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Semilinear Equations at Resonance with Non-symmetric Linear Part

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1. INTRODUCTION

Consider the equation

$$Lu = Nu, \tag{1.1}$$

where L is a densely defined unbounded linear operator on a Hilbert space H, and N is nonlinear. The aim of this paper is to extend the already well-known existence results for the case where L is self-adjoint, to a general situation, where no symmetry at all is required on L.

Throughout this paper, we will assume L to be closed with a compact resolvent and N to grow at most linearly at infinity. We are interested in problems at resonance, i.e., with $\ker L \neq \{0\}$. It is well known, in the self-adjoint case, that Eq. (1.1) has a solution whenever N asymptotically lies between 0 (which is an eigenvalue of L) and the first positive eigenvalue λ_1, with no interference with those values, a condition which can be expressed by asking that, for some constants $\mu \in [0, \lambda_1/2[$ and $\nu \geqslant 0$,

$$\left\| Nu - \frac{\lambda_1}{2} u \right\| \leqslant \mu \|u\| + \nu, \qquad \text{for all} \quad u \in H. \tag{1.2}$$

Some interference can even be allowed at one or both sides provided that some extra conditions are added (e.g., Landesman–Lazer type conditions; cf. [LL], [AM], [BN], [FF_2]).

When the operator L is no longer self-adjoint, possible interference of the nonlinearity with complex eigenvalues of L should be taken into account. However, as explained below, more fundamental modifications are then required concerning the conditions imposed on the nonlinearity.

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If L is normal, the situation does not depart much from the self-adjoint case. Indeed, we can then always find a sufficiently small $\rho > 0$ so that Eq. (1.1) has a solution if there exist $\mu \in [0, \rho/2[$ and $v \geqslant 0$ such that

$$\left\| Nu - \frac{\rho}{2} u \right\| \leqslant \mu \|u\| + v, \qquad \text{for all} \quad u \in H. \tag{1.3}$$

This follows from the fact that, in this case,

$$\exists \rho > 0 \text{ such that, } \forall u \in \operatorname{dom} L, \qquad \left\| Lu - \frac{\rho}{2} u \right\| \geqslant \frac{\rho}{2} \|u\|. \tag{1.4}$$

The existence proof for the nonlinear equation is an easy application of Leray–Schauder theory. Actually, condition (1.4) does not really require L to be normal; it is sufficient that

$$\ker L = \ker L^*, \tag{1.5}$$

L^* being the adjoint of L (see [BN], [OZ], [AOZ]).

We will show that the assumption (1.5) is crucial for obtaining (1.4). Moreover, we will provide situations where (1.5) is not satisfied and show that no solutions of (1.1) exist under assumption (1.3) alone, no matter how small ρ is taken. Such a phenomenon has been already observed by Gaudenzi and Zanolin [GZ] for nth order boundary value problems.

At this point, the question arises of finding how to modify condition (1.3) in order to adapt it to the case where $\ker L \neq \ker L^*$. A fairly natural idea is to introduce a continuous linear operator J which maps $\ker L$ into $\ker L^*$. With such an operator, we will show that (1.4) generalizes to

$$\exists \rho > 0 \text{ such that, } \forall u \in \operatorname{dom} L, \qquad \left\| Lu - \frac{\rho}{2} Ju \right\| \geqslant \frac{\rho}{2} \|Ju\|. \tag{1.6}$$

The associated condition for N becomes

$$\left\| Nu - \frac{\rho}{2} Ju \right\| \leqslant \mu \|Ju\| + v, \tag{1.7}$$

with $\mu \in [0, \rho/2[$, $v \geqslant 0$. We will prove, under the provision that J is an isomorphism between $\ker L$ and $\ker L^*$, that (1.1) is solvable under condition (1.7) (assuming, of course, that ρ comes from (1.6)). Once the role of the operator J is understood, it is an easy matter to treat various existence problems, letting, for instance, N interfere with the eigenvalue 0, while satisfying a generalized Landesman–Lazer type condition.

The operator J will not in general be unique, and it may be difficult to identify a convenient one, when dealing with applications to differential equations. However, we will provide some examples where J can be canonically defined in a simple way.

The paper is organized as follows. In Section 2, we define the linear setting and introduce the operator J, which will play a crucial role throughout the paper. In relation with (1.6), we define the set

$$\mathscr{A}_J = \left\{ \rho \in \mathbb{R} \mid \forall u \in \operatorname{dom} L, \ \left\| Lu - \frac{\rho}{2} Ju \right\| \geq \left\| \frac{\rho}{2} Ju \right\| \right\};$$

we will show that $\mathscr{A}_J$ is a closed interval whose interior contains 0, when J is an isomorphism between $\ker L$ and $\ker L^*$.

In Section 3, we prove two existence results, the first one being established under condition (1.7). For the second result, condition (1.7) is somehow relaxed, allowing N to interfere with the eigenvalue 0; a generalized Landesman–Lazer type condition (see [Ce], [Fi], [Ne], [Ma_1]) appears in that theorem.

In Section 4, we prove two non-existence results, showing that the use of the operator J is really necessary in concrete examples. We thus generalize a result of Gaudenzi and Zanolin [GZ].

In Section 5, we develop our results in a semi-abstract setting. In particular, in Theorem 5.4, we present a Landesman–Lazer type condition for non-selfadjoint problems, which was already considered, e.g., by Shaw [Sh] for bounded nonlinearities.

In Section 6, we illustrate our approach by two applications to boundary value problems. We first consider a scalar problem, and generalize a result of Ahmad [Ah]. Second, we consider the periodic boundary value problem for a system of differential equations. Both applications are rather simple, and we did not search for greater generality. An example of a boundary value problem for an eighth order ordinary differential equation is also included, for which the authors are indebted to M. Gaudenzi.

2. The Linear Setting

We will study problem (1.1) in a Hilbert space H equipped with the scalar product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. A more general setting could be considered, as in [AOZ] for instance, but we prefer keeping things simple at this stage.

Let $L : \operatorname{dom} L \subset H \to H$ be a densely defined Fredholm operator with zero index and nontrivial kernel, and consider projections $P, Q : H \to H$ such that

$$\operatorname{Im} P = \ker L, \qquad \ker Q = \operatorname{Im} L$$

(see [Ma$_2$] for the terminology). To P and Q, we associate the right inverse of L, defined by $K_{P,Q} = (L|_{\text{dom } L \cap \ker P})^{-1} (I-Q)\colon H \to H$; we assume that $K_{P,Q}$ is a compact operator.

Let $J\colon H \to H$ be a continuous linear operator, and define the set

$$\mathscr{A}_J = \left\{ \rho \in \mathbb{R} \mid \forall u \in \text{dom } L,\ \left\| Lu - \frac{\rho}{2} Ju \right\| \geqslant \left\| \frac{\rho}{2} Ju \right\| \right\}.$$

We denote by L^* and J^* the adjoint operators of L and J, respectively. Recall that $\ker L^* = (\text{Im } L)^\perp$. The following lemma describes the properties of the set $\mathscr{A}_J$. Notice that $\mathscr{A}_J$ could have been defined equivalently by

$$\mathscr{A}_J = \{ \rho \in \mathbb{R} \mid \forall u \in \text{dom } L,\ \|Lu\|^2 \geqslant \rho \langle Lu, Ju \rangle \}.$$

LEMMA 2.1. *The set $\mathscr{A}_J$ is a closed interval containing* 0. *If $J(\ker L) \subset \ker L^*$, then* 0 *is an interior point of $\mathscr{A}_J$. Otherwise, $\mathscr{A}_J = \{0\}$.*

Proof. It is easy to see that $\mathscr{A}_J$ is an interval containing 0 and that $\inf \mathscr{A}_J$ and $\sup \mathscr{A}_J$, when finite, belong to $\mathscr{A}_J$. Assume that $J(\ker L) \subset \ker L^* = (\text{Im } L)^\perp$. By the continuity of $K_{P,Q}$, there exists $\delta > 0$ such that $\|(I-P)\,u\| \leqslant \delta \|Lu\|$ for every $u \in \text{dom } L$. Then, for every $u \in \text{dom } L$,

$$\begin{aligned} |\langle Lu, Ju \rangle| &= |\langle Lu, J(I-P)\,u \rangle| \\ &\leqslant \delta \|J\|_{\mathscr{L}(H)} \|Lu\|^2, \end{aligned}$$

where $\|\cdot\|_{\mathscr{L}(H)}$ denotes the operator norm on the space $\mathscr{L}(H)$ of linear continuous operators from H into H. Setting $\rho = (\delta \|J\|_{\mathscr{L}(H)})^{-1}$, one sees that $[-\rho, \rho] \subset \mathscr{A}_J$, so that 0 is an interior point of $\mathscr{A}_J$.

Assume now that there exists $u \in J(\ker L) \setminus (\text{Im } L)^\perp$. Let $v \in \ker L$ be such that $Jv = u$, and $w \in \text{dom } L$ such that $\langle Lw, u \rangle \neq 0$. Arguing by contradiction, let $\rho \in \mathscr{A}_J \setminus \{0\}$. Then, for any $n \in \mathbb{Z}$, we have

$$\begin{aligned} \|Lw\|^2 = \|L(w+nv)\|^2 &\geqslant \rho \langle L(w+nv), J(w+nv) \rangle \\ &\geqslant \rho \langle Lw, Jw \rangle + \rho n \langle Lw, u \rangle, \end{aligned}$$

which clearly is impossible. ∎

In the following lemma, we present a possible way of determining the set $\mathscr{A}_J$.

PROPOSITION 2.2. *Assume that $J(\text{dom } L) \subset \text{dom } L^*$ and $J(\ker L) \subset \ker L^*$. Then, $\sup \mathscr{A}_J$ (resp. $\inf \mathscr{A}_J$) is the least positive (resp. greatest negative) eigenvalue λ of*

$$\begin{gathered} 2L^* Lu = \lambda (J^* Lu + L^* Ju) \\ u \in \text{dom } L \cap \ker P. \end{gathered} \tag{2.1}$$

Remark. If (2.1) has no positive (resp. negative) eigenvalue, then $\sup \mathscr{A}_J = +\infty$ (resp. $\inf \mathscr{A}_J = -\infty$).

Proof. The set $\operatorname{dom} L \cap \ker P$, equipped with the scalar product $a(u, v) := \langle Lu, Lv \rangle$, is a Hilbert space. Let us introduce the symmetric bilinear form

$$b(u, v) = \tfrac{1}{2}(\langle Lu, Jv \rangle + \langle Lv, Ju \rangle).$$

Set

$$\alpha = \sup_{u \in \operatorname{dom} L \cap \ker P} \frac{b(u, u)}{a(u, u)} = \sup_{u \in \operatorname{dom} L \cap \ker P} \frac{\langle Lu, Ju \rangle}{\|Lu\|^2};$$

if $\alpha \leqslant 0$, it is easily seen that $\sup \mathscr{A}_J = +\infty$, whereas, if $\alpha > 0$, we will have $\sup \mathscr{A}_J = \alpha^{-1}$. We will relate α to the eigenvalues of the problem

$$b(u, v) = \mu a(u, v), \qquad \forall v \in \operatorname{dom} L \cap \ker P. \tag{2.2}$$

The spectral theorem we use [Wi, p. 91] requires the bilinear form b to be weakly continuous for the topology induced by a. Let us show that this is the case. Let (u_k), (v_k) be weakly convergent sequences in $\operatorname{dom} L \cap \ker P$, i.e., for every $w \in \operatorname{Im} L$,

$$\langle Lu_k, w \rangle \to \langle Lu, w \rangle, \qquad \langle Lv_k, w \rangle \to \langle Lv, w \rangle,$$

for some $u, v \in \operatorname{dom} L \cap \ker P$. Then, writing

$$b(u_k, v_k) = \tfrac{1}{2}(\langle Lu_k, JK_{P,Q} Lv_k \rangle + \langle Lv_k, JK_{P,Q} Lu_k \rangle),$$

we see that $b(u_k, v_k)$ converges towards $b(u, v)$, since $K_{P,Q}$ is compact. The spectral theorem then tells us that, for $\alpha \leqslant 0$, the problem (2.2) has no positive eigenvalue, whereas, for $\alpha > 0$, the number $\alpha = (\sup \mathscr{A}_J)^{-1}$ is the greatest eigenvalue.

To conclude, it remains to relate the eigenvalues of the problems (2.1) and (2.2). Let us first observe that if α is an eigenvalue for the problem (2.2), α^{-1} is an eigenvalue for

$$2\langle Lu, Lv \rangle = \lambda(\langle Lu, Jv \rangle + \langle Lv, Ju \rangle), \qquad \forall v \in \operatorname{dom} L \cap \ker P, \tag{2.3}$$

and conversely. Since $J(\operatorname{dom} L) \subset \operatorname{dom} L^*$, the eigenvalue problem (2.3) can also be written as

$$2\langle Lu, Lv \rangle = \lambda(J^* Lu, v \rangle + \langle v, L^* Ju \rangle), \qquad \forall v \in \operatorname{dom} L \cap \ker P. \tag{2.4}$$

Any solution u of the above equation must clearly be such that $Lu \in \operatorname{dom} L^*$. Hence the eigenvalue problem (2.4) is equivalent to (2.1). We conclude that if $\alpha \neq 0$ is an eigenvalue of (2.2), α^{-1} is an eigenvalue of (2.1) (and conversely). ∎

We now turn our attention to an injectivity result which will be used in the sequel.

PROPOSITION 2.3. *Assume that* $J(\ker L) \subset \ker L^*$. *Let* $\rho \in [0, \sup \mathscr{A}_J[$. *If* $B: H \to H$ *is a continuous linear operator such that*

$$\left\| Bu - \frac{\rho}{2} Ju \right\| \leqslant \frac{\rho}{2} \|Ju\|, \tag{2.5}$$

for every $u \in \operatorname{dom} L$, *then the only solutions of*

$$Lu = Bu \tag{2.6}$$

are in $\ker L \cap \ker B$.

Proof. Let u be a solution of (2.6). Take $\rho' \in]\rho, \sup \mathscr{A}_J[$. Then,

$$\|Lu\|^2 = \|Bu\|^2 \leqslant \rho \langle Bu, Ju \rangle = \rho \langle Lu, Ju \rangle \leqslant \frac{\rho}{\rho'} \|Lu\|^2,$$

and the result follows. ∎

The reader is invited to sketch a geometrical proof of Proposition 2.3, using the uniform convexity of H.

3. THE SEMILINEAR EQUATION

In this section, we consider the semilinear equation

$$Lu = Nu, \tag{3.1}$$

where $N: H \to H$ is a continuous, not necessarily linear, operator which transforms bounded sets into bounded sets. The linear operator L satisfies the assumptions of the previous section. We introduce a linear continuous operator $J: H \to H$ and assume that

$$J_{|\ker L}: \ker L \to \ker L^* \text{ is an isomorphism.}$$

The first theorem we present is a "non-resonance" result, in the sense that if N verifies the conditions of Theorem 3.1, so does $N + h$, for any $h \in H$. The result provides thus surjectivity conditions for $L - N$.

THEOREM 3.1. *In the above defined setting, choose* $\rho \in]0, \sup \mathscr{A}_J[$ *and assume that*

$$\left\| Nu - \frac{\rho}{2} Ju \right\| \leqslant \mu \|Ju\| + \nu \tag{3.2}$$

for some $\mu \in [0, \rho/2[$, $\nu \geqslant 0$, *and for all* $u \in H$. *Then, Eq.* (3.1) *has a solution.*

Although a direct proof of Theorem 3.1 can easily be written, using Schauder's fixed point theorem, it is worth showing that it can be obtained as a corollary of Theorem 3.2, which also permits us to deal with resonance situations. By this, we refer to problems for which the conditions of the theorem are verified by the function N, but may not be verified by $N + h$, for some $h \in H$.

THEOREM 3.2. *In the above setting, assume*

(i) *there exist* $\rho \in [0, \sup \mathscr{A}_J[$ *and* $\alpha \in [0, 1[$ *such that*

$$\left\| Nu - \frac{\rho}{2} Ju \right\|^2 \leqslant \left\| \frac{\rho}{2} Ju \right\|^2 + O(\|u\|^{2\alpha}),$$

for $u \in H$, $\|u\| \to \infty$;

(ii) *for any sequence* $(u_n) \subset \operatorname{dom} L$ *such that* $\|u_n\| \to \infty$ *and* $\|Lu_n\| = O(\|u_n\|^\alpha)$ *for* $n \to \infty$, *one has, for n sufficiently large,*

$$\langle Nu_n, JPu_n \rangle \geqslant 0.$$

Then, Eq. (3.1) *has a solution.*

Proof. Let us consider, for $\lambda \in [0, 1[$, the equation

$$Lu = (1 - \lambda) \frac{\rho}{2} Ju + \lambda Nu. \tag{3.3}$$

Notice first that, for $\lambda = 0$, Proposition 2.3 together with the fact that $\ker(J_{|\ker L}) = \{0\}$ yields that (3.3) only has the trivial solution. By the theory of the coincidence degree (cf. [Ma_2]), the theorem will be proven if we are able to show that the solutions of (3.3) are a priori bounded in H, for $\lambda \in]0, 1[$. Assume, by contradiction, that there exist sequences $(\lambda_n) \subset]0, 1[$ and $(u_n) \subset \operatorname{dom} L$, solutions of (3.3). Take $\rho' \in]\rho, \sup \mathscr{A}_J[$. Since $\rho' \in \mathscr{A}_J$, it is easily shown, using the definition of $\mathscr{A}_J$, that for all $u \in \operatorname{dom} L$,

$$\left\| Lu - \frac{\rho}{2} Ju \right\|^2 \geqslant \left(1 - \frac{\rho}{\rho'}\right) \|Lu\|^2 + \left\| \frac{\rho}{2} Ju \right\|^2.$$

But, u_n being a solution of (3.3) for $\lambda = \lambda_n$, we have

$$\left\| Lu_n - \frac{\rho}{2} Ju_n \right\| = \lambda_n \left\| Nu_n - \frac{\rho}{2} Ju_n \right\|,$$

from which we deduce, using (i), that $\|Lu_n\| = O(\|u_n\|^\alpha)$ for $n \to \infty$ and, since $\alpha < 1$, $\|Lu_n\| = o(\|u_n\|)$ for $n \to \infty$. Using the continuity of $K_{P,Q}$ and J, and the fact that $J_{|\ker L}$ is an isomorphism, we then see that $\|J(I-P)\, u_n\| = o(\|JPu_n\|)$ for $n \to \infty$. On the other hand, multiplying (3.3) by JPu_n and using the equality $\langle Lu_n, JPu_n \rangle = 0$ (since $J(\ker L) = \operatorname{Im} L^\perp$), we obtain

$$\begin{aligned}
\frac{\langle Nu_n, JPu_n \rangle}{\|JPu_n\|^2} &= \frac{(1-\lambda_n)\, \rho \langle Ju_n, JPu_n \rangle}{2\lambda_n \|JPu_n\|^2} \\
&\leqslant -\frac{(1-\lambda_n)\, \rho}{2\lambda_n} \left[1 - \frac{\|J(I-P)\, u_n\|}{\|JPu_n\|} \right].
\end{aligned}$$

But, for n large, the second hand member in the above relation will be negative. This contradicts (ii). ∎

Proof of Theorem 3.1. Let us prove that assumptions (i) and (ii) of Theorem 3.2 are satisfied. Squaring both sides of (3.2), one easily obtains (i) with $\alpha = 1/2$. In order to show that (ii) holds, let $(u_n) \subset \operatorname{dom} L$ be such that $\|u_n\| \to \infty$ and $\|Lu_n\| = O(\|u_n\|^{1/2})$ for $n \to \infty$. A constant $c > 0$ can be found such that $\|(I-P)\, u_n\| \leqslant c\|u_n\|^{1/2}$. Take $\varepsilon \in \,]0, 1/\rho[$. We then have

$$\begin{aligned}
\langle Nu_n, JPu_n \rangle &= \langle Nu_n, Ju_n \rangle - \langle Nu_n, J(I-P)\, u_n \rangle \\
&\geqslant \langle Nu_n, Ju_n \rangle - c\|J\|\, \|Nu_n\|\, \|u_n\|^{1/2} \\
&\geqslant \langle Nu_n, Ju_n \rangle - \varepsilon \|Nu_n\|^2 - \frac{c^2 \|J\|^2}{4\varepsilon} \|u_n\|.
\end{aligned}$$

Using (3.2), one finally gets

$$\langle Nu_n, JPu_n \rangle \geqslant \frac{1}{\rho} \|Nu_n\|^2 + \eta \|Ju_n\|^2 - \eta', \tag{3.4}$$

for some positive constants η, η'. Since $(I-P)\, u_n = o(\|u_n\|)$ for $n \to \infty$ and since $J_{|\ker L}$ is an isomorphism, it is clear that $\|Ju_n\| \to \infty$ for $n \to \infty$. Hence, relation (3.4) implies that (ii) holds, and the conclusion then follows from Theorem 3.2. ∎

4. The Case $\mathscr{A}_J = \{0\}$

In the previous section, the assumptions made on the operator J were such to guarantee that the set $\mathscr{A}_J$ would not reduce to $\{0\}$. Let us now examine this degeneracy situation. Having in mind Theorems 3.1 and 3.2, we raise the following two questions.

Question 1. If $\mathscr{A}_J = \{0\}$, does there exist a positive ρ such that, whenever N satisfies (3.2) for some $\mu \in [0, \rho/2[$ and $\nu \geq 0$, Eq. (3.1) has a solution?

Question 2. If $\mathscr{A}_J = \{0\}$, N has a bounded range, and satisfies condition (ii) of Theorem 3.2 with $\alpha = 0$, does Eq. (3.1) always have a solution?

We will answer negatively to both questions, exhibiting situations, with $J = I$, where the equations fail to have a solution. For that purpose, assume $\Omega \subset \mathbb{R}^n$ is a bounded domain. We will take $H = L^2(\Omega; \mathbb{R})$, and consider an operator $L: \operatorname{dom} L \subset H \to H$, which is densely defined, closed, has a compact resolvent, and a one-dimensional kernel. Let $\ker L = \mathbb{R}\varphi$ and $\ker L^* = \mathbb{R}\psi$; we assume that the functions φ, ψ are continuous on $\bar{\Omega}$ and that the sets $\{x \in \Omega \mid \varphi(x)\,\psi(x) > 0\}$ and $\{x \in \Omega \mid \varphi(x)\,\psi(x) < 0\}$ both have positive measure. In that case, taking $J = I$, it is clear that J does not map $\ker L$ into $\ker L^*$; hence, by Lemma 2.1, $\mathscr{A}_J = \{0\}$. The following proposition will allow us to give a negative answer to Question 1. We will denote by $\mathscr{C}(\bar{\Omega}; \mathbb{R})$ the space of continuous real-valued functions defined on $\bar{\Omega}$.

Proposition 4.1. *In the setting defined above, for every $\rho > 0$, there exists $p \in \mathscr{C}(\bar{\Omega}; \mathbb{R})$ such that, for almost every $x \in \Omega$,*

$$0 < p(x) < \rho$$

and the equation

$$Lu = p(x)\,u$$

has a nontrivial solution.

Proof. The assumptions on φ and ψ yield the possibility of constructing two positive functions $p_1, p_2 \in \mathscr{C}(\bar{\Omega}; \mathbb{R})$ such that

$$\int_\Omega p_1 \varphi\psi < 0 < \int_\Omega p_2 \varphi\psi. \tag{4.1}$$

The proof will be complete once the following result has been established.

Claim. There exists $\delta > 0$ such that, for every $s \in]0, \delta[$, we can find a $\lambda \in [0, 1]$ for which the equation

$$Lu = s[\lambda p_1(x) + (1 - \lambda)\,p_2(x)]\,u \tag{4.2}$$

has a nontrivial solution.

To prove the claim, assume by contradiction that there is a positive sequence (s_n) converging to 0, such that, for every $\lambda \in [0, 1]$, Eq. (4.2) with $s = s_n$ only has the trivial solution. Taking, if necessary, n sufficiently large, we can assume that the operator $(L - s_n[\lambda p_1 + (1-\lambda) p_2] I)$ still has a compact resolvent (see [Ka, Theorem 3.17, p. 214]), and since we are assuming that 0 is not an eigenvalue, it is invertible. In particular, the equation

$$Lu - s_n[\lambda p_1 + (1-\lambda) p_2] u = s_n \psi \tag{4.3}$$

has a solution $u = u_{n,\lambda}$. Without loss of generality, we may assume that $\|\varphi\| = \|\psi\| = 1$. Multiplying both sides of (4.3) by ψ and integrating, we get

$$-\int_\Omega [\lambda p_1 + (1-\lambda) p_2] u_{n,\lambda} \psi = 1, \tag{4.4}$$

so that, for some $\eta > 0$, we have that $\|u_{n,\lambda}\| \geqslant \eta$ for every n and $\lambda \in [0, 1]$. Set $v_{n,\lambda} = u_{n,\lambda} / \|u_{n,\lambda}\|$. Since $u_{n,\lambda}$ is a solution of (4.3), we can write

$$Lv_{n,\lambda} - s_n[\lambda p_1 + (1-\lambda) p_2] v_{n,\lambda} = s_n \psi / \|u_{n,\lambda}\|.$$

From this, we deduce that

$$\lim_{n \to \infty} Lv_{n,\lambda} = 0, \qquad \text{uniformly in } \lambda \in [0, 1].$$

and, consequently,

$$\lim_{n \to \infty} (I - P) v_{n,\lambda} = 0, \qquad \text{uniformly in } \lambda \in [0, 1]$$

(recall that P is a projection on $\ker L$ and that the right inverse of L is continuous). Then, for any $\lambda \in [0, 1]$, the sequence $(v_{n,\lambda})$ must contain a subsequence which converges to some element of $\ker L$; since $v_{n,\lambda}$ is of norm 1, the limit must be either φ or $-\varphi$. But, by (4.4), we have

$$\int_\Omega [\lambda p_1 + (1-\lambda) p_2] v_{n,\lambda} \psi < 0.$$

It then results from (4.1) that $v_{n,0}$ converges to $-\varphi$ and $v_{n,1}$ converges to φ for $n \to \infty$. Define the number $r_{n,\lambda}$ by $Pv_{n,\lambda} = r_{n,\lambda} \varphi$; if $\bar{n}$ is taken large enough, the following relations will clearly hold:

$$\|(I - P) v_{\bar{n},\lambda}\| \leqslant 1/2 \qquad (\lambda \in [0, 1]),$$

$$r_{\bar{n},0} \leqslant -1/2, \qquad r_{\bar{n},1} \geqslant 1/2.$$

Since $u_{n,\lambda}$ depends continuously on $\lambda \in [0, 1]$ (see [Ka, p. 417]), so do $v_{n,\lambda}$ and $r_{n,\lambda}$. Consequently, there must exist a $\bar{\lambda} \in]0, 1[$ such that $r_{\bar{n},\bar{\lambda}} = 0$. Then

$$1 = \|v_{n,\bar{\lambda}}\| = \|(I-P)\, v_{\bar{n},\bar{\lambda}}\| \leqslant 1/2,$$

a contradiction. ∎

We are now able to answer Question 1. For any $\rho > 0$, consider a function p given by Proposition 4.1 (i.e., such that $0 < p(x) < \rho$ for a.e. $x \in \Omega$); we can assume p to be small enough so that the operator $L - pI$ has compact resolvent. Take $h \in L^2(\Omega; \mathbb{R})$ and let $Nu = pu + h$, $J = I$. Then, condition (3.2) is satisfied for some $\mu \in [0, \rho/2[$ and $v \geqslant 0$, but (3.1) has no solution, unless $h \in \ker(L - pI)^*$.

In order to answer Question 2, consider the same setting as above with $n = 1$, and let $(Nu)(x) = \arctan u(x) + h(x)$, $J = I$. We will show that a function $h \in L^2(\Omega; \mathbb{R})$ can be found such that condition (ii) of Theorem 3.2 is satisfied, whereas Eq. (3.1) has no solution. Because of the assumption on φ and ψ, it is possible to construct a function $h \in L^2(\Omega; \mathbb{R})$ such that

$$\left(\int_\Omega h\varphi\right) \Big/ \left(\int_\Omega |\varphi|\right) \in]-\pi/2, \pi/2[\tag{4.5}$$

and

$$\left(\int_\Omega h\psi\right) \Big/ \left(\int_\Omega |\psi|\right) \notin [-\pi/2, \pi/2]. \tag{4.6}$$

Let us prove that condition (ii) of Theorem 3.2 holds with $\alpha = 0$. By contradiction, if not,

$$\frac{1}{\|u_n\|} \int_\Omega (\arctan u_n(x) + h(x))\, Pu_n(x)\, dx < 0, \tag{4.7}$$

for some sequence $(u_n) \subset \operatorname{dom} L$ such that $\|u_n\| \to \infty$ for $n \to \infty$ and $(\|(I-P)\, u_n\|)$ is bounded. Assuming $\|\varphi\| = 1$, this implies that $(u_n/\|u_n\|)$ converges either to φ or to $-\varphi$. Let us consider the second case, for example. By Fatou's lemma, we deduce from (4.7) that

$$\int_\Omega h\varphi \geqslant \int_\Omega \liminf_{n\to\infty} \left(\operatorname{artan} u_n(x) \frac{Pu_n(x)}{\|u_n\|}\right) dx.$$

Then, computing the limit, we get

$$\int_\Omega h\varphi \geqslant \int_{\varphi > 0} \frac{\pi}{2}\varphi + \int_{\varphi < 0} -\frac{\pi}{2}\varphi = \frac{\pi}{2}\int_\Omega |\varphi|,$$

in contradiction with (4.5). An analogous contradiction is reached when $(u_n/\|u_n\|)$ converges to φ, so that condition (ii) of Theorem 3.1 holds with $\alpha = 0$. Let us now prove that (3.1) has no solutions. For, if u were a solution of (3.1), we would have

$$\int_\Omega (\arctan u(x) + h(x))\, \psi(x)\, dx = 0,$$

from which it follows that

$$-\frac{\pi}{2}\int_\Omega |\psi| \leqslant \int_\Omega h\psi \leqslant \frac{\pi}{2}\int_\Omega |\psi|,$$

in contradiction with (4.6).

5. Semi-abstract Existence Results

Let $\Omega \in \mathbb{R}^n$ be a bounded domain, and take $H = L^2(\Omega; \mathbb{R}^m)$. As above, the linear operator $L: \operatorname{dom} L \subset H \to H$ is assumed to be densely defined, closed, with a compact resolvent and a nontrivial kernel.

The linear operator $J: H \to H$ will be taken of the form

$$(Ju)(x) = \chi_u(x)\, \mathscr{J} u(x),$$

where $\mathscr{J}$ is an $m \times m$ matrix, and $\chi_u \in L^\infty(\Omega; \mathbb{R})$ is such that, for some positive constants a, b,

$$a \leqslant \chi_u(x) \leqslant b, \tag{5.1}$$

for every $u \in H$ and a.e. $x \in \Omega$. We will assume that J maps $\ker L$ isomorphically onto $\ker L^*$. Consequently, by Lemma 2.1, we will have $\sup \mathscr{A}_J > 0$. Although the hypotheses on J are a serious restriction, we will show cases below where such a J can be found. The following result is an immediate consequence of Proposition 2.3. The norm in $\mathbb{R}^m$ is denoted by $|\cdot|$, the scalar product by $(\cdot, \cdot)$.

Proposition 5.1. *Let $M \in L^\infty(\Omega; \mathbb{R}^{m \times m})$ be such that, for a given $\gamma \in\,]0, a \sup \mathscr{A}_J[$,*

$$\left| M(x)\, u - \frac{\gamma}{2} \mathscr{J} u \right| < \frac{\gamma}{2} |\mathscr{J} u| \tag{5.2}$$

for every $u \in \mathbb{R}^m \setminus \{0\}$ and a.e. $x \in \Omega$. Then, the equation

$$Lu = M(x)\, u \tag{5.3}$$

has only the trivial solution.

Proof. Rewrite (5.2) as

$$(M(x)\,u, \mathscr{J}u) > \frac{1}{\gamma}\,|M(x)\,u|^2,$$

and set $B: H \to H: (Bu)(x) = M(x)\,u(x)$. We then have, using (5.1),

$$\langle Bu, Ju\rangle = \int_\Omega (M(x)\,u(x), \chi_u(x)\,\mathscr{J}u(x))\,dx$$

$$> \frac{a}{\gamma}\int_\Omega |M(x)\,u(x)|^2 = \frac{a}{\gamma}\,\|Bu\|^2.$$

By Proposition 2.3, it follows that, if u is a solution of (5.3), then $u \in \ker L \cap \ker B$. Hence, for such a function u, $M(x)\,u(x) = 0$ for a.e. $x \in \Omega$, which, by (5.2), implies $Ju = 0$ and, consequently, $u = 0$, since J is one-to-one on $\ker L$. ▮

COROLLARY 5.2. *Let* $\gamma \in\,]0, a \sup \mathscr{A}_J[$. *Then, for any function* $q \in L^\infty(\Omega; \mathbb{R})$ *such that* $0 < q(x) < \gamma$ *for a.e.* $x \in \Omega$, *the equation*

$$Lu = q(x)\,\mathscr{J}u$$

only has the trivial solution.

Proof. It suffices to take $M(x) = q(x)\,\mathscr{J}$ and to apply Proposition 5.1. ▮

Let $g: \Omega \times \mathbb{R}^m \to \mathbb{R}^m$ be a L^2-Carathéodory function, by which we mean that

(i) for each $u \in \mathbb{R}^m$, g is measurable in x;

(ii) for almost every $x \in \Omega$, g is continuous in u;

(iii) for every $R > 0$, there exists a function $h_R \in L^2(\Omega; \mathbb{R})$ such that $|g(x, u)| \leqslant h_R(x)$ for a.e. $x \in \Omega$ and for all $u \in \mathbb{R}^m$ with $|u| \leqslant R$.

The hypotheses on g made in the sequel will imply that g is growing at most linearly in u. In that case, the Nemitzkii operator $N: H \to H: u \mapsto g(\cdot, u(\cdot))$ is well defined. Using Theorem 3.1, we are able to prove the following existence result for the equation

$$Lu = g(x, u). \tag{5.4}$$

THEOREM 5.3. *Let* $\gamma \in\,]0, a \sup \mathscr{A}_J[$ *and assume that*

$$\left|g(x, u) - \frac{\gamma}{2}\,\mathscr{J}u\right| \leqslant \mu|\mathscr{J}u| + \nu, \tag{5.5}$$

for some $\mu \in [0, \gamma/2[$, $\nu \geqslant 0$, *for every* $u \in \mathbb{R}^m$ *and a.e.* $x \in \Omega$. *Then, Eq.* (5.4) *has a solution.*

Proof. Squaring both sides of (5.5), we get

$$\gamma(g(x, u), \mathscr{J} u) \geqslant |g(x, u)|^2 + c|\mathscr{J} u|^2 - c',$$

for some $c \in]0, \gamma^2/4[$ and $c' \geqslant 0$. Set $\rho = \gamma/a$. Then, for some constant c'',

$$\begin{aligned}\rho\langle Nu, Ju\rangle &= \rho \int_\Omega (g(x, u(x)), \chi_u(x) \mathscr{J} u(x))\, dx \\ &\geqslant \rho \int_\Omega \frac{\chi_u(x)}{\gamma} [|g(x, u(x))|^2 + c|\mathscr{J} u(x)|^2 - c']\, dx \\ &\geqslant \int_\Omega \left[|g(x, u(x))|^2 + c|\mathscr{J} u(x)|^2 - \frac{\rho c' b}{\gamma}\right] dx \\ &\geqslant \|Nu\|^2 + \frac{c}{b^2} \|Ju\|^2 - c''.\end{aligned}$$

From this, we deduce that

$$\begin{aligned}\left\| Nu - \frac{\rho}{2} Ju \right\|^2 &= \|Nu\|^2 - \rho\langle Nu, Ju\rangle + \frac{\rho^2}{4} \|Ju\|^2 \\ &\leqslant \left(\frac{\rho^2}{4} - \frac{c}{b^2}\right) \|Ju\|^2 + c''\end{aligned}$$

and, consequently,

$$\left\| Nu - \frac{\rho}{2} Ju \right\| \leqslant \left(\frac{\rho^2}{4} - \frac{c}{b^2}\right)^{1/2} \|Ju\| + \sqrt{c''}.$$

The conclusion then follows from Theorem 3.1. ∎

We now consider the scalar case, i.e., $m = 1$. Without loss of generality, we may take $\mathscr{J} = 1$. By the hypotheses made on J, to each $\varphi \in \ker L$ is associated a $\psi \in \ker L^*$, such that $\psi(x) = \chi_\varphi(x)\, \varphi(x)$. Moreover, because of the hypotheses made on χ_u, $\varphi(x)$ and $\psi(x)$ will have the same sign.

Theorem 5.4. *Let* g *be a* L^2*-Carathéodory function such that, for some* $\gamma \in [0, a \sup \mathscr{A}_J[$,

$$0 \leqslant \liminf_{|u| \to \infty} \frac{g(x, u)}{u} \leqslant \limsup_{|u| \to \infty} \frac{g(x, u)}{u} \leqslant \gamma$$

Let us assume that there exists $h \in L^2(\Omega; \mathbb{R})$ *such that*

$$\operatorname{sgn} u g(x, u) \geqslant h(x)$$

for every $u \in \mathbb{R}$ *and almost every* $x \in \Omega$. *For every* $\psi \in \ker L^*$, *assume that*

$$\int_{\psi<0} (\limsup_{u \to -\infty} g(x, u))\, \psi(x)\, dx$$

$$+ \int_{\psi>0} (\liminf_{u \to +\infty} g(x, u))\, \psi(x)\, dx > 0.$$

Then, Eq. (5.4) *has a solution.*

Proof. We will apply Theorem 3.2. It can be shown (see, e.g., [FF_1]) that, under the above assumption, we can write

$$g(x, u) = q(x, u)\, u + r(x, u), \tag{5.6}$$

where $0 \leqslant q(x, u) \leqslant \gamma' < a \sup \mathscr{A}_J$, the function r being bounded by a L^2-function. It is not difficult to check that assumption (i) of Theorem 3.2 is satisfied for $\rho = \gamma'/a$ and $\alpha = 1/2$. In order to verify condition (ii) of Theorem 3.2, we assume, by contradiction, that there exists a sequence $(u_k) \subset \operatorname{dom} L$ such that $\|u_k\| \to \infty$, $\|Lu_k\| = O(\|u_k\|^{1/2})$, and

$$\liminf_{k \to \infty} \frac{1}{\|u_k\|} \int_\Omega g(x, u_k(x))\, \chi_{Pu_k}(x)\, Pu_k(x)\, dx \leqslant 0. \tag{5.7}$$

In order to apply Fatou's lemma, we must prove the existence of a function $\eta \in L^1(\Omega; \mathbb{R})$ such that

$$\frac{1}{\|u_k\|} g(u, u_k(x))\, \chi_{Pu_k}(x)\, Pu_k(x) \geqslant \eta(x) \tag{5.8}$$

for a.e. $x \in \Omega$. Referring to the function q in (5.6), we have, using the inequality (5.1),

$$\frac{1}{\|u_k\|} q(x, u_k(x))\, u_k(x)\, \chi_{Pu_k}(x)\, Pu_k(x)$$

$$\geqslant -\frac{\gamma' b((I-P)\, u_k(x))^2}{4\|u_k\|}.$$

But, by hypothesis, $\|Lu_k\| = O(\|u_k\|^{1/2})$ for $k \to \infty$ and, as L has a compact resolvent, $((I-P)\, u_k/\|u_k\|^{1/2})$ will converge in $L^2(\Omega; \mathbb{R})$; hence, that sequence is bounded by a L^2-function. We conclude from this that (5.8) holds when $g(x, u)$ is replaced by $q(x, u)\, u$. On the other hand, concerning the term r in the decomposition (5.6), we recall that it is bounded by a L^2-function. Hence we can assume, by passing if necessary to a subsequence, that

$$\left(\frac{1}{\|u_k\|} r(x, u_k(x))\, \chi_{Pu_k}(x)\, Pu_k(x)\right)$$

is bounded below by a L_1-function. Therefore, it is legitimate to apply Fatou's lemma in (5.7). Since $(Pu_k/\|u_k\|)$ converges to a function $\varphi \in \ker L$ and since J maps $\ker L$ onto $\ker L^*$, we have that $(\chi_{Pu_k}(x)\, Pu_k(x)/\|u_k\|)$ converges to a function $\psi \in \ker L^*$. According to a remark made above, $\varphi(x)$ and $\psi(x)$ have the same sign; consequently, $u_k(x)$ has the same sign as $\psi(x)$ for k sufficiently large. It is then seen that the application of Fatou's lemma to (5.7) leads to a contradiction with respect to the hypotheses of the theorem. ∎

6. Applications to Differential Equations

We first consider a scalar boundary value problem on a smooth bounded domain in $\mathbb{R}^n$:

$$\begin{aligned} \mathscr{D}u &= g(x, u) && \text{in } \Omega \\ \mathscr{B}u &= 0 && \text{in } \partial\Omega. \end{aligned} \tag{P}$$

We take $H = L^2(\Omega; \mathbb{R})$ and assume that $\mathscr{D}$, $\mathscr{B}$ are linear operators, to which we associate the linear operator $L\colon \operatorname{dom} L \subset H \to H$, where $\operatorname{dom} L = \{u \in \operatorname{dom} \mathscr{D} \mid \mathscr{B}u = 0\}$. Again, we assume that L is densely defined, closed, and has a compact resolvent. Moreover, we will assume that the kernel is one-dimensional, so that for some functions φ, $\psi \in L^2(\Omega; \mathbb{R})$, we have $\ker L = \mathbb{R}\varphi$ and $\ker L^* = \mathbb{R}\psi$. We then suppose that positive constants a, b exist, such that

$$a \leqslant \frac{\psi(x)}{\varphi(x)} \leqslant b \tag{6.1}$$

for a.e. $x \in \Omega$. In that case, there is a natural choice for the operator $J\colon H \to H$, namely

$$(Ju)(x) = \frac{\psi(x)}{\varphi(x)}\, u(x). \tag{6.2}$$

Clearly, J is continuous and maps $\ker L$ isomorphically onto $\ker L^*$. Consequently, Theorem 5.4 can be applied. Notice that the last assumption of the theorem can now be made more explicit, taking advantage of the fact that $\dim \ker L^* = 1$. Indeed, setting

$$g_+(x) = \liminf_{u \to +\infty} g(x, u), \qquad g^-(x) = \limsup_{u \to -\infty} g(x, u),$$

the condition becomes

$$\int_{\psi < 0} g_+ \psi + \int_{\psi > 0} g^- \psi < 0 < \int_{\psi < 0} g^- \psi + \int_{\psi > 0} g_+ \psi.$$

Remark. The hypotheses on $\ker L$, $\ker L^*$, and condition (6.1) are verified, e.g., for Dirichlet boundary conditions, if $\mathscr{D}$ is an elliptic partial differential operator having 0 as its first nonnegative eigenvalue. In that case, both φ and ψ can be taken to be positive in Ω and, on $\partial\Omega$, one has $(\partial\varphi/\partial\nu) < 0$, $(\partial\psi, \partial\nu) < 0$, where ν is the outward unit normal to $\partial\Omega$. This situation has been considered by Ahmad [Ah]. Other examples can be found. For instance, the conditions are fulfilled, if $n = 1$, $\Omega =]a, b[$, $\mathscr{D}u = u^{(8)} - \lambda_1 u$ with the boundary conditions

$$u(a) = u''(a) = u^{(5)}(a) = u^{(6)}(a) = 0,$$
$$u(b) = u''(b) = u^{(5)}(b) = u^{(6)}(b) = 0,$$

λ_1 being the first positive eigenvalue of the operator $u \mapsto u^{(8)}$, subject to the above boundary conditions. It is interesting to notice that, for this problem, (6.1) is verified, although φ and ψ change sign in $]a, b[$. But the problem is not self-adjoint, nor is $\ker L$ equal to $\ker L^*$.

As a second application of our general theory, we consider a periodic problem for a system of differential equations

$$\begin{aligned} u' + Au &= g(t, u) \\ u(0) &= u(T). \end{aligned} \tag{P'}$$

Here, A is a $m \times m$ matrix. We will show that an operator J, with the desired properties, can be found which is a matrix operator. More precisely, there is an invertible $m \times m$ matrix $\mathscr{J}$ such that the operator J, defined by $(Ju)(t) = \mathscr{J}u(t)$, maps $\ker L$ isomorphically onto $\ker L^*$. In fact, the elements of $\ker L$ are of the form

$$a_0 + \sum_{j=1}^{q} \left(a_j \cos\left(\frac{2\pi k_j}{T} t\right) + b_j \sin\left(\frac{2\pi k_j}{T} t\right)\right),$$

and those of $\ker L^*$ of the form

$$\tilde{a}_0 + \sum_{j=1}^{q} \left(\tilde{a}_j \cos\left(\frac{2\pi k_j}{T} t\right) + \tilde{b}_j \sin\left(\frac{2\pi k_j}{T} t\right)\right),$$

where a_0 and $\tilde{a}_0$ may be (simultaneously) zero vectors, while $a_0, a_1, ..., a_q$, $b_1, ..., b_q$ (resp. $\tilde{a}_0, \tilde{a}_1, ..., \tilde{a}_q, \tilde{b}_1, ..., \tilde{b}_q$) are linearly independent (a_0 and $\tilde{a}_0$ are of course to be dropped from the lists when they are zero vectors). It is clear that an invertible matrix $\mathscr{J}$ can be found which transforms a_j into $\tilde{a}_j$ and b_j into $\tilde{b}_j$. The matrix $\mathscr{J}$ has the required properties and we can then apply Theorem 5.3 with $a = 1$ to get an existence result for problem (P'). Actually, we have written a computer program which computes a matrix $\mathscr{J}$ from the matrix A of the system.

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