SIAM J. MATH. ANAL.
Vol. 24, No. 5, pp. 1294–1311, September 1993

010

SUBHARMONIC SOLUTIONS FOR SOME SECOND-ORDER DIFFERENTIAL EQUATIONS WITH SINGULARITIES*

ALESSANDRO FONDA†, RAÚL MANÁSEVICH‡, AND FABIO ZANOLIN§

Abstract. The existence of infinitely many subharmonic solutions is proved for the periodically forced nonlinear scalar equation $u'' + g(u) = e(t)$, where g is a continuous function that is defined on a open proper interval $(A, B) \subset \mathbb{R}$. The nonlinear restoring field g is supposed to have some singular behaviour at the boundary of its domain. The following two main possibilities are analyzed:

(a) The domain is unbounded and g is sublinear at infinity. In this case, via critical point theory, it is possible to prove the existence of a sequence of subharmonics whose amplitudes and minimal periods tend to infinity.

(b) The domain is bounded and the periodic forcing term $e(t)$ has minimal period $T > 0$. In this case, using the generalized Poincaré–Birkhoff fixed point theorem, it is possible to show that for any $m \in \mathbb{N}$, there are infinitely many periodic solutions having mT as minimal period.

Applications are given to the dynamics of a charged particle moving on a line over which one has placed some electric charges of the same sign.

Key words. periodic solutions, subharmonics, repulsive singularities, saddle point theorem, critical levels, twist maps, generalized Poincaré–Birkhoff theorem

AMS subject classifications. 34C15, 34C25, 58E05, 70D05

1. Introduction.

A scalar equation of the form

$$u'' + g(u) = e(t) \tag{1.1}$$

can be viewed as a model for a system with one degree of freedom subject to an internal force given by the nonlinear restoring field $g(u)$ and an external time-dependent perturbation represented by $e(t)$.

In this paper we are interested in situations where $g(u)$ is a field having one or more singularities, all of which are of repelling type, and $e(t)$ will be supposed to be a periodic forcing, with period $T > 0$. We prove the existence of subharmonic solutions of (1.1), i.e., periodic solutions whose periods are integer multiples of T.

A simple physical model for this type of equation can be given by the dynamics of a charged particle moving on a line, over which one has placed some electric charges of the same sign. Since we consider only trajectories which do not collide with the singularity points, we can reduce our study to two different cases: the case of one singularity, with the particle moving on one side, and the case of two singularities, with the particle in between.

In §2 we deal with the one-singularity case. This case has been already considered by Lazer and Solimini in [17] (see also [14], [18]). They proved the existence of at least

*Received by the editors February 19, 1992; accepted for publication (in revised form) November 30, 1992.

†Dipartimento di Scienze Matematiche, Università di Trieste, P. le Europa 1, 34127, Trieste, Italy. The research of this author was sponsored by MURST 60%.

‡Departamento de Ingenieria Matemáticas, F.C.F.M. Universidad de Chile, Casilla 170, Correo 3, Santiago, Chile. The research of this author was sponsored by FONDECYT (1209-91) and by the European Economic Community.

§Dipartimento di Matematica e Informatica, Università di Udine, Via Zanon 6 - 33100, Udine, Italy. The research of this author was sponsored by the European Economic Community.

one T-periodic solution of (1.1) for the model described above. Under some general assumptions on g, we will show that, besides the T-periodic solutions, there is a whole sequence of subharmonic solutions whose minimal period is an arbitrarily large integer multiple of T. The proofs of the results of this section will use variational arguments providing critical points of saddle type for the action functional.

The forced two-singularities case for (1.1), as far as we know, has not been considered explicitly in the literature. Some work has been done for systems with a potential well (see [2], [1], [4]), but without a forcing term.

In §3, we will consider this case. Using a generalized version of the Poincaré–Birkhoff fixed point theorem we will prove that, for any fixed period which is an integer multiple of T, there are infinitely many periodic solutions with such a minimal period.

To be more specific, we will consider $g(u)$ to be defined on an open interval (A, B), which may be bounded or unbounded. This will permit us to deal simultaneously with various different qualitative situations. Intuitively, we may think of the extreme points A and B of the domain of g as "singularities" for the field g. With this in mind, it is reasonable to look for conditions on g such that g grows faster than linear at the singularities. Such requirement is satisfied when A (respectively, B) is finite, and $\lim_{x\to A^+} g(x) = -\infty$ (respectively, $\lim_{x\to B^-} g(x) = +\infty$) while, for $A = -\infty$ (respectively, $B = +\infty$), we will assume that $g(x)/x \to +\infty$ as $x \to A^+$ (respectively, $x \to B^-$).

In this setting, the search of T-periodic solutions and subharmonic solutions in the case $A = -\infty$ and $B = +\infty$ has already been considered in several papers starting with Morris [19], [20] who proved in [20] the existence of infinitely many subharmonics of any order for e smooth and $g(x) = 2x^3$. Extensions of Morris's result were obtained in [9], [7] for any g continuous and such that $g(x)/x \to +\infty$ for $x \to \pm\infty$, using a generalization of the Poincaré–Birkhoff fixed point theorem due to W. Ding [9]. Namely, the existence of fixed points for the iterates of the Poincaré map associated to (1.1) is obtained in [8], by showing that there are circular annuli in the plane (u, u') where the twist condition at the boundaries (which are circumferences) is satisfied.

In [5] Del Pino and Manásevich considered the case $A \in \mathbb{R}$ and $B = +\infty$ for a variant of (1.1) motivated by a problem in nonlinear elasticity. They proved the existence of infinitely many T-periodic solutions using the more refined version of W. Ding's theorem in [10], where fixed points of an area-preserving homeomorphism twisting the boundaries of an annulus are obtained for annuli with star-shaped boundaries. Note that in this case, the singularity in A modifies the geometry of the planar flow and now the twist property has to be checked on the boundary of some annular regions which are "deformations" of circular annuli through a non-Euclidean metric. For another recent application of the Poincaré–Birkhoff theorem to (1.1), see also [6].

In §4, we apply our results to the dynamics of an electric charge moving in a Coulombian field with one or two singularities.

2. Sublinear case and one singularity. We consider the equation

$$u'' + g(t, u) = e(t), \tag{2.1}$$

where $g : \mathbb{R} \times (0, +\infty) \to \mathbb{R}$ is a Caratheodory function, T-periodic in its first variable, such that for every positive constants $r < R$ there is a $\nu = \nu_{r,R} \in L^1(0, T)$ with $|g(t, x)| \le \nu(t)$ for almost every $t \in [0, T]$ and all $x \in [r, R]$. Moreover, $e : \mathbb{R} \to \mathbb{R}$ is locally integrable and T-periodic ($T > 0$). We denote by $\bar{e}$ the mean value of $e(t)$, i.e., $\bar{e} = (1/T) \int_0^T e(t)\, dt$.

THEOREM 2.1. *Let $F : (0, +\infty) \to \mathbb{R}$ be a continuously differentiable function satisfying the following two properties:*

$$\text{(k}_1\text{)} \qquad \lim_{x \to +\infty} \frac{F(x)}{x^2} = 0,$$

$$\text{(k}_2\text{)} \qquad \lim_{x \to 0^+} F(x) = +\infty.$$

Assume that

$$\text{(k}_3\text{)} \qquad g(t,x) \le F'(x),$$

and

$$\text{(k}_4\text{)} \qquad g(t,x)\mathrm{sgn}(x-1) \ge -h(t),$$

for all $x > 0$ and almost every $t \in [0,T]$, where $h \in L^1([0,T], \mathbb{R}^+)$. If, moreover,

$$\text{(k}_5\text{)} \qquad \frac{1}{T}\int_0^T \limsup_{x \to 0^+} g(t,x)\, dt \ < \bar{e} < \frac{1}{T}\int_0^T \liminf_{x \to +\infty} g(t,x)\, dt,$$

then (2.1) has a sequence $(x_k)_{k \ge 1}$ of positive kT-periodic solutions whose minimal periods tend to infinity.

We first define a truncation function. Thus, for $r > 0$, let us define $g_r : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ as follows:

$$g_r(t,x) = \begin{cases} g(t,x) & \text{if } x \ge r; \\ g(t,r) & \text{if } x < r. \end{cases}$$

PROPOSITION 2.1. *For every $k \in \mathbb{N}$ there exist positive r_k, R_k such that for any $s \in (0, r_k]$ and any kT-periodic solution u of*

$$u'' + g_s(t,u) = e(t), \tag{2.2}$$

we have that $r_k \le u(t) \le R_k$ for all $t \in \mathbb{R}$. In particular, any kT-periodic solution of (2.2) with $s = r_k$ is a solution of (2.1).

Proof. Without loss of generality, we can assume that $\bar{e} = 0$ (just subtract $\bar{e}$ to both sides of (2.2)).

We argue by contradiction. Fix $k \in \mathbb{N}$ and assume that for every $n \in \mathbb{N}$, there are $s_n \in (0, 1/n)$ and a kT-periodic function u_n that satisfy

$$u_n'' + g_{s_n}(t, u_n) = e(t), \tag{2.3}$$

and such that $\{u_n(t) \mid t \in \mathbb{R}\} \not\subset [1/n, n]$.

In the following, we denote by $\|\cdot\|_q$ the usual L^q-norm on $(0, kT)$.

We claim there exists a $d \ge 1$ such that for every n there is $t_n^{(1)} \in [0, kT]$ with $u_n(t_n^{(1)}) \in [1/d, d]$.

Indeed, suppose for instance that, for a subsequence, $\max u_n \to c \in [-\infty, 0]$. Since

$$0 = \int_0^{kT} g_{s_n}(t, u_n(t))\, dt, \tag{2.4}$$

by Fatou's lemma we have

$$0 \leq \int_0^{kT} \limsup_{n\to\infty} [g(t, u_n(t))\chi_{[u_n > s_n]} + g(t, s_n)\chi_{[u_n \leq s_n]}]\ dt \ \leq \int_0^{kT} \limsup_{x\to 0^+} g(t, x)\ dt$$

contradicting the Landesman-Lazer like condition (k_5). A similar contradiction is obtained if we let $\min u_n \to +\infty$, and the claim follows.

From now on we fix the constant $d \geq 1$ such that (according to (k_3) and (k_5)), $F'(x) > 0$ for each $x \in [d, +\infty)$.

Next, let us prove that there exists a positive constant R such that $\max u_n \leq R$ for every n. We will use some ideas from [22]. By contradiction, assume there exists a subsequence, still denoted by (u_n) for which $\max u_n \to +\infty$. Then we can find an interval $[\alpha_n, \beta_n]$, containing a point $t_n^{(2)}$ with $u_n(t_n^{(2)}) = \max u_n$, such that $(\beta_n - \alpha_n) \leq kT$ and

$$u_n(\alpha_n) = d = u_n(\beta_n);$$
$$d \leq u_n(t) \leq u_n(t_n^{(2)}) \quad \text{for all } t \in [\alpha_n, \beta_n].$$

For $t \in [\alpha_n, \beta_n]$, we have that (2.3) can be written as

$$\text{(2.5)} \qquad \begin{aligned} u_n' &= v_n + \int_{\alpha_n}^{t} e(s)\ ds \\ v_n' &= -g(t, u_n). \end{aligned}$$

Since $v_n(t) - \int_{\alpha_n}^{t} h(s)\ ds$ is decreasing in this interval, using (2.5) we obtain

$$\text{(2.6)} \qquad \max u_n - d \leq kT(v_n(\alpha_n) + ||e||_1 + ||h||_1),$$

so that, for n large enough, $v_n(\alpha_n) > ||e||_1$. On the other hand, again from (2.5) we find that $v_n(t_n^{(2)}) \leq ||e||_1$. Thus there exists a $t_n^{(3)} \in (\alpha_n, t_n^{(2)}]$ such that $v_n(t_n^{(3)}) = ||e||_1$. For t in the interval $[\alpha_n, t_n^{(3)}]$ we have

$$\begin{aligned} &\frac{d}{dt}\left[F(u_n(t)) + \frac{1}{2}(v_n(t) - ||e||_1)^2\right] \\ &= F'(u_n(t))\left[v_n(t) + \int_{\alpha_n}^{t} e(s)\ ds\right] + (v_n(t) - ||e||_1)(-g(t, u_n(t))) \\ &\geq F'(u_n(t))(v_n(t) - ||e||_1) + (v_n(t) - ||e||_1)(-g(t, u_n(t))) \\ &\geq (F'(u_n(t)) - g(t, u_n(t)))[v_n(t) - ||e||_1] \geq 0. \end{aligned}$$

Thus $F(u_n(\cdot)) + \frac{1}{2}(v_n(\cdot) - ||e||_1)^2$ is increasing in this interval and hence

$$F(d) + \tfrac{1}{2}(v_n(\alpha_n) - ||e||_1)^2 \leq F(u_n(t_n^{(3)})).$$

From assumption (k$_1$) we find that for any $\varepsilon > 0$ there is $C'_\varepsilon > 0$ such that

$$F(u) \leq \varepsilon u^2 + C'_\varepsilon \quad \text{for every} \quad u \geq d.$$

Hence

$$\begin{aligned} F(d) + \tfrac{1}{2}(v_n(\alpha_n) - ||e||_1)^2 &\leq \varepsilon(u_n(t_n^{(3)}))^2 + C'_\varepsilon \\ &\leq \varepsilon(\max u_n)^2 + C'_\varepsilon. \end{aligned}$$

Now choosing ε small enough and calling on (2.6) we obtain a contradiction when $n \to \infty$. Thus, we have proved that there exists $R > 0$ such that $\max u_n \leq R$ for every n.

Next, from (2.4) and $s_n \leq 1/d$ we have

$$\int_{[u_n<1/d]} |g_{s_n}(t,u_n(t))|\,dt \leq \int_{[u_n<1/d]} (-g_{s_n}(t,u_n(t))+h(t))\,dt + ||h||_1$$
$$\leq \int_{[1/d\leq u_n\leq R]} |g(t,u_n(t))|\,dt + 2||h||_1 \leq C,$$

and we obtain that $||g_{s_n}(\cdot,u_n(\cdot))||_1 \leq 2C$; hence

$$||u_n'||_\infty \leq C_1 := 2C + ||e||_1.$$

Now define $\tilde{g}_{s_n}(t,x) := g_{s_n}(t,x) - h(t)$. Then (2.3) can be written as

$$u_n'' + \tilde{g}_{s_n}(t,u_n) = e(t) - h(t). \tag{2.7}$$

Set

$$f_{s_n}(x) = \begin{cases} F'(x) & \text{if } x \geq s_n \\ F'(s_n) & \text{if } x < s_n, \end{cases}$$

and

$$\eta_{s_n}(x) = \min\{0, f_{s_n}(x)\}.$$

Since $h(t) \geq 0$, we have that, for $x \leq 1$, $\tilde{g}_{s_n}(t,x) \leq \eta_{s_n}(x)$. Assume next there is a $t_n^{(4)} > t_n^{(1)}$ such that $u(t_n^{(4)}) < 1/n$. Then there are $t_n^{(5)} < t_n^{(6)}$ such that $[t_n^{(5)}, t_n^{(6)}]$ is contained in $[t_n^{(1)}, t_n^{(4)}]$, and such that $u_n(t_n^{(5)}) = 1/d$, $u_n(t_n^{(6)}) = 1/n$ and $1/n \leq u_n(t) \leq 1/d$ for all $t \in [t_n^{(5)}, t_n^{(6)}]$. Note that $t_n^{(6)} - t_n^{(5)} \leq kT$.

Then, multiplying (2.7) by $(u_n' - C_1)$ and integrating over $[t_n^{(5)}, t_n^{(6)}]$, we get

$$\frac{1}{2}(u_n'(t_n^{(6)}) - C_1)^2 - \frac{1}{2}(u_n'(t_n^{(5)}) - C_1)^2 + \int_{t_n^{(5)}}^{t_n^{(6)}} \tilde{g}_{s_n}(t,u_n(t))(u_n'(t) - C_1)\,dt$$
$$\leq 2C_1(||e||_1 + ||h||_1).$$

Thus, since $(u_n' - C_1) \leq 0$ and $\eta_{s_n}(u_n(t)) \leq 0$, we obtain

$$\int_{t_n^{(5)}}^{t_n^{(6)}} \eta_{s_n}(u_n(t))u_n'(t)\,dt \leq \int_{t_n^{(5)}}^{t_n^{(6)}} \eta_{s_n}(u_n(t))(u_n'(t) - C_1)\,dt \leq \tilde{C},$$

where $\tilde{C} := 2C_1(C_1 + ||e||_1 + ||h||_1)$. Setting $H_{s_n}(x) = \int_{1/d}^x \eta_{s_n}(\xi)\,d\xi$ (a primitive of η_{s_n}), it follows that

$$H_{s_n}(1/n) = H_{s_n}(u_n(t_n^{(6)})) - H_{s_n}(u_n(t_n^{(5)})) \leq \tilde{C}.$$

But

$$H_{s_n}(1/n) \geq \int_{1/d}^{1/n} F'(x)\,dx \geq F(1/n) - F(1/d) \to +\infty \quad \text{as} \quad n \to \infty,$$

and hence we have a contradiction. Thus the proposition is proved. □

Proof of Theorem 2.1. We consider (2.2) with $s = r_k$, i.e.,

$$u'' + g_{r_k}(t,u) = e(t). \tag{2.8}$$

At this point we are in the same situation as in the proof of Theorem 2.5 in [12]. This is why we prefer to give only the main lines of the proof, the details being available from [12]. We take r_k sufficiently small, so that $(1/T)\int_0^T g(t,r_k)\,dt < \bar{e}$, and in such a way that $r_k \to 0$ as $k \to \infty$. Denoting by G_{r_k} a primitive of g_{r_k} with respect to its second variable, we have that for every k we are in the situation of [12, Thm. 2.1]. Thus we can apply the Saddle Point theorem to the functional $\phi_k : H^1_{kT} \to \mathbb{R}$ defined by

$$\phi_k(u) = \int_0^{kT} \left(\frac{1}{2}(u')^2 - G_{r_k}(t,u) + eu\right)\,dt,$$

whose critical points correspond to the kT-periodic solutions of (2.8). We find that for each k there is a $\rho_k > 0$ sufficiently large and a critical point u_k of ϕ_k such that

$$\phi_k(u_k) = \inf_{\gamma\in\Gamma_k}\max_{\xi\in[-\rho_k,\rho_k]} \phi_k(\gamma(\xi)),$$

where $\Gamma_k = \{\gamma \in C([-\rho_k,\rho_k], H^1_{kT}) \mid \gamma(\pm\rho_k) = \pm\rho_k\}$. By Fatou's lemma, we have

$$\liminf_{|x|\to\infty,k\to\infty} \operatorname{sgn}(x)\int_0^T\int_0^1 g_{r_k}(t,xs)\,ds\,dt > \bar{e}\,T,$$

and since $\int_0^T G_{r_k}(t,x)\,dt = x\int_0^T\int_0^1 g_{r_k}(t,xs)\,ds\,dt$,

$$\liminf_{|x|\to\infty,k\to\infty} \int_0^T G_{r_k}(t,x)\,dt\ -\bar{e}xT = +\infty.$$

Reasoning next as in the proof of [12, Thms. 2.1 and 2.5] (see also [11]), we can show that

$$\lim_{k\to\infty}\frac{1}{k}\phi_k(u_k) = -\infty. \tag{2.9}$$

Now we can prove that the minimal periods of the kT-periodic solutions u_k tend to infinity as $k \to \infty$. If not, for a subsequence there would be a subsequence with a common period, say $\bar{k}T$. Noting that from Proposition 2.1 the set of $\bar{k}T$-periodic solutions of (2.8) is bounded in H^1_{kT}, independently of $k \geq \bar{k}$, we get a contradiction with (2.9). □

As a consequence of Theorem 2.1 we have the following (cf. [12, Cor. 2.6]).

COROLLARY 2.1. *Suppose $g : \mathbb{R}\times(0,+\infty) \to \mathbb{R}$ to be continuous, and that*

$$\lim_{x\to+\infty}\frac{g(t,x)}{x} = 0, \tag{m$_1$}$$

$$\limsup_{x\to 0^+} xg(t,x) \leq c < 0, \tag{m$_2$}$$

uniformly with respect to t. If moreover

(m3) $$\bar{e} < \frac{1}{T}\int_0^T \liminf_{x\to+\infty} g(t,x)\,dt,$$

the conclusion of Theorem 2.1 *holds.*

If we substitute (k4)–(k5) with the more restrictive sign condition

(k'5) $$\limsup_{x\to0^+} g(t,x) \le c_1 < \bar{e} < c_2 \le \liminf_{x\to+\infty} g(t,x),$$

uniformly with respect to t, it is possible to get more precise information about the subharmonic solutions of (2.1), as follows.

THEOREM 2.2. *Assume* (k1), (k2), (k3), *and* (k'5). *Then there exists an integer* $m^* \in \mathbb{N}$ *such that for every* $m \ge m^*$, *equation* (2.10) *has at least one positive periodic solution having minimal period* mT.

The proof of this statement is omitted since it can be achieved via the generalized Poincaré–Birkhoff fixed point theorem arguing as in [8] (proof of Theorem 1.1). On the other hand, we prefer to present a different application of this theorem to the case of two singularities in the next section.

Remark 2.1. It is possible to see that, in Theorem 2.1, we can replace conditions (k4) and (k5) by

(k''4) there is a $d \ge 1$ such that $g(t,x)\mathrm{sgn}(x-1) > \bar{e}$ for all $x \in (0,1/d) \cup (d,+\infty)$;

(k''5) $$\lim_{x\to+\infty}\int_0^T (G(t,x) - \bar{e}x)\,dt = +\infty;$$

where G is a primitive of g with respect to the x variable. We are then led to the following.

COROLLARY 2.2. *Assume that* $g(t,x) = g(x)$ *and that the following conditions hold:*

(j1) $$\lim_{x\to+\infty}\frac{G(x)}{x^2} = 0;$$

(j2) *There is a* $d \ge 1$ *such that* $g(x)\mathrm{sgn}(x-1) > \bar{e}$ *for all* $x \in (0,1/d) \cup (d,+\infty)$;

(j3) $$\lim_{x\to0^+} G(x) = \lim_{x\to+\infty}(G(x) - \bar{e}x) = +\infty.$$

Then the same conclusion of Theorem 2.1 *holds.*

Remark 2.2. By a suitable change of variables, we can easily restate the analogous version of the results of this section in the case where $g(t,x)$ is defined on $\mathbb{R}\times(A,+\infty)$ or on $\mathbb{R}\times(-\infty,B)$, the singularity being at $A \in \mathbb{R}$ or at $B \in \mathbb{R}$, respectively.

3. Superlinear case and two singularities. Consider again equation

(3.1) $$u'' + g(u) = e(t),$$

where $g : (A,B) \to \mathbb{R}$ is continuous and $e : \mathbb{R} \to \mathbb{R}$ is T-periodic ($T > 0$), with $e \in L^1_{\mathrm{loc}}$.

Here we suppose that

$$-\infty \le A < B \le +\infty$$

and fix any $c \in (A,B)$.

Our goal is to prove the existence of infinitely many subharmonics of any order for (3.1) with $(u(t), u'(t))$ lying in the open strip

$$\mathcal{S} := (A, B) \times \mathbb{R}$$

and giving a precise statement about the nodal properties of $u(t) - c$.

To this end we consider also the equivalent system

$$u' = v + E(t), \qquad v' = -g(u) + \bar{e}, \tag{3.2}$$

where $E(t) := \int_0^t (e(\xi) - \bar{e})\, d\xi$ and $\bar{e} := (1/T)\int_0^T e(t)\, dt$. Note that $E : \mathbb{R} \to \mathbb{R}$ is continuous and T-periodic with mean value zero. Let G be a primitive of g; e.g., the one defined by

$$G(x) := \int_c^x g(s)\, ds.$$

To describe our result, we assume further the next conditions.

$$\lim_{x\to A^+} G(x) = \lim_{x\to B^-} G(x) = +\infty. \tag{i$_1$}$$

$$\lim_{x\to A^+} \frac{g(x)}{(x-c)} = \lim_{x\to B^-} \frac{g(x)}{(x-c)} = +\infty. \tag{i$_2$}$$

Remark 3.1. Note that condition (i_2) reads as follows:

If $A = -\infty$, then $\lim_{x\to-\infty} g(x)/x = +\infty$, while if $A \in \mathbb{R}$, then $\lim_{x\to A^+} g(x) = -\infty$.

If $B = +\infty$, then $\lim_{x\to+\infty} g(x)/x = +\infty$, while if $B \in \mathbb{R}$, then $\lim_{x\to B^-} g(x) = +\infty$.

We also observe that (i_1) is always satisfied at $A = -\infty$ or at $B = +\infty$ when (i_2) is assumed.

Finally, we remark that (i_2) is independent upon the choice of the point $c \in (A, B)$.

We further introduce the following terminology (see, e.g., [15, p.17]).

For $f : \mathcal{O} \to \mathcal{S}$, $\alpha \mapsto f(\alpha)$, with $\alpha \in \mathcal{O}$, where $(\mathcal{O}, \prec)$ is a directed set, we write

$$f(\alpha) \to \partial\mathcal{S}$$

if, for every compact set $\mathcal{K} \subset \mathcal{S}$, there is $\alpha_{\mathcal{K}} \in \mathcal{O}$ such that $f(\alpha) \notin \mathcal{K}$, for all $\alpha \in \mathcal{O}$ with $\alpha \succ \alpha_{\mathcal{K}}$.

The case in which $e(t)$ is a constant, and hence $E(t) \equiv 0$, can be completely analyzed in terms of energy levels arguments. Indeed, we can prove that if $\Gamma(z_0)$ is the orbit of

$$u' = v, \qquad v' = -g(u) + \bar{e},$$

with $(u(0), v(0)) = z_0$, then, for $z_0 \to \partial\mathcal{S}$, $\Gamma(z_0)$ is a periodic orbit with minimal period tending to zero. Therefore we assume henceforth that $e(t) \neq \bar{e}$ on a set of positive measure, so that $E(t)$ is nonconstant and it has a positive minimal period. Without loss of generality, we can suppose that T is the minimal period of $E(t)$.

Now we are in position to state our main result.

THEOREM 3.1. *Assume* (i_1) *and* (i_2) *and let* $m \geq 1$ *be any fixed integer. Then* (3.1) *has infinitely many periodic solutions with minimal period* mT. *More precisely, for each* $m \geq 1$, *there is an integer* $\nu_m^* \geq 0$ *such that for every* $p \in \mathbb{N}$ *with* p *prime with* m *and* $p > \nu_m^*$, *equation* (3.1) *has at least one periodic solution* $u = u_{m,p}(\cdot)$, *with minimal period* mT *and such that* $u(t) - c$ *has exactly* $2p$ *simple zeros in the interval* $[0, mT)$. *Moreover,* $(u_{m,p}(t), u'_{m,p}(t)) \to \partial S$, *as* $p \to +\infty$, *uniformly with respect to* $t \in [0, mT]$.

Remark 3.2. Note that if we are interested only in the existence of T-periodic solutions, then we can apply the above theorem taking $m = 1$, and we find ν_1^* such that for every $p \in \mathbb{N}$ with $p > \nu_1^*$ there is at least one T-periodic solution $u_p(\cdot)$ with $u_p(t) - c$ having exactly $2p$ simple zeros in the interval $[0, T)$.

This remark is true even in the case when T is not the minimal period of $E(t)$; however, in such a situation, we cannot guarantee that the periodic solutions we find have T as minimal period.

The proof of Theorem 3.1 is based on W. Ding's generalized Poincaré–Birkhoff fixed point theorem [10] which provides fixed points for the Poincaré map (and its mth iterates) associated to system (3.2). To do this, we need to have such an operator well defined. Hence, a first requirement is to have the uniqueness of the solutions for the Cauchy problems associated to (3.2). This difficulty can be overcome by a standard smoothing of the field g as briefly described in [23]. Of course, then it will be necessary to prove that the fixed points related to the approximating equations are all contained in the same annulus in order to pick up a sequence of these fixed points converging to a fixed point representing the initial value of an mT-periodic solution to (3.2). Here we do not follow such a program which has been already accomplished with all the details in various preceding papers. Accordingly, from now on, *we assume the uniqueness of the solutions for the Cauchy problems associated to system* (3.2) leaving the interested reader to complete the missing details following, e.g., [7].

We also remark that if we assume condition (i_1)–(i_2) then we have that the same is satisfied for the function $g(x) - \bar{e}$. Hence, calling $g(x)$ what was written before as $g(x) - \bar{e}$ and $e(t)$ instead of $e(t) - \bar{e}$, from now on we can assume, without loss of generality, that

$$(i_0) \qquad \frac{1}{T}\int_0^T e(t)dt = 0$$

holds. Notice that system (3.2) takes now the form

$$(3.3) \qquad u' = v + E(t), \qquad v' = -g(u).$$

Our first step is to prove the global existence in the past and in the future of the solutions to (3.3). In this direction we have the following result which is proved under some more general conditions than in Theorem 3.1.

PROPOSITION 3.1. *Assume* (i_0), (i_1) *and suppose that there are constants* a, b *with* $A < a < b < B$ *such that*

$$(3.4) \qquad g(x) < 0 \quad \text{for } A < x \leq a, \qquad g(x) > 0 \quad \text{for } b \leq x < B.$$

Then any noncontinuable solution $z = (u, v)$ *of* (3.3) *is defined in* $(-\infty, +\infty)$.

Proof. From (i_1) it is clear that $G(x) \geq G_{\min} > -\infty$, for all $x \in (A, B)$.

Let $\eta : (A, B) \to \mathbb{R}$, be a C^1 function defined as follows (cf. [16, p.120]): $\eta(x) = -||E||_\infty$ for $x \in (A, a)$, $\eta(x) = ||E||_\infty$ for $x \in (b, B)$, and η increasing in $[a, b]$. We thus have that $0 \le \eta'(x) \le L_\eta$ for all x in (A, B), where L_η is a suitable constant depending on the function η.

Now, define $V : (A, B) \times \mathbb{R} \to \mathbb{R}$ by

$$V(x,y) := \tfrac{1}{2}(y + \eta(x))^2 + G(x) - G_{\min}.$$

It is clear that for any $D > 0$ there is a compact set $\mathcal{B}_D$ contained in $\mathcal{S}$ such that

$$(x,y) \in \mathcal{S} \setminus \mathcal{B}_D \quad \text{implies that} \quad V(x,y) > D.$$

Next, let $z(t) = (u(t), v(t))$ be a solution of (3.3) defined in a maximal interval (α, β), with $\alpha < t_0 < \beta$ and $z(t_0) \in \mathcal{S}$. For $t \in [t_0, \beta)$, we have

$$\begin{aligned}\frac{d}{dt}V(z(t)) =&[\eta'(u)(v + \eta(u)) + g(u)]u' + [v + \eta(u)]v' \\ =&\eta'(u)[v + \eta(u)]^2 + \eta'(u)[v + \eta(u)][-\eta(u) + E(t)] - g(u)[\eta(u) - E(t)].\end{aligned}$$

Noting that $-g(x)[\eta(x) - E(t)] \le 0$ for all $x \in (A, a) \cup (b, B)$, we find that there is a constant $R_0 > 0$ such that $-g(x)[\eta(x) - E(t)] \le R_0$ for all $x \in (A, B)$ and $t \in \mathbb{R}$.

Thus

$$\begin{aligned}\frac{d}{dt}V(z(t)) &\le L_\eta[v + \eta(u)]^2 + 2||E||_\infty L_\eta|v + \eta(u)| + R_0 \\ &\le \frac{3}{2}L_\eta[v + \eta(u)]^2 + R_1,\end{aligned}$$

where R_1 is a constant depending on $||E||_\infty$ and R_0. We obtain

$$\frac{d}{dt}V(z(t)) \le 3L_\eta V(z(t)) + R_1.$$

We claim that β must be $+\infty$. Otherwise from the last inequality and Gronwall's lemma,

$$V(z(t)) \le \left[\frac{R_1}{3L_\eta} + V(z(t_0))\right] \exp(3L_\eta(\beta - t_0)) := \text{constant} := R_2.$$

Then

$$z(t) \in \mathcal{B}_{R_2} \quad \text{for all} \quad t \in [t_0, \beta).$$

But, this contradicts the global continuation theorem and the claim is proved.

Global continuability to the left follows from the above argument and by changing $t - t_0$ for $t_0 - t$ in (3.3). □

According to Proposition 3.1 and the uniqueness assumption for the Cauchy problems, we have that any solution to (3.3) is uniquely defined on $(-\infty, +\infty)$ by its initial conditions.

Remark 3.3. It is obvious that hypothesis (i_2) implies the existence of suitable constants a and b with

$$A < a < c < b < B \tag{3.5}$$

such that (3.4) holds. *Henceforth* (3.4) *and* (3.5) (*as well as* (i_0)) *will be constantly assumed in connection with* (i_2).

We define now the compact set

$$\mathcal{M} := [a, b] \times [-||E||_\infty, ||E||_\infty]$$

and observe that $(c, 0) \in \mathcal{M}$. A corollary of Proposition 3.1 is the following.

PROPOSITION 3.2. *Assume* (i_1) *and* (i_2). *Then for every compact set* $\mathcal{K} \subset \mathcal{S}$ *and each* $m \in \mathbb{N}$, *there is a compact set* $\mathcal{B} = \mathcal{B}(\mathcal{K}, m) \subset \mathcal{S}$ *with* $\mathcal{K} \subset \mathcal{B}$ *such that for each solution* $z = (u, v)$ *of system* (3.3), *the following inference holds:*

$$z(0) \notin \mathcal{B} \Rightarrow z(t) \notin \mathcal{K} \quad \forall t \in [-mT, mT].$$

In particular, for each $m \in \mathbb{N}$, *there is a compact set* $\mathcal{R}_m$ *with* $\mathcal{M} \subset \mathcal{R}_m \subset \mathcal{S}$ *such that*

$$z(0) \notin \mathcal{R}_m \Rightarrow z(t) \notin \mathcal{M} \quad \forall t \in [-mT, mT] \tag{3.6}$$

holds for any solution z *to* (3.3).

By the second part of Proposition 3.2 and (3.6) we have that given any solution $z(\cdot) = (u(\cdot), v(\cdot))$ of equation (3.3) with $z(0) \notin \mathcal{R}_m$, it follows that if $t \in [0, mT]$ is such that $u(t) - c = 0$, or, respectively, $v(t) = 0$, then $u'(t) \neq 0$, respectively, $v'(t) \neq 0$. Then, according to [5], we can define the *rotation number* $\psi_m(u)$ as

$$\psi_m(u) = k\pi + \lim_{t \to 0^+} \tan^{-1}\left(\frac{v(t)}{u(t) - c}\right) - \lim_{t \to mT^-} \tan^{-1}\left(\frac{v(t)}{u(t) - c}\right), \tag{3.7}$$

where k is the number of zeros of $u(t) - c$ in $(0, mT)$. Geometrically, $\psi_m(u)$ represents the total angle the vector from the origin to the point $(u(t) - c, v(t))$ describes as t goes from zero to mT, positive angles measured clockwise.

On the other hand, if $z(0) \notin \mathcal{R}_m$, then $z(t) \neq (c, 0)$, for all $t \in [0, mT]$ and therefore we can use polar coordinates with center in $(c, 0)$ to express $z(t)$ via Prüfer transformation as

$$u(t) = c + \rho(t) \cos\theta(t), \qquad v(t) = \rho(t) \sin\theta(t).$$

By standard facts,

$$-\theta'(t) = \frac{g(u(t))(u(t) - c) + v(t)^2 + E(t)v(t)}{(u(t) - c)^2 + v(t)^2} \tag{3.8}$$

and by the definition of the rotation number, we have

$$\theta(0) - \theta(mT) = \psi_m(u). \tag{3.9}$$

We recall that, from Proposition 3.1 and the uniqueness of the solutions to the Cauchy problems associated to system (3.3), we have that for every $z_0 = (x_0, y_0) \in \mathcal{S}$ there is a unique solution $z(t) = z(t; z_0) = \big(u(t; z_0), v(t; z_0)\big)$ of (3.3) with $z(0) = z_0$, which is defined on $\mathbb{R}$. Hence the Poincaré map

$$\phi : \mathcal{S} \to \mathcal{S}, \qquad \phi(z_0) := z(T; z_0)$$

is defined and it is continuous on $\mathcal{S}$. By the Liouville theorem it follows that ϕ is an area—preserving homeomorpism of the strip $\mathcal{S}$ onto itself. Clearly, all of these properties of ϕ hold true for any of the maps

$$\phi^k : \mathcal{S} \to \mathcal{S}, \quad \phi^k(z_0) := z(kT; z_0), \quad k \in \mathbb{Z}.$$

In particular, we note that z_0 is the initial point of a mT-periodic solutions $z(\cdot)$ of (3.3), with $m \in \mathbb{N}$, if and only if z_0 is a fixed point of the mth iterate ϕ^m of the Poincaré map ϕ.

A consequence of Proposition 3.2 which is crucial for the next application of the Poincaré–Birkhoff theorem is given by

$$\phi^{-m}((c,0)) = (\phi^m)^{-1}((c,0)) \in \mathcal{R}_m. \tag{3.10}$$

Finally, we set

$$\Psi_m(z_0) := \frac{\psi_m(u(\cdot; z_0))}{2\pi} \quad \text{for } z_0 \in \mathcal{S} \setminus \mathcal{R}_m$$

and observe that the map

$$\Psi_m : \mathrm{dom}\Psi_m \supset \mathcal{S} \setminus \mathcal{R}_m \to \mathbb{R}$$

is continuous. Notice that if $u(\cdot; z_0)$ is mT-periodic, then

$$\#_m u = 2\Psi_m(z_0), \tag{3.11}$$

where $\#_m u$ is the number of zeros of $u(t) - c$ in the interval $[0, mT)$. In this case, the simplicity of the zeros of $u(t) - c$ implies that $\#_m u$ is always an even number.

With the above notation, we can prove the next result.

PROPOSITION 3.3. *Assume* (i_1), (i_2). *Then*

$$\Psi_m(z_0) \to +\infty \quad \text{as } z_0 \to \partial\mathcal{S}.$$

Proof. In order to simplify the notation in the proof, we choose $c = (a+b)/2$.

From assumption (i_2) and (3.5), for any constant $R > 0$ we can find two numbers, d_R^- and d_R^+, with

$$A < d_R^- < a < c < b < d_R^+ < B$$

such that

$$g(x)(x-c) \geq 3R(x-c)^2 \quad \text{for all} \quad x \in (A, d_R^-] \cup [d_R^+, B).$$

Moreover, we can assume without loss of generality, by taking d_R^- smaller and d_R^+ larger, if necessary, that

$$\mathcal{R}_m \subset (d_R^-, d_R^+) \times \mathbb{R}.$$

Thus, for $x \in (A, d_R^-] \cup [d_R^+, B)$, and by choosing $R \geq \frac{1}{2}(2||E||_\infty/b - a)^2$, we have

$$\begin{aligned} g(x)(x-c) + y^2 + E(t)y &\geq 3R(x-c)^2 + \frac{1}{2}y^2 - \frac{1}{2}||E||_\infty^2 \\ &\geq 2R(x-c)^2 + \frac{1}{2}y^2 + R\left(\frac{b-a}{2}\right)^2 - \frac{1}{2}||E||_\infty^2 \\ &\geq 2R(x-c)^2 + \frac{1}{2}y^2. \end{aligned}$$

Now let us take $\gamma_R > 0$ so that

$$g(x)(x-c) \geq 3R(x-c)^2 - \gamma_R \quad \text{for all} \quad x \in (A,B).$$

Thus, for every $x \in [d_R^-, d_R^+]$, we obtain

$$g(x)(x-c) + y^2 + E(t)y \geq 2R(x-c)^2 + \tfrac{1}{2}y^2,$$

provided that $|y| \geq ||E||_\infty + (||E||_\infty^2 + 2\gamma_R)^{\frac{1}{2}} := D_R$.

In conclusion from this last argument, from Proposition 3.2, and from (3.8) we have that for all $m \in \mathbb{N}$, for all $R > 0$, there is a compact set $\mathcal{W}(R,m) \supset [d_R^-, d_R^+] \times [D_R^-, D_R^+]$ such that

$$\mathcal{R}_m \subset \mathcal{W}(R,m) \subset \mathcal{S}$$

and if $z(0) \in \mathcal{S} \setminus \mathcal{W}(R,m)$, then $z(t) \notin [d_R^-, d_R^+] \times [D_R^-, D_R^+]$ for $t \in [-mT, mT]$, and

$$-\theta'(t) \geq \frac{2R(u(t)-c)^2 + \frac{1}{2}v(t)^2}{(u(t)-c)^2 + v(t)^2}.$$

Thus, if $z(0) \notin \mathcal{W}(R,m)$ and $t \in [-mT, mT]$, we obtain

$$\frac{-\theta'(t)}{2R\cos^2\theta(t) + \frac{1}{2}\sin^2\theta(t)} \geq 1,$$

that implies

$$\int_{\theta(mT)}^{\theta(0)} \frac{d\theta}{2R\cos^2\theta + \frac{1}{2}\sin^2\theta} \geq mT$$

(see [3] for analogous computations). Using the fact that

$$\int_0^{2\pi} \frac{d\theta}{2R\cos^2\theta + \frac{1}{2}\sin^2\theta} = \frac{2\pi}{\sqrt{R}}$$

and recalling that

$$\frac{\theta(0) - \theta(mT)}{2\pi} \leq k+1,$$

where k denotes here the integer part of $\Psi_m(z(0))$, we obtain

$$k + 1 \geq mT\sqrt{R}/2\pi$$

and hence

$$\Psi_m(z(0)) \geq \left(mT\sqrt{R}/2\pi\right) - 1.$$

Letting R go to $+\infty$, we end the proof of the proposition. □

At this point we have all the tools to prove Theorem 3.1.

Proof of Theorem 3.1. We define the function

$$W : \mathcal{S} \to \mathbb{R}, \qquad W(x,y) := G(x) + \tfrac{1}{2}y^2$$

and observe that from (i_1) and (3.4) it follows that there is a constant $L_m > 0$ such that, for each $L \geq L_m$, the set $W^{-1}(L)$ is a simple closed curve which is star-shaped with respect to the point $(c,0)$ and it is contained in $\mathcal{S} \setminus \mathcal{R}_m$. Then, for any r_1, r_2 with $L_m \le r_1 < r_2$, we can consider the annulus

$$\mathcal{A} = \mathcal{A}(r_1,r_2) := \{(x,y) \in \mathcal{S} \mid r_1 \le W(x,y) \le r_2\} = W^{-1}([r_1,r_2])$$

and the inner disc

$$\mathcal{D} = \mathcal{D}(r_1) := \{(x,y) \in \mathcal{S} \mid W(x,y) < r_1\} = W^{-1}((-\infty,r_1)).$$

By the choice of r_1 and r_2 we have that the boundary of $\mathcal{A}$ is the union of two simple closed curves $\partial^-\mathcal{A}$ and $\partial^+\mathcal{A}$, named respectively *the inner boundary* and *the outer boundary* of $\mathcal{A}$, such that

$$\partial\mathcal{D} = \partial^-\mathcal{A} = W^{-1}(r_1)$$

is star-shaped around $(c,0)$. Moreover, as $\mathcal{D} \supset \mathcal{R}_m$, from (3.10) we obtain

$$(c,0) \in \phi^m(\mathcal{D}).$$

Now we argue as follows.

At first we fix any constant $r = r(m) \ge L_m$ and, using the continuity of Ψ_m, define

$$\nu_m^* := \text{int}\left[\max\{\Psi_m(z_0) \mid z_0 \in \partial\mathcal{D}(r)\}\right], \tag{3.12}$$

where int$[\xi]$ denotes the integer part of the number $\xi \in \mathbb{R}$.

Second, we choose any number $p \in \mathbb{N}$, with p prime with m and

$$p > \nu_m^*. \tag{3.13}$$

Then, using Proposition 3.3 and the continuity of Ψ_m, we can find another constant $R = R(m,p)$ with $R > r$ such that

$$\min\{\Psi_m(z_0) \mid z_0 \in W^{-1}(R)\} > p. \tag{3.14}$$

Now we observe that (3.12), (3.13), and (3.14) imply that on the boundaries of the annulus

$$\mathcal{A} = \mathcal{A}(r,R),$$

the twist condition

$$\Psi_m(z_0) < p \quad \text{for } z_0 \in \partial^-\mathcal{A}, \qquad \Psi_m(z_0) > p \quad \text{for } z_0 \in \partial^+\mathcal{A}$$

is satisfied.

Thus we have met all the conditions in order to apply W. Ding's generalization of the Poincaré–Birkhoff fixed point theorem [10] and hence we can conclude that the map ϕ^m has a fixed point, say $z^*_{m,p}$, belonging to the annulus $\mathcal{A}$. Furthermore, we also obtain

$$\Psi_m(z^*_{m,p}) = p. \tag{3.15}$$

The continuity of Ψ_m implies that $\Psi_m(z_0)$ is bounded for z_0 belonging to a compact subset of $\mathcal{S} \setminus \mathcal{R}_m$, thus, as

$$\Psi_m(z^*_{m,p}) \to +\infty$$

for $p \to +\infty$, we have

$$\mathrm{z}^*_{m,p} \to \partial S \quad \text{as} \quad p \to +\infty.$$

This last property, in connection with Proposition 3.2, finally implies that

$$\left(u^*_{m,p}(t), \frac{d}{dt}u^*_{m,p}(t)\right) \to \partial S \quad \text{as} \quad p \to +\infty,$$

uniformly with respect to $t \in [0, mT]$, where $(u^*, v^*) = z^*$ is the solution to (3.3) starting at $\mathrm{z}^* = \mathrm{z}^*_{m,p}$ at the time $t = 0$.

Our last goal now is to prove that the solution we find does have minimal period mT. To this aim it is sufficient to prove that

$$\phi^k(\mathrm{z}^*_{m,p}) \neq \mathrm{z}^*_{m,p} \quad \text{for each} \;\; 1 \leq k \leq m-1,$$

holds.

Assume by contradiction (cf. [7, §6]) that there is some k with $1 \leq k < m$ such that $\mathrm{z}^* = \mathrm{z}^*_{m,p}$ is a fixed point of ϕ^k. Then, by (3.9), (3.11), and the definition of Ψ_m, we obtain that

$$\exists\, \ell \in \mathbb{N},\ \ell < p: \quad \Psi_k(\mathrm{z}^*) = \ell.$$

Observing now that

$$(\phi^m)^k(\mathrm{z}^*) = (\phi^k)^m(\mathrm{z}^*) = \mathrm{z}^*,$$

we have

$$pk = \Psi_{m\cdot k}(\mathrm{z}^*) = \ell m$$

which yields

$$\frac{m}{p} = \frac{k}{\ell},$$

a contradiction with the assumption that p is prime with m.

In conclusion, we observe that if the local uniqueness of the solutions for the Cauchy problems associated to (3.3) is not guaranteed, we have to repeat the above argument for a sequence of approximating equations of the form

$$u' = v + E(t), \qquad v' = -g_n(u),$$

where $g_n : (A, B) \to \mathbb{R}$ is smooth and $g_n \to g$ uniformly on compact sets. It is possible to check that, for n sufficiently large, all the fixed points of the iterates of the Poincaré operator of the approximating equations belong to the same annulus $\mathcal{A}$ and the rotation number Ψ_m of all these fixed points is the same and equal to p. Hence we can pass to the limit for a subsequence and get a fixed point of ϕ^m (which now could be a multivalued function). For the missing technical details concerning such an approximation approach, we refer the reader to [7, §6].

At this step, all the assertions in Theorem 3.1 are justified and the proof is now complete. □

4. Examples. In this section we consider two examples for the applicability of our main results.

First, we examine the case with one singularity at a point $A \in \mathbb{R}$.

Example 4.1. Let an electric charge Q be placed at the fixed point $A \in \mathbb{R}$ and suppose that $y > A$ (or $y < A$) denotes the position of an electric charge q, having the same sign of Q, which lives in a one-dimensional space.

The Coulombian force h acting on q at time t is given by

$$h(y(t)) = \kappa qQ \frac{y(t)-A}{|y(t)-A|^3} := \kappa_0 \frac{y(t)-A}{|y(t)-A|^3},$$

where κ is a suitable constant and $\kappa_0 = \kappa qQ > 0$.

Let $e : \mathbb{R} \to \mathbb{R}$ be a T-periodic external forcing term acting on the system. We assume that $e \in L^1_{\text{loc}}$ and denote by $\bar{e} = \frac{1}{T}\int_0^T e(s)\,ds$ the mean value of the function e.

Then the Newton law for the dynamics of the charge q yields the following differential equation (unitary mass is assumed).

$$y''(t) = h(y(t)) + e(t). \tag{4.1}$$

Now, from Theorem 2.1, we have the following proposition.

PROPOSITION 4.1. *Equation* (4.1) *has periodic solutions if and only if* $\bar{e} \neq 0$. *If* $\bar{e} < 0$ (*respectively,* $\bar{e} > 0$), *all periodic solutions of* (4.1) *lie in* $(A, +\infty)$ (*respectively,* $(-\infty, A)$), *and besides having at least one* T*-periodic solution,* (4.1) *also has subharmonic solutions with minimal period* mT, *for every sufficiently large integer* m.

Proof. We rewrite (4.1) as

$$u''(t) + g(u(t)) = e(t), \tag{4.2}$$

with $u(t) = y(t) - A$ and $g(x) := -h(x + A)$. Let G be a primitive of g, e.g.,

$$G(x) = \frac{\kappa_0}{|x|}.$$

Note that

$$\lim_{|x|\to\infty} g(x) = 0, \qquad \lim_{|x|\to\infty} \frac{G(x)}{x^2} = 0 \tag{4.3}$$

and that

$$\lim_{x\to 0} g(x)\operatorname{sgn}(-x) = \lim_{x\to 0} G(x) = +\infty. \tag{4.4}$$

At first we claim that $\bar{e} \neq 0$ is a necessary condition. This follows from [17]. Namely, assume for instance that (4.2) has a τ-periodic solution $\tilde{u}(t)$, with $\tilde{u}(t) > 0$ for all $t \in \mathbb{R}$. Integrating both sides of (4.2) on the interval $[0, \tau]$, we obtain

$$\frac{1}{\tau}\int_0^\tau g(\tilde{u}(t))\,dt = \frac{1}{\tau}\int_0^\tau e(t)\,dt = \bar{e}.$$

Since $g(x) < 0$ for all $x > 0$, we obtain $\bar{e} < 0$. Analogously, if (4.2) has a negative solution, $\bar{e}$ has to be positive. Thus the claim is achieved.

Now, we suppose $\bar{e} < 0$, According to (4.3) and (4.4) all the assumptions of Theorem 2.2 are satisfied and we conclude with the result. If $\bar{e} > 0$, we can reduce by a change of variables to the previous case (see Remark 2.2). □

Next we present an example with two singularities.

Example 4.2. Let Q_1 and Q_2 be two electric charges placed at the fixed points $A \in \mathbb{R}$ and $B \in \mathbb{R}$ with $A < B$. Suppose that u, with $A < u < B$ denotes the position of an electric charge q where we assume that Q_1, Q_2 and q have all the same sign.

Now the Coulombian force l acting on q at the time t takes the form

$$\begin{aligned} l(u(t)) &= \kappa \frac{qQ_1}{(u(t)-A)^2} - \kappa \frac{qQ_2}{(u(t)-B)^2} \\ &= \frac{\kappa_1}{(u(t)-A)^2} - \frac{\kappa_2}{(u(t)-B)^2}, \end{aligned}$$

where $\kappa_i = \kappa q Q_i > 0$ for $i = 1, 2$.

Let $e : \mathbb{R} \to \mathbb{R}$ be a T-periodic forcing term as above and consider the differential equation

$$u''(t) = l(u(t)) + e(t). \tag{4.5}$$

We assume also that $\int_0^t (e(s) - \bar{e})\, ds$ is not constant. Then from Theorem 3.1 we have the following proposition.

PROPOSITION 4.2. *For every* $m \geq 1$, *equation* (4.5) *has infinitely many* mT-*periodic solutions lying in* (A, B), *all having minimal period* mT.

Proof. We write (4.5) as

$$u''(t) + g(u(t)) = e(t). \tag{4.6}$$

with $g(x) := -l(x)$. Note that

$$\lim_{x \to A^+} g(x) = -\infty, \qquad \lim_{x \to B^-} g(x) = +\infty$$

and

$$\lim_{x \to A^+} G(x) = \lim_{x \to B^-} G(x) = +\infty,$$

where, for a fixed $c \in (A, B)$,

$$G(x) = \int_c^x g(s)\, ds = \frac{\kappa_1}{x-A} + \frac{\kappa_2}{B-x} - \frac{\kappa_1}{c-A} - \frac{\kappa_2}{B-c}.$$

Since $e(t)$ is T-periodic and nonconstant, its minimal period equals to T/γ, for some $\gamma \in \mathbb{N}$. We apply now Theorem 3.1 and have that for every $m \geq 1$ equation (4.6) has at least infinitely many periodic solutions having $mT = m\gamma(T/\gamma)$ as minimal period. □

Remark 4.1. Since the nonlinearities in the above examples are locally Lipschitz continuous, arguing as in [21] we could claim that for any subharmonic solution we found there is a second one with the same minimal period and the same number of zeros, which is not a shift of the previous one.

REFERENCES

[1] A. AMBROSETTI AND V. COTI ZELATI, *Solutions with minimal period for Hamiltonian systems in a potential well*, Ann. Inst. H. Poincaré Anal. Non Lineaire, 4 (1987), pp. 275–296.

[2] V. BENCI, *Normal modes of a Lagrangian system constrained in a potential well*, Ann. Inst. H. Poincaré Anal. Non Lineaire, 1 (1984), pp. 379–400.

[3] A. Capietto, J. Mawhin, and F. Zanolin, *A continuation approach to superlinear periodic boundary value problems*, J. Differential Equations, 88 (1990), pp. 347–395.

[4] V. Coti Zelati, S. Li, and S. Wu, *Periodic solutions of a class of singular nonautonomous second order systems in potential well*, SISSA 1991/M, preprint.

[5] M. A. Del Pino and R. F. Manásevich, *Infinitely many T-periodic solutions for a problem arising in nonlinear elasticity*, J. Differential Equations, to appear.

[6] M. A. Del Pino, R. F. Manásevich, and A. E. Murua, *On the number of 2π-periodic solutions for $u'' + g(u) = s(1 + h(t))$ using the Poincaré-Birkhoff theorem*, J. Differential Equations, 95 (1992), pp. 240–258.

[7] T. Ding and F. Zanolin, *Periodic solutions of Duffing's equation with superquadratic potential*, J. Differential Equations, 97 (1992), pp. 328–378.

[8] T. Ding and F. Zanolin, *Subharmonic solutions of second order nonlinear equations: a time-map approach*, Nonlinear Analysis, TMA, 20 (1993), pp. 509–532.

[9] W. Ding, *Fixed points of twist mappings and periodic solutions of ordinary differential equations*, Acta Math. Sinica, 25 (1982), pp. 227–235. (In Chinese.)

[10] ———, *A generalization of the Poincaré-Birkhoff theorem*, Proc. Amer. Math. Soc., 88 (1983), pp. 341–346.

[11] A. Fonda and A. C. Lazer, *Subharmonic solutions of conservative systems with non-convex potential*, Proc. Amer. Math. Soc., 115 (1992), pp. 183–190.

[12] A. Fonda and M. Ramos, *Large amplitude subharmonic oscillations for scalar second order differential equations with asymmetric nonlinearities*, J. Differential Equations, to appear.

[13] A. Fonda, M. Ramos, and M. Willem, *Subharmonic solutions for second order differential equations*, Topological Methods in Nonlinear Anal., 1 (1993), pp. 41–57.

[14] P. Habets and L. Sanchez, *Periodic solutions of some Lienard equations with singularities*, Proc. Amer. Math. Soc., 109 (1989), pp. 1035–1046.

[15] J.K. Hale, *Ordinary Differential Equations*, R. E. Krieger Publishing, Huntington, NY, 1980.

[16] J.P. La Salle and S. Lefschetz, *Stability by Liapunov's Direct Method with Applications*, Academic Press, New York, 1963.

[17] A.C. Lazer and S. Solimini, *On periodic solutions of nonlinear differential equations with singularities*, Proc. Amer. Math. Soc., 88 (1987), pp. 109–114.

[18] J. Mawhin, *Topological degree and boundary value problems for nonlinear differential equations*, CIME Lectures, Montecatini, June 1991, to appear.

[19] G. R. Morris, *A differential equation for undamped forced oscillations* III, Proc. Cambridge Phil. Soc., 61 (1965), pp. 133–155.

[20] ———, *An infinite class of periodic solutions of $x'' + 2x^3 = p(t)$*, Proc. Cambridge Phil. Soc., 61 (1965), pp. 157–164.

[21] W. D. Neumann, *Generalizations of the Poincaré-Birkhoff fixed point theorem*, Bull. Austral. Math. Soc., 17 (1977), pp. 375–389.

[22] P. Omari, G. Villari, and F. Zanolin, *Periodic solutions of the Liénard equation with one-sided growth restrictions*, J. Differential Equations, 67 (1987), pp. 278–293.

[23] M. Struwe, *Multiple solutions of anticoercive boundary value problems for a class of ordinary differential equations of second order*, J. Differential Equations, 37 (1980), pp. 285–295.