1}{2} \arcsin\left(\frac{s}{S_n}\right)\right)}{\sqrt{S_n + \eta} - \sqrt{S_n - \eta} \tan\left(\frac{1}{2} \arcsin\left(\frac{s}{S_n}\right)\right)} \right| .$$

It is then not hard to see that

$$\lim_{n \to \infty} \omega_n^{(1)} = \frac{\pi}{2\sqrt{\mu}} > \frac{T}{2}.$$

Consequently,

$$\liminf_{n \to \infty} (\beta_n^{(1)} - \alpha_n^{(1)}) > \frac{T}{2}.$$

In an analogous way we can prove that

$$\liminf_{n \to \infty} (\alpha_n^{(2)} - \beta_n^{(2)}) > \frac{T}{2},$$

and we reach a contradiction with the fact that $\alpha_n^{(2)} - \alpha_n^{(1)} < T$.

So, there is a sufficiently large $\bar{n}$ such that, for any solution u of (P_λ) satisfying (2.4), one has $\max(u) \neq S_{\bar{n}}$. We set $S = S_{\bar{n}}$.

Now we will prove that there is $R \in (0,A)$ such that, for any solution u of (P_λ) satisfying $\max(u) \leq S$ and (2.4) one has $\min(u) > R$. Assume by contradiction that there is a sequence (u_n) of solutions of (P_λ) for some $\lambda = \lambda_n \in (0,1)$ such that $\min(u_n) \leq 1/n$, $\max(u_n) \leq S$ and $\{u_n(t) : t \in [0,T]\} \cap [A,B] \neq \emptyset$. Since $\int_0^T g_{\lambda_n}(u_n(t))\,dt = 0$, we have by (2.3),

$$\int_{[u_n<M]} |\,g_{\lambda_n}(u_n(t))\,|\,dt \leq \int_{[u_n<M]} (|\,g_{\lambda_n}(u_n(t)) - c_1\,| + c_1)\,dt$$

$$= \int_{[u_n<M]} (2c_1 - g_{\lambda_n}(u_n(t)))\,dt$$

$$\leq 2c_1 T + \int_{[M\leq u_n<S]} g_{\lambda_n}(u_n(t))\,dt$$

$$\leq T(2c_1 + \max\{|\,g_\lambda(x)\,| : M \leq x \leq S, 0 \leq \lambda \leq 1\}.$$

In conclusion, $(\,\|\,g_{\lambda_n}(\cdot, u_n(\cdot))\,\|_1\,)$ is bounded, and then $(\,\|\,u_n'\,\|_\infty)$ is bounded, as well. Take $t_n^{(1)}$, $t_n^{(2)}$ such that

$$u_n(t_n^{(1)}) = \frac{1}{n} < A = u_n(t_n^{(2)}).$$

Multiplying the equation in (P_{λ_n}) by u_n' and integrating over $[t_n^{(1)}, t_n^{(2)}]$, we get

$$\frac{1}{2}(u_n'(t_n^{(2)}))^2 - \frac{1}{2}(u_n'(t_n^{(1)}))^2 + G_{\lambda_n}(A) - G_{\lambda_n}\left(\frac{1}{n}\right) = (1-\lambda_n)\int_{t_n^{(1)}}^{t_n^{(2)}} e u_n',$$

where

$$G_{\lambda_n}(x) = \int_1^x g_{\lambda_n}(\xi)\,d\xi = \lambda_n(x - 1 - \log x) + (1-\lambda_n)\,G(x).$$

Since $(\| u_n' \|_\infty)$ is bounded, $(G_{\lambda_n}\left(\frac{1}{n}\right))$ has to be bounded, too. But this clearly is impossible, by *(jjj)*.

We thus proved that no solution of (P_λ) can belong to $\partial\mathcal{A}$, for any $\lambda \in (0,1)$. Further, we notice that there exists a constant $Q > 0$ such that, for any $u \in \bar{\mathcal{A}}$, if (u, v) is a solution of (P_λ'), then $\| v \|_\infty < Q$. This follows from the fact that the derivatives of the solutions of (P_λ) belonging to $\bar{\mathcal{A}}$ have to be bounded. Define the following open, bounded set :

$$\mathcal{B} = \{ z = (u,v) \in C([0,T],\mathbb{R}^2) : u \in \mathcal{A}, \| v \|_\infty < Q \}.$$

By the above, no solution of (P_λ') can be in $\partial\mathcal{B}$.

Denoting by $\deg_B$ the Brouwer topological degree, the field

$$\gamma(u,v) = \left(v, \frac{u - M}{u}\right),$$

corresponding to $\lambda = 1$, is such that

$$\deg_B(\gamma, \mathcal{B} \cap \mathbb{R}^2, 0) = \deg_B(\gamma, (A,B) \times (-Q,Q),0) = \det J_\gamma(1,0) = -1.$$

Let $X = L^\infty(0,T; \mathbb{R}^2)$, and define

$$D(L) = \{ z \in C^1([0,T]; \mathbb{R}^2) : z(0) = z(T) \},$$
$$L : D(L) \subset X \to X, \quad Lz = z' ;$$
$$N : \bar{\mathcal{B}} \to X, \quad Nz = N(u,v) = (v(\cdot) + E(\cdot), -g(u(\cdot))).$$

By Lemma 1 in [CMZ] and the homotopy invariance of the coincidence degree, we have that

$$D_L(L - N,\mathcal{B}) = \deg_B(\gamma, \mathcal{B} \cap \mathbb{R}^2,0).$$

So, $D_L(L - N,\mathcal{B}) \neq 0$, and the result is now a consequence of degree theory (cf. [$Ma_{1,3}$] ; see also the Appendix).

Case 2 : g is bounded from above.

Define, for $\gamma \in (0,1)$, the truncated functions $g_\gamma : \mathbb{R} \to \mathbb{R}$,

$$g_\gamma(x) = \begin{cases} g(x) & \text{if } x \geq \gamma \\ g(\gamma) & \text{if } x < \gamma. \end{cases}$$

Set $G_\gamma(x) = \int_1^x g_\gamma(\xi)\, d\xi$ and consider the functionals $\phi_\gamma : H_T^1 \to \mathbb{R}$,

$$\phi_\gamma(u) = \int_0^T \left[\frac{1}{2}(u'(t))^2 - G_\gamma(u(t)) + e(t)u(t)\right] dt,$$

whose critical points correspond to solutions of

$$(P_\gamma) \qquad \begin{cases} u'' + g_\gamma(u) = e(t) \\ u(0) - u(T) = u'(0) - u'(T) = 0. \end{cases}$$

Define the set

$$\mathcal{S} = \{u \in H_T^1 : u(t) > 1 \text{ for every } t \in [0,T]\}.$$

Clearly, its boundary is given by

$$\partial\mathcal{S} = \{u \in H_T^1 : u(t) \geq 1 \text{ for every } t \in [0,T], \text{ and } \exists t_u \in [0,T] : u(t_u) = 1\}.$$

Since g is bounded from above, we have $\lim_{x\to+\infty} \frac{2G_\gamma(x)}{x^2} = 0$, uniformly in $\gamma \in (0,1)$. Extending $e(t)$ and the functions in H_T^1 by T-periodicity, taking $u \in \mathcal{S}$, we can use Poincaré's inequality for $u(\cdot) - 1$, and find a constant $m > 0$ such that

$$\inf_{\partial\mathcal{S}} \phi_\gamma \geq -m, \tag{2.6}$$

for every $\gamma \in (0,1)$. We want to prove the following.

Claim. There exists $\gamma_0 \in (0,1)$ such that, for every $\gamma \in (0,\gamma_0)$, any solution u of (P_γ) verifying $\phi_\gamma(u) \geq -m$ is such that $\min(u) \geq \gamma_0$, and hence is a solution of (P).

Assume by contradiction that there are sequences (γ_n) (u_n) such that $\gamma_n \leq 1/n$, u_n is a solution of (P_{γ_n}), $\phi_{\gamma_n}(u_n) \geq -m$ and $min(u_n) < 1/n$.

Since $\int_0^T g_{\gamma_n}(u_n(t))\, dt = 0$ and g is bounded from above by a constant $c_2 > 0$, we have

$$\int_{[g_{\gamma_n}<0]} |g_{\gamma_n}(u_n(t))|\, dt = \int_{[0\leq g_{\gamma_n}\leq c_2]} g_{\gamma_n}(u_n(t))\, dt \leq c_2 T.$$

So, $\| g_{\gamma_n}(u_n(\cdot)) \|_1 \le 2c_2T$, and hence also $\| u_n' \|_\infty \le 2c_2T$. Since $(\phi_{\gamma_n}(u_n))$ is bounded below and $(\| u_n' \|_\infty)$ is bounded, there must exist two positive constants R_1, R_2 such that

$$\max (u_n) \in [R_1, R_2].$$

Otherwise, $u_n(t)$ would go uniformly to 0 or to $+\infty$, and then $\phi_{\gamma_n}(u_n)$ would go to $-\infty$. Let $\tau_n^{(1)}$, $\tau_n^{(2)}$ be such that, for n sufficiently large,

$$u_n(\tau_n^{(1)}) = \frac{1}{n} < R_1 = u_n(\tau_n^{(2)}).$$

Multiplying the equation in (P_{γ_n}) by u_n' and integrating on $[\tau_n^{(1)}, \tau_n^{(2)}]$, being $\gamma_n \le 1/n$, we get

$$\frac{1}{2}(u_n'(\tau_n^{(2)}))^2 - \frac{1}{2}(u_n'(\tau_n^{(1)}))^2 + G(R_1) - G\left(\frac{1}{n}\right) = \int_{\tau_n^{(1)}}^{\tau_n^{(2)}} e u_n' .$$

Since $(\| u_n' \|_\infty)$ is bounded, we find a contradiction with *(jjj)*. This proves the Claim.

We now fix $\gamma \in (0,\gamma_0]$ such that $g(\gamma) < 0$ and, by *(jjj)*, $G_\gamma(0) > m/T$. Using *(jj)*, we can find a sufficiently large $R > 1$ for which $G(R) > m/T$. Hence, we are in the situation of the Mountain Pass Theorem, since $\mathcal{S}$ is a neighborhood of R and

$$\max \{\phi_\gamma(0), \phi_\gamma(R),\} < \inf_{\partial S} \phi_\gamma$$

(see the Appendix).

We now show that ϕ_γ satisfies the Palais-Smale condition. Let (u_k) be a sequence in H_T^1 such that $(\phi_\gamma(u_k))$ is bounded and $\phi_\gamma'(u_k) \to 0$ as $k \to \infty$. Then there is a constant $\varepsilon > 0$ such that, for every $v \in H_T^1$,

$$\left| \int_0^T [u_k' v' - g_\gamma(u_k) v + e v] \, dt \right| < \varepsilon \| v \|_{H^1}. \tag{2.7}$$

Taking $v \equiv 1$, we get

$$\left| \int_0^T g_\gamma(u_k(t)) \, dt \right| \le \varepsilon \sqrt{T},$$

hence

$$\int_{[g_\gamma<0]} |\, g_\gamma(u_k(t)) \,|\, dt \le \varepsilon\sqrt{\mathrm{T}} + \int_{[g_\gamma\ge 0]} g_\gamma(u_k(t))\, dt.$$

Since g_γ is bounded from above, we conclude that $(\|g_\gamma(u_k(\cdot))\|_1)$ is bounded. Taking $v = \tilde{u}_k := (u_k - \frac{1}{\mathrm{T}}\int_0^{\mathrm{T}} u_k(t)\, dt)$ in (2.7), by the use of Wirtinger and Sobolev inequalities we have that $(\tilde{u}_k)$ has to be bounded. Assume by contradiction that, for a subsequence, $\| u_k \|_{\mathrm{H}^1} \to \infty$, as $k \to \infty$. Then, $|\, u_k(\mathrm{t}) \,| \to \infty$, uniformly in t. This leads to a contradiction, since $(\tilde{u}_k)$ and $(\phi_\gamma(u_k))$ are bounded, and, by (jj) and $\mathrm{g}(\gamma) < 0$,

$$\lim_{|x|\to\infty} \mathrm{G}_\gamma(x) = +\infty.$$

The Palais-Smale condition then holds (cf. [R*a*]), and we can conclude that ϕ_γ has a critical point u_γ such that

$$\phi_\gamma(u_\gamma) = \inf_{\gamma\in\Gamma} \max_{s\in[0,1]} \phi_\gamma(\eta(s)) \ge \inf_{\partial \mathrm{S}} \phi_\gamma,$$

where $\Gamma = \{\eta \in \mathscr{C}([0,1], \mathrm{H}^1_\mathrm{T}) : \eta(0) = 0, \eta(1) = \mathrm{R}\}$. By (2.6) and the Claim above, u_γ will be a solution of (P), and the theorem is proved.

3. Subharmonics : a variational approach

We consider the equation

$$u'' + g(u) = e(t), \tag{3.1}$$

where $g : (0, +\infty) \to \mathbb{R}$ is a continuous function, and $e : \mathbb{R} \to \mathbb{R}$ is a locally integrable periodic function with period $T > 0$.

Theorem 3.1. — Assume the following conditions :

(*i*) $$\lim_{x \to +\infty} \frac{G(x)}{x^2} = 0 ;$$

(*ii*) $\exists d \geq 1 : [x \in (0,d^{-1}) \cup (d, +\infty) \Rightarrow (g(x) - \bar{e})(x - 1) > 0]$;

(*iii*) $$\lim_{x \to 0^+} G(x) = \lim_{x \to +\infty} [G(x) - \bar{e}x] = +\infty.$$

Then equation (3.1) *has a sequence* $(x_k)_{k \geq 1}$ *of* kT*-periodic solutions whose minimal periods tend to infinity. In particular, if* T *is the minimal period of* $e(t)$, (3.1) *has solutions with minimal period* kT, *for every sufficiently large prime integer* k.

Proof. — Without loss of generality, we can assume that $\bar{e} = 0$. Define, for $r > 0$, the truncated functions $g_r : \mathbb{R} \to \mathbb{R}$ *as follows :*

$$g_r(x) = \begin{cases} g(x) & \text{if } x \geq r \\ g(r) & \text{if } x < r. \end{cases}$$

We will prove the following.

Claim. For every positive integer k there exist r_k, R_k, $0 < r_k < \frac{1}{d} \leq d < \mathrm{R}_k$, *such that, for any* $s \in (0,r_k]$ *and any* $k\mathrm{T}$*-periodic solution* u *of*

$$u'' + g_s(u) = e(t), \tag{3.2$_s$}$$

one has $r_k \leq u(t) \leq \mathrm{R}_k$, *for all* $t \in \mathbb{R}$. *In particular, any* $k\mathrm{T}$*-periodic solution of* $(3.2)_{r_k}$ *is a solution of* (3.1).

In order to prove the above claim, we argue by contradiction. Let us fix a positive integer k and assume that, for every integer $n > d$, there exists a $s_n \in \left(0, \frac{1}{n}\right]$ and a $k\mathrm{T}$-periodic function u_n verifying

$$u_n'' + g_{s_n}(u_n) = e(t), \tag{3.3}$$

and such that $\{ u_n(t) : t \in \mathbb{R} \} \not\subset \left[\frac{1}{n}, n\right]$. Notice that, integrating (3.3), one has

$$\int_0^{k\mathrm{T}} g_{s_n}(u_n(t))\, dt = 0, \tag{3.4}$$

so that, by (*ii*) and the fact that $n > d$, there must exist a $t_n^{(1)} \in [0,k\mathrm{T}]$ such that $u_n(t_n^{(1)}) \in \left[\frac{1}{d}, d\right]$. After this remark, we prove that there must exist $\mathrm{R} > 0$ such that $\max(u_n) \leq \mathrm{R}$ for every n. For, otherwise, there would exist a subsequence, still denoted (u_n), for which $\max(u_n) \to +\infty$ as $n \to \infty$. We can then find an interval $[a_n,\beta_n]$, containing a point $t_n^{(2)}$ at which $u_n(t_n^{(2)}) = \max(u_n)$, such that $(\beta_n - a_n) \leq k\mathrm{T}$, and

$$u_n(a_n) = d = u_n(\beta_n),$$

$$d \leq u_n(t) \leq u_n(t_n^{(2)}), \quad \text{for all } t \in [a_n,\beta_n].$$

Let us consider the interval $[a_n,\beta_n]$. Equation (3.3) can be written as

$$u_n' = v_n + \int_{a_n}^{t} e(s)\, \mathrm{ds} \tag{3.5}$$

$$v_n' = -g(u_n). \tag{3.6}$$

Since v_n is decreasing, one has, using (3.5),

$$\max(u_n) - d \leq kT(v_n(a_n) + \| e \|_l), \tag{3.7}$$

so that, for n large enough, $v_n(a_n) > \| e \|_1$. On the other hand, by (3.5) one has $v_n(t_n^{(2)}) \leq \| e \|_1$. So, there exists a $t_n^{(3)} \in (a_n, t_n^{(2)}]$ such that $v_n(t_n^{(3)}) = \| e \|_1$. We will restrict our attention to the interval $[a_n, t_n^{(3)}]$. One has :

$$\frac{d}{dt}[G(u_n(t)) + \frac{1}{2}(v_n(t) - \| e \|_1)^2] =$$

$$= g(u_n(t))[v_n(t) + \int_{a_n}^{t} e(s)\, ds] + (v_n(t) - \| e \|_1)(-g(u_n(t)))$$

$$= g(u_n(t))[\int_{a_n}^{t} e(s)\, ds + \| e \|_1]$$

$$\geq 0.$$

Being $(G(u_n(\cdot)) + \frac{1}{2}(v_n(\cdot) - \| e \|_1)^2)$ increasing on $[a_n, t_n^{(3)}]$, one has :

$$G(d) + \frac{1}{2}(v_n(a_n) - \| e \|_1)^2 \leq G(u_n(t_n^{(3)})).$$

Using assumption (*i*), for any $\varepsilon > 0$ there is $C_\varepsilon > 0$ such that

$$G(x) \leq \varepsilon x^2 + C_\varepsilon, \quad \text{for every } x \geq d.$$

Hence,

$$G(d) + \frac{1}{2}(v_n(a_n) - \|e\|_1)^2 \leq \varepsilon(u_n(t_n^{(3)}))^2 + C_\varepsilon.$$

Chosing ε small enough, the above, together with (3.7), leads to a contradiction with the fact that $\max(u_n) \to +\infty$ as $n \to \infty$. So, we proved that there exists $R > 0$ such that $\max(u_n) \leq R$ for every n.

Using assumption (*ii*) and (3.4), one has

$$\int_{[u_n<1/d]} |g_{s_n}(u_n(t))|\, dt = -\int_{[u_n<1/d]} g_{s_n}(u_n(t))\, dt$$

$$= \int_{[u_n\geq 1/d]} g_{s_n}(u_n(t))\, dt$$

$$\leq \int_{[u_n\geq 1/d]} |g_{s_n}(u_n(t))|\, dt$$

$$\leq k\mathrm{T} \max\{\, |g(x)| : d^{-1} \leq x \leq \mathrm{R}\} := \mathrm{C}.$$

Consequently, $\| g_{s_n}(u_n) \|_1 \leq 2\mathrm{C}$, and we have

$$\| u_n' \|_\infty \leq \| u_n'' \|_1 \leq 2\mathrm{C} + \| e \|_1.$$

Now, for $n \geq \mathrm{R}$, one that $u_n(t) \leq n$ for every t. There must then be a $t_n^{(4)} > t_n^{(1)}$ such that $(t_n^{(4)} - t_n^{(1)}) < k\mathrm{T}$ and $0 < u_n(t_n^{(4)}) < 1/n$. Multiplying (3.3) by u_n' and integrating over $[t_n^{(1)}, t_n^{(4)}]$, we get

$$\frac{1}{2}(u_n'(t_n^{(4)}))^2 - \frac{1}{2}(u_n'(t_n^{(1)}))^2 + \mathrm{G}_{s_n}(u_n(t_n^{(4)})) - \mathrm{G}_{s_n}(u_n(t_n^{(1)})) = \int_{t_n^{(1)}}^{t_n^{(4)}} e u_n',$$

where $\mathrm{G}_{s_n}(x) = \int_1^x g_{s_n}(\xi)\, d\xi$. So,

$$\mathrm{G}_{s_n}(u_n(t_n^{(4)})) \leq \mathrm{G}_{s_n}(u_n(t_n^{(1)})) + \frac{1}{2}(u_n'(t_n^{(1)}))^2 + \|e\|_1 \|u_n'\|_\infty \leq \mathrm{C}'.$$

On the other hand,

$$\mathrm{G}_{s_n}(u_n(t_n^{(4)})) = \int_1^{1/n} g_{s_n}(\xi)\, d\xi + \int_{1/n}^{u_n(t_n^{(4)})} g_{s_n}(\xi)\, d\xi \geq \mathrm{G}(1/n).$$

Since, by assumption (*iii*), $\mathrm{G}(1/n) \to +\infty$ as $n \to \infty$, we get a contradiction, and the Claim is proved.

We will now deal with equation $(3.2)_{r_k}$. Denoting by G_{r_k} a primitive of g_{r_k}, we have :

$$(i)' \qquad \liminf_{|x|\to\infty} \frac{G_{r_k}(x)}{x^2} = 0 \,;$$

$$(ii)' \qquad x \in (0,d^{-1}) \cup (d, +\infty) \Rightarrow g_{r_k}(x)(x-1) > 0 \,;$$

$$(iii)' \qquad \liminf_{|x|\to\infty} G_{r_k}(x) = +\infty.$$

Let us consider the functional $\varphi_k : H^1_{kT} \to \mathbb{R}$ defined as

$$\varphi_k(u) = \int_0^{kT} \left[\frac{1}{2}(u')^2 - G_{r_k}(u) + eu\right] dt,$$

whose critical points correspond to kT-periodic solutions of $(3.2)_{r_k}$. It is straightforward to see that, writing $H^1_{kT} = \mathbb{R} \oplus \tilde{H}^1_{kT}$, $(i)'$ and $(iii)'$ give us the geometry of the Saddle Point Theorem (see the Appendix). Accordingly, each $u \in H^1_{kT}$ will be written as $u = \bar{u} + \tilde{u}$, where $\bar{u} \in \mathbb{R}$ and $\tilde{u} \in \tilde{H}^1_{kT}$. We will denote the norm in H^1_{kT} simply by $\|\cdot\|$.

We now prove that the Palais-Smale condition holds for the functionals φ_k. Fix k and let (u_n) be a sequence in H^1_{kT} such that $(\varphi_k(u_n))$ is bounded and $\varphi_k'(u_n) \to 0$ as $n \to \infty$. Assume by contradiction that, for a subsequence, $\| u_n \| \to \infty$ as $n \to \infty$.

First of all we prove that, in this case, $| \bar{u}_n | \to \infty$ and

$$\lim_{n\to\infty} \frac{\| \tilde{u}_n \|}{| \bar{u}_n |} = 0. \tag{3.8}$$

By $(i)'$, for any $\varepsilon > 0$ there is a $c_\varepsilon \geq 0$ such that

$$G_{r_k}(x) \leq \varepsilon u^2 + c_\varepsilon,$$

for every $x \in \mathbb{R}$. Choosing ε small enough, using Wirtinger's inequality we get

$$\varphi_k(u_n) \geq \frac{1}{2}\int_0^{kT} (\tilde{u}_n)^2 - \varepsilon \int_0^{kT} \tilde{u}_n^2 - kT\varepsilon\bar{u}_n^2 - kTc_\varepsilon - \left(\int_0^{kT} e^2\right)^{1/2}\left(\int_0^{kT} \tilde{u}_n^2\right)^{1/2}$$

$$\geq \frac{1}{4}\int_0^{kT} (\tilde{u}_n')^2 - kT\varepsilon\bar{u}_n^2 - c_\varepsilon',$$

for some $c'_\varepsilon \geq 0$. This implies that $|\bar{u}_n| \to \infty$ as $n \to \infty$. In fact, if this were not true, a subsequence of $(\bar{u}_n)$ would be bounded. Since $(\varphi_k(u_n))$ is bounded, $(\tilde{u}_n)$ would be bounded, too, contradicting the assumption that $(\|u_n\|)$ is unbounded. From the above, we have

$$\frac{1}{\bar{u}_n^2}\int_0^{kT}(\tilde{u}'_n)^2 \leq 4kT\varepsilon + \frac{c''_\varepsilon}{\bar{u}_n^2},$$

for some $c''_\varepsilon \geq 0$. Using Wirtinger's inequality, (3.8) easily follows.

As a consequence of (3.8), we have that

$$\min_{t\in[0,kT]} |u_n(t)| \to \infty, \tag{3.9}$$

as $n \to \infty$. Since $\varphi'_k(u_n) \to 0$, there exist a constant $C > 0$ such that $|\varphi_k(u_n)v| \leq C\|v\|$, for every $v \in H^1_{kT}$. In particular, taking $v \equiv 1$, we get

$$\left|\int_0^{kT} g_{r_k}(u_n(t))\,dt\right| \leq CkT. \tag{3.10}$$

On the other hand, taking $v = \tilde{u}_n$, we have

$$\left|\int_0^{kT}\left[\frac{1}{2}(\tilde{u}'_n)^2 - g_{r_k}(u_n(t))\,\tilde{u}_n + e\tilde{u}_n\right]dt\right| \leq C\|\tilde{u}_n\|. \tag{3.11}$$

By (3.9) and $(ii)'$, for n large enough we have

$$\left|\int_0^{kT} g_{r_k}(u_n(t))\,dt\right| = \int_0^{kT} |g_{r_k}(u_n(t))|\,dt,$$

and from (3.10) and (3.11) we can conclude that $(\tilde{u}_n)$ is bounded. Since $(\varphi_k(u_n))$ is bounded, we have that $\left(\int_0^{kT} G_{r_k}(u_n(t))\,dt\right)$ has to be bounded, as well. But this is in contradiction with (3.9) an $(iii)'$. Hence, (u_n) has to be bounded, and the Palais-Smale condition holds.

We can conclude that there is a sufficiently large $\rho_k > 0$ and a critical point u_k of φ_k such that

$$\varphi_k(u_k) = \inf_{\eta\in\Gamma_k} \max_{\xi\in[-\rho_k,\rho_k]} \varphi_k(\eta(\xi)),$$

where $\Gamma_k = \{\eta \in C([-\rho_k,\rho_k], H^1_{kT}) : \eta(\pm\rho_k) = \pm\rho_k\}$. We will take $\rho_k \geq k$.

We want to prove that

$$\lim_{k\to\infty} \frac{1}{k} \varphi_k(u_k) = -\infty. \tag{3.12}$$

In order to do this, define $\eta_k \in C([-\rho_k,\rho_k], H^1_{kT})$ as

$$\eta_k(\xi)(t) = \xi + 2k\left(1 - \frac{|\xi|}{\rho_k}\right) \sin\left(\frac{2\pi t}{kT}\right).$$

Since, by a Fourier series' argument,

$$\int_0^{kT} e(t)\eta_k(\xi)(t)\, dt = -\int_0^{kT} E(t) \frac{d}{dt} \eta_k(\xi)(t)\, dt = 0$$

$\left(\text{here } E(t) = \int_0^t e(s)\, ds\right)$, we have

$$\frac{1}{k} \max_{\xi\in[-\rho_k,\rho_k]} \varphi_k(\eta_k(\xi)) \leq \frac{4\pi^2}{T} -$$

$$\min_{\xi\in[-\rho_k,\rho_k]} \frac{T}{2\pi} \int_0^{2\pi} G_{r_k}\left(\xi + 2k\left(1 - \frac{|\xi|}{\rho_k}\right) \sin s\right) ds.$$

It is not difficult to see that, for $\xi \in [-\rho_k,\rho_k]$, one has

$$|\xi + 2k\left(1 - \frac{|\xi|}{\rho_k}\right) \sin s| \geq k$$

on a subset of $[0,2\pi]$ of measure at least $\left(\frac{2}{3}\pi\right)$. Since, by *(iii)'*, G_{r_k} can be chosen to be positive on $\mathbb{R}$, we get

$$\frac{1}{k} \max_{\xi\in[-\rho_k,\rho_k]} \varphi_k(\eta_k(\xi)) \leq \frac{4\pi^2}{T} - \left(\frac{T}{2\pi}\right)\left(\frac{2}{3}\pi\right) \min\{G_{r_k}(-k), G_{r_k}(k)\}.$$

The right hand side tends to $-\infty$ as $k \to \infty$, and (3.12) follows.

Assume now by contradiction that the minimal periods of the solutions (u_k) do not tend towards infinity. Then, for a subsequence, there would be a common period, say $\bar{k}$T. By the Claim proved above, the set of $\bar{k}$T-periodic solutions of the equation $(3.2)_{r_k}$ is bounded away from zero and from above, uniformly with respect to k. This easily gives a contradiction with (3.12). If k is a prime integer, the solutions u_k which are not T-periodic must have minimal period kT, and the proof is thus completed.

4. Appendix

4.1 Upper and lower solutions

Let $T > 0$ be a given period, and α, β be two functions in the Sobolev space $W^{2,1}([0,T],\mathbb{R})$, such that

$$\alpha(t) \leq \beta(t),$$

for every $t \in [0,T]$. Let $\Omega = \{(t,x) \in [0,T] \times \mathbb{R} : \alpha(t) \leq x \leq \beta(t)\}$, and let $g : \Omega \to \mathbb{R}$ be a Caratheodory function.

Consider the periodic problem

$$(P_1) \qquad \begin{cases} -u'' = g(t,u) \\ u(0) - u(T) = u'(0) - u'(T) = 0. \end{cases}$$

Definition 4.1.1. *We say that* α *is a lower solution for problem* (P_1) *if*

$$\begin{cases} -\alpha''(t) \leq g(t,\alpha(t)) \quad a.e.\ in\ [0,T] \\ \alpha(0) = \alpha(T)\,,\ \alpha'(0) \geq \alpha'(T). \end{cases}$$

In an analogous way, β *is an upper solution for* (P_1) *if*

$$\begin{cases} -\beta''(t) \geq g(t,\beta(t)) \quad a.e.\ in\ [0,T] \\ \beta(0) = \beta(T)\,,\ \beta'(0) \leq \beta'(T). \end{cases}$$

The following theorem was proved under more regularity assumptions by Knobloch [Kn]. A modern treatment can be found in [Ad], [HS] and [Ma_2].

THEOREM 4.1.1. *Assume that problem* (P_1) *has a lower solution* α *and an upper solution* β *satisfying*

$$\alpha(t) \leq \beta(t),$$

for every $t \in [0,T]$. *Then problem* (P_1) *has a solution* u *such that*

$$\alpha(t) \leq u(t) \leq \beta(t),$$

for every $t \in [0,T]$.

4.2 THE COINCIDENCE DEGREE

Let X and Z be real normed vector spaces. A linear mapping $L : D(L) \subset X \to Z$ is called *Fredholm* if the following conditions hold :

(*i*) $\ker L = L^{-1}(0)$ has finite dimension ;
(*ii*) $R(L) = L(D(L))$ is closed and has finite codimension.

The *index* of L is the integer $\dim \ker L - \operatorname{codim} R(L)$. We will assume L to be Fredholm of index zero.

There exists a continuous projector $P : X \to X$, a projector $Q : Z \to Z$ such that $R(P) = \ker L$, $\ker Q = R(L)$, and a bijection $J : \ker L \to R(Q)$. It is then easy to verify that $L + JP : D(L) \to Z$ is a bijection.

Let $\Delta \subset X$ and $N : \Delta \to Z$ a mapping. It is clear that the equation

$$Lx + Nx = 0 \tag{4.2.1}$$

is equivalent to the equation

$$(L + JP)x + (N - JP)x = 0,$$

and hence to the fixed point problem

$$x + (L + JP)^{-1} (N - JP)x = 0. \tag{4.2.2}$$

It is not difficult to see that we have the equivalence between (4.2.1) considered in $D(L) \cap \Delta$ and (4.2.2) considered in Δ. Notice also that $(L + JP)^{-1} JP$ is a linear, continuous operator of finite rank in X, and hence a compact operator.

Let E be a metric space and $G : E \to Z$ be a mapping.

DEFINITION 4.2.1. *We say that* $G : E \to Z$ *is* L-compact on E *if* $(L + JP)^{-1} G : E \to X$ *is compact on* E.

For $E \subset X$, $X = Z$ and $L = I$, this concept reduces to the classical one of compact mapping introduced by Schauder.

Using the projectors P and Q introduced above and letting

$$K_{PQ} = (L|_{D(L)\cap \ker P})^{-1}(I - Q),$$

(K_{PQ} is the *right inverse of* L associated to P and Q), it is easy to verify that $G : E \to Z$ is L-compact on E if and only if $QG : E \to Z$ is continuous, QG(E) is bounded and $K_{PQ}G : E \to X$ is compact. Of course, if $L : D(L) \to Z$ is invertible, the L-compactness of G on E is equivalent to the compactness of $L^{-1}G$ on E.

If $G : X \to Z$ is L-compact on each bounded set $B \subset X$, we shall say that G is L-*completely continuous on X*. The following useful property of linear L-completely continuous mappings can be proved using the definition of L-compactness.

PROPOSITION 4.2.1. *If* $A : X \to Z$ *is linear,* L-*completely continuous on* X *and if* $ker(L + A) = \{0\}$, *then* $L + A : D(L) \to Z$ *is bijective and, for each* L-*compact mapping* $G : E \to Z$, *the mapping* $(L + A)^{-1}G : E \to X$ *is* L-*compact on* E.

Let us denote by C_L the set of couples (F,Ω) where the mapping $F : D(L) \cap \bar{\Omega} \to Z$ is of the form $F = L + N$ with $N : \bar{\Omega} \to Z$ L-compact and Ω is an open bounded subset of X, satisfying the condition

$$0 \notin F(D(L) \cap \partial\Omega).$$

A mapping D_L from C_L into $\mathbb{Z}$ will be called a *degree relatively to* L if it is not identically zero and satisfies the following axioms.

1. ADDITION-EXCISION PROPERTY. If $(F,\Omega) \in C_L$ and Ω_1 and Ω_2 are disjoint open subsets in Ω such that

$$0 \notin F[D(L) \cap (\bar{\Omega} \setminus (\Omega_1 \cap \Omega_2))],$$

then (F,Ω_1) and (F,Ω_2) belong to C_L and

$$D_L(F,\Omega) = D_L(F,\Omega_1) + D_L(F,\Omega_2).$$

2. Homotopy invariance property. If Γ is open and bounded in $X \times [0,1]$, $\mathcal{H} : (D(L) \cap \bar{\Gamma}) \times [0,1] \to Z$ has the form

$$\mathcal{H}(x,\lambda) = Lx + \mathcal{N}(x,\lambda),$$

where $\mathcal{N} : \bar{\Gamma} \to Z$ is L-compact on $\bar{\Gamma}$, and if

$$\mathcal{H}(x,\lambda) \neq 0$$

for each $x \in D(L) \cap (\partial\Gamma)_\lambda$ and each $\lambda \in [0,1]$, where

$$(\partial\Gamma)_\lambda = \{x \in X : (x,\lambda) \in \partial\Gamma\},$$

then the mapping $\lambda \to D_L(\mathcal{H}(.,\lambda),\Gamma\lambda)$ is constant on $[0,1]$, where Γ_λ denotes the set

$$\{x \in X : (x,\lambda) \in \Gamma\}.$$

The following properties are simple consequences of the axioms.

Excision property. *If* $(F,\Omega) \in C_L$ *and if* $\Omega_1 \subset \Omega$ *is an open set such that*

$$0 \notin F(D(L) \cap (\bar{\Omega} \setminus \Omega_1)),$$

then $(F,\Omega_1) \in C_L$ *and* $D_L(F,\Omega) = D_L(F,\Omega_1)$.

Existence property. *If* $(F,\Omega) \in C_L$ *is such that* $D_L(F,\Omega) \neq 0$, *then* F *has at least one zero in* Ω.

Boundary value dependence. *If* $(F,\Omega) \in C_L$ *and* $(G,\Omega) \in C_L$ *are such that* $Fx = Gx$ *for each* $x \in D(L) \cap \partial\Omega$, *then* $D_L(F,\Omega) = D_L(G,\Omega)$.

A degree D_L will be said to be *normalized* if it satisfies the following third axiom.

3. Normalization property. *If* $(F,\Omega) \in C_L$, with F the restriction to $\bar{\Omega}$ of a linear one-to-one mapping from D(L) into Z, then $D_L(F - b,\Omega) = 0$ if $b \notin F(D(L) \cap \Omega)$ and $|D_L(F - b,\Omega)| = 1$ if $b \in F(D(L) \cap \Omega)$.

A mapping degree D_o was first constructed by Kronecker in 1869 when $X = Z = \mathbb{R}^n$, F is of class C^1, Ω has a regular boundary and $0 \notin F(\partial\Omega)$, and then by Brouwer in 1912 when X and Z are finite dimensional oriented vector spaces, F is continuous and

$0 \notin F(\partial\Omega)$ (it is called the *Brouwer degree* and usually denoted by $\deg(F,\Omega,0)$). In 1934, Leray and Schauder constructed a mapping degree D_I when $X = Z$ is a Banach space, $F = I + N$ with $N : \bar{\Omega} \to X$ compact and $0 \notin (I + N)(\partial\Omega)$ (it is called the *Leray-Schauder degree,* and, as it reduces to the Brouwer degree when X is finite-dimensional, it will also be denoted by $\deg(F,\Omega,0)$). Those degree mappings satisfy the three axioms above and further properties that we will use freely in the sequel.

We will define now a generalization of the Leray-Schauder degree which was introduced by Mawhin in 1972. A systematic exposition will be found in [$Ma_{1,3}$].

Let $(F,\Omega) \in C_L$, with $F = L + N$. Denote by $\mathscr{C}(L)$ the set of linear completely continuous mappings $A : X \to Z$ such that $\ker(L + A) = \{0\}$. By Proposition 4.2.1, $L + A : D(L) \to Z$ is bijective and $(L + A)^{-1}G$ compact over $E \subset X$ whenever $G : E \to Z$ is L-compact.

One can prove that if $A \in \mathscr{C}(L)$ and $B \in \mathscr{C}(L)$, and if we set $\Delta_{B,A} = (L + B)^{-1}(A - B)$,then $\Delta_{B,A}$ is completely continuous on X and

$$I + (L + B)^{-1}(N - B) = (I + \Delta_{B,A})[I + (L + A)^{-1}(N - A)].$$

It is easy to check that $\ker(I + \Delta_{B,A}) = \ker(L + A) = \{0\}$, and hence, by the product formula of Leray-Schauder degree, we obtain

$$D_I(I + (L + B)^{-1}(N - B),\Omega)$$
$$= D_I(I + \Delta_{B,A},B(r)).D_I(I +(L + A)^{-1}(N - A),\Omega),$$

where $r > 0$ is arbitrary and $B(r) = \{x \in X : \|x\| < r\}$. We can now define a relation in $\mathscr{C}(L)$ by $B \sim A$ if and only if $D_I(I + \Delta_{B,A},B(r)) = +1$. It is an equivalence relation over $\mathscr{C}(L)$. If we fix an orientation on ker L and on coker L = Z/R(L), we can for example define $\mathscr{C}_+(L)$ as the class containing the application A of the form $\pi_Q^{-1} \Lambda P$, where $\Lambda : \ker L \to \operatorname{coker} L$ is an orientation preserving isomorphism and π_Q is the restriction to R(Q) of the canonical projection $\pi : Z \to \operatorname{coker} L$. Setting $J = \pi_Q^{-1} \Lambda : \ker L \to R(Q)$, it is easy to compute that

$$(L + JP)^{-1} = J^{-1}Q + K_{P,Q},$$

and hence

$$I + (L + JP)^{-1}(N - JP) = I - P + J^{-1}QN + K_{P,Q}N.$$

The following definition is therefore justified.

DEFINITION 4.2.2. *If* $(F,\Omega) \in C_L$, *the degree of* F *in* Ω *with respect to* L *is defined by*

$$D_L(F,\Omega) = D_I(I + (L + A)^{-1}(N - A),\Omega)$$
$$= deg(I + (L + A)^{-1}(N - A),\Omega,0),$$

for any $A \in \mathscr{C}_+(L)$.

It is easy to see, using the properties of the Leray-Schauder degree, that D_L satisfies the three axioms of the previous section and reduces to $D_I = \deg$ if $X = Z$ and $L = I$. The existence and homotopy invariance properties of the degree easily lead to interesting existence theorems.

THEOREM 4.2.1. *Let* $(H,\Omega) \in C_L$ *and* $F = L + N$ *with* $N : \bar{\Omega} \to Z$ *L-compact and* Ω *open and bounded in* X. *Assume that the following conditions are satisfied.*

(*i*) $\lambda Fx + (1 - \lambda)Hx \neq 0$ *for each* $(x,\lambda) \in (D(L) \cap \partial\Omega) \times]0,1[$.
(*ii*) $D_L(H,\Omega) \neq 0$.

Then the equation $Lx + Nx = 0$ *has at least one solution in* $D(L) \cap \bar{\Omega}$.

In order to compute the degree in practical situations, we now consider the case of an autonomous ordinary differential equation.

Let $\omega > 0$ be fixed, $\mathscr{C}_\omega = \{x \in \mathscr{C}(\mathbb{R},\mathbb{R}^n) : x(t + \omega) = x(t), t \in \mathbb{R}\}$ with the sup norm. Define the linear operator $\mathscr{L}$ in $\mathscr{C}_\omega$ by $D(\mathscr{L}) = \{x \in \mathscr{C}_\omega : x \text{ is of class } C^1\}$ and $(\mathscr{L}x)(t) = x'(t)$ $(t \in \mathbb{R})$. If $q : \mathbb{R}^n \to R^n$ is continuous, we shall consider the ω-periodic solutions of the autonomous differential equation

$$x'(t) = q(x(t)). \tag{4.2.3}$$

If $\mathscr{Q} : \mathscr{C}_\omega \to \mathscr{C}_\omega$ is the continuous mapping defined by

$$(\mathscr{Q}x)(t) = q(x(t)),$$

then finding the ω-periodic solutions of (4.2.3) is equivalent to solving the abstract equation

$$\mathscr{L}x = \mathscr{Q}x$$

in $D(\mathscr{L})$. Now, $x \in ker\,\mathscr{L}$ *if and only if* $x \in D(\mathscr{L})$ and

$$x(t) = c \in \mathbb{R}^n$$

for all $t \in \mathbb{R}^n$, so that $\ker \mathscr{L} \simeq \mathbb{R}^n$.

The following result is proved in [CMZ, Lemma 1].

THEOREM 4.2.2. *Assume that* $q : \mathbb{R}^n \to \mathbb{R}^n$ *is continuous and that there exists an open bounded set* $\Omega \in \mathscr{C}_\omega$ *such that (4.2.3) has no solution on* $\partial\Omega$. *Then*

$$D_{\mathscr{L}}(\mathscr{L} - \mathscr{Q},\Omega) = (-1)^n \, deg(\mathscr{Q}|_{\mathbb{R}^n}, \Omega \cap \mathbb{R}^n, 0)$$

where $\mathbb{R}^n$ *is identified with the subspace of constant functions of* $\mathscr{C}_\omega$.

4.3 Mountain pass and saddle points

Let E be a Banach space, and $\varphi : E \to \mathbb{R}$ be a continuously differentiable functional.

DEFINITION 4.3.1. *We say that* φ *satisfies the Palais-Smale condition if every sequence* (u_n) *in* E *for which* $(\varphi(u_n))$ *is bounded and* $\varphi'(u_n) \to 0$ *as* $n \to \infty$ *has a converging subsequence.*

The following result is due to Ambrosetti and Rabinowitz [AR], and is known as the Mountain Pass Theorem.

THEOREM 4.3.1. *Assume that* $\varphi : E \to \mathbb{R}$ *satisfies the Palais-Smale condition. Let* u_0 *and* u_1 *be two points in* E *and let* $\mathscr{S}$ *be a neighborhood of* u_0 *which does not contain* u_1 *and is such that*

$$\max\{\varphi(u_0),\varphi(u_1)\} < \inf_{\partial\mathscr{S}} \varphi.$$

Then φ *has critical point* $u \in E$ *such that*

$$\varphi(u) = \inf_{\eta\in\Gamma} \max_{s\in[0,1]} \varphi(\eta(s)),$$

where $\Gamma = \{\eta \in C([0,1],E) : \eta(0) = u_0, \eta(1) = u_1\}$.

We denote by B_R the open ball centered at 0 with radius $R > 0$. The following result of Rabinowitz (cf. [Ra]) is known as the Saddle Point Theorem.

THEOREM 4.3.2. *Let* $E = V \oplus W$, *where* V *is finite dimentional nontrivial subspace of* E. *Assume that* $\varphi : E \to \mathbb{R}$ *satisfies Palais-Smale condition, and that there is a constant* $R > 0$ *such that*

$$\max_{\partial B_R \cap V} \varphi < \inf_{W} \varphi.$$

Then φ *has a critical point* $u \notin \mathrm{E}$ *such that*

$$\varphi(u) = \inf_{\eta\in\Gamma} \max_{\xi\in\bar{B}_R\cap V} \varphi(\eta(\xi)),$$

where $\Gamma = \{\eta \in C(\bar{B}_R \cap V,E) : \eta_{|\partial B_R \cap V} = id\}$.

The above results, very often used in the applications, have their origin in the work of Palais and Smale, who succeded in extending to infinite dimensional spaces the theories of Morse and Lusternik-Schnirelman. Actually, one could write a single minimax theorem generalizing both the above results. However, we prefer to state therm in this simple way, which makes them easy to apply.

When considering a periodic problem

$$(P_1) \qquad \begin{cases} -u'' = g(t,u) \\ u(0) - u(T) = u'(0) - u'(T) = 0. \end{cases}$$

where $g : [0,T] \times \mathbb{R} \to \mathbb{R}$ is a Caratheodory function, we can define a functional on a Hilbert space whose critical points correspond to solutions of (P_1). Let $H^1_T = \{u \in H^1([0,T],\mathbb{R}) : u(0) = u(T)\}$, where $H^1([0,T],\mathbb{R})$ is the Sobolev space of functions with a square integrable derivative. Set $G(t,x) = \int_0^x g(t,\xi)\, d\xi$, and define the functional $\varphi : H^1_T \to \mathbb{R}$ as follows :

$$\varphi(u) = \int_0^T \left[\frac{1}{2}(u'(t))^2 - G(t,u(t))\right] dt.$$

It can be proved that functional φ is continuously differentiable and that its critical points are the T-periodic solutions of (P_1). One has

$$\varphi'(u_n)v = \int_0^T [u'_n(t)v'(t) - g(t,u_n(t))v(t)]\, dt,$$

for every $v \in H^1_T$. Concerning the Palais-Smale condition, for this particular type of functional we have the following.

PROPOSITION 4.3.3. *The functional* φ *defined above satisfies the Palais-Smale condition if every sequence* (u_n) *in* E *for which*

$(\varphi(u_n))$ is bounded and $\varphi'(u_n) \to 0$ as $n \to \infty$ has a bounded subsequence.

Proof. It is sufficient to show that if (u_n) is a bounded sequence such that $\varphi'(u_n) \to 0$ *as* $n \to \infty$, then (u_n) has a convergent subsequence. Being (u_n) bounded, there is a $u \in H^1_T$ such that a subsequence (u_{n_k}) converges weakly in H^1 and uniformly to u. We have

$$\varphi'(u_{n_k})(u_{n_k} - u) - \varphi'(u)(u_{n_k} - u) = \int_0^T |u'_{n_k}(t) - u'(t)|^2 \, dt -$$

$$- \int_0^T [(g(t,u_{n_k}(t)) - g(t,u(t)))(u_{n_k}(t) - u(t))] \, dt.$$

Since $\varphi'(u_{n_k}) \to 0$, the left hand side converges to zero. The same is true for the second term of the right hand side, by the uniform convergence. It follows that $\int_0^T |u'_{n_k} - u'|^2 \, dt$ converges to zero, and hence (u_{n_k}) converges strongly in H^1 to u.

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