The Problem of Plateau (pp. 111–128)
ed. Th. M. Rassias

CRITICAL POINT THEORY AND MULTIPLE PERIODIC SOLUTIONS OF CONSERVATIVE SYSTEMS WITH PERIODIC NONLINEARITY

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ABSTRACT

We use a cohomological minimax theorem of Chang to prove multiplicity results for the periodic solutions of some Hamiltonian systems of the second and or the first order with nonlinearity verifying some periodicity condition. Applications are given to systems of coupled pendulums, discretization of Josephson equations and extensions of the Conley-Zehnder results about Arnold's conjecture.

1. Introduction

The first aim of this paper is to extend the results obtained by Mawhin [19] concerning the multiplicity of periodic solutions for a second order system of the form

$$(M(t)u')' + Au + D_uF(t,u) = h(t), \tag{1.1}$$

where M(t) and A are symmetric matrices, M(t) being positive definite, F and D_uF are bounded and satisfy a periodicity condition along the directions of the null-space of A, and h belongs to a suitable subspace of L^1.

The special case M(t) ≡ Id and A = 0 has been studied in [21]. The existence of two distinct solutions was proved, generalizing a result of [20] for the pendulum equation (see also [17], [22]).

In [19], Mawhin proved,whenever A is semi-negative definite, the existence of (m + 1) distinct periodic solutions of (1), where m is the dimension of the null-space of A. His result unifies and completes previous existence theorems for the satellite equation (cf. [18]), the linearly coupled pendulum (cf. [15]) and the Josephson multipoint system (cf. [14]). See also [12].

Here we will prove the existence of at least (m + 1) distinct periodic solutions of (1), without requiring A to be semi-negative definite. As already shown in [19], if all the solutions are non-degenerate, then there are at least 2^m of them. The proof consists in applying an abstract minimax theorem of Chang [6], which is based on algebraic topology methods and Morse theory. Minimax techniques of this type have been applied by Morse-Tompkins and Shiffman, to the problem of the multiplicity of solutions of the Plateau problem , a few years after the fundamental contributions of Jesse Douglas and Tibor Rado to the existence question for this problem. Nowadays, the Plateau problem continues to be a source of inspiration for the development of modern tools in critical point theory (see e.g. [29]).

Our second objective is to give a multiplicity result for the periodic solutions of a Hamiltonian system of the form

$$J\dot{u} + Au + D_uH(t,u) = h(t), \qquad (1.2)$$

where J is the standard symplectic matrix, A is a symmetric matrix, H, D_uH and $D_{uu}H$ are bounded and satisfy a periodicity condition along the directions of the null-space of A, and h belongs to a suitable subspace of L^2.

The situation differs from the above one for the fact that the functional associated to (1.2) is strongly indefinite. A finite dimensional reduction will be used in order to overpass this difficulty.

The special case A = 0 has been studied by Conley and Zehnder [9, 10] and Chang [6]. Here we extend their results to the case $A \neq 0$, and prove a theorem which is the perfect analogous to the one we have for system (1.1). When the paper (which is summarized in [28]) was written, we have received a preprint of Chang [27] which contains very close results.

The paper is organized as follows. In section 2 we recall some concepts of algebraic topology, two deformation lemmas and some results of Morse Theory. In section 3 we present two abstract existence theorems by Chang [6, 7]. The proofs are also carried out for the reader's convenience. In sections 4 and 5 we apply the abstract results of section 3 to prove multiplicity results for periodic solutions of (1.1) and (1.2), respectively.

2. Some preliminaries

Given a topological space X and a subspace $A \subset X$, we denote by $H_n(X,A)$ $[H^n(X,A)]$ the n^{th} singular homology [cohomology] vector space of X relative to A, with respect to a given field (e.g. $\mathbb{R}$).

We recall that the elements of $H_n(X,A)$ are equivalence classes of singular chains having zero boundary. These elements are invariant under every continuous deformation $\tau : X \to X$ such that $\tau|_A = id_A$, in the sense that τ induces an isomorphism τ_* of $H_n(X,A)$ into itself, and we identify each element of $H_n(X,A)$ with its image under τ_*. The analogous is true for cohomologies, as well.

We state the following two properties of homologies, which hold for cohomologies as well.

(a) If X' is a strong deformation retract of X and $A' \subset X'$ is a strong deformation retract of A, then $H_n(X,A) \approx H_n(X',A')$.

(b) (Künneth formula) For any topological space Y,

$$H_n(X \times Y, A \times Y) \approx \oplus_{p+q=n} [H_p(X,A) \otimes H_q(Y)].$$

Let us now briefly recall the concepts of cup and cap products. Suppose A and B are subspaces of X such that, for example, one of the following three situations is true :

$$A = \varnothing, \quad A = B, \quad B = \varnothing.$$

Then there exist an operation, called the "cup product"

$$\begin{aligned} H^n(X,B) \times H^m(X,A) &\to H^{n+m}(X,A \cup B) \\ (\omega,\rho) &\mapsto \omega \cup \rho \end{aligned} \tag{2.1}$$

and an operation, called the "cap product"

$$\begin{aligned} H_{n+m}(X, A \cup B) \times H^n(X,B) &\to H_m(X,A) \\ (z,\omega) &\mapsto z \cap \omega, \end{aligned} \tag{2.2}$$

which are bilinear and invariant under continuous deformations leaving $A \cup B$ fixed (in the sense we saw above).

The cup and cap products are naturally induced by two operators, defined on the corresponding spaces of singular chains and cochains, which are denoted by the same symbols $\cup$ and $\cap$. If $z \in H_{n+m}(X, A \cup B)$, $\omega \in H^n(X,B)$ and $\rho \in H^m(X,A)$, then for all $\eta \in z$, $c \in \omega$ and $d \in \rho$ one has

$$\langle \eta, c \cup d \rangle = \langle \eta \cap c, d \rangle, \tag{2.3}$$

where $\langle .,. \rangle$ denotes the pairing between singular chains and cochains. Moreover, denoting by $|\eta|$ the support of the chain η, one has

$$|\eta \cap c| \subset |\eta|. \tag{2.4}$$

We now consider a Riemannian manifold $\mathcal{M}$ of class C^2 and a C^1 functional $f : \mathcal{M} \to \mathbb{R}$. We will use the following notations.

$$f^a = \{x \in \mathcal{M} : f(x) \leq a\}$$

$$K_c = \{x \in \mathcal{M} : f(x) = c, df(x) = 0\}.$$

That is, K_c is the set of critical points at the level c.

The Palais-Smale condition, in short (PS), plays a fundamental role in the following deformation lemmas. Recall that (PS) holds iff every sequence (x_n) in $\mathcal{M}$ such that $f(x_n)$ is bounded and $df(x_n) \to 0$ possesses a convergent subsequence.

Assume that at every point x of the boundary of $\mathcal{M}$, df(x) points inwards in $\mathcal{M}$. Then the following lemmas hold.

First Deformation Lemma. *Assume* (PS) *holds for* f. *Fix* $c \in \mathbb{R}$ *and let* N *be a closed neighborhood of* K_c. *Then there is a continuous map* $\eta : [0,1] \times \mathcal{M} \to \mathcal{M}$, *as well as numbers* $\bar{\varepsilon} > \varepsilon > 0$ *such that*

(1) $\eta|_{\mathcal{M} \setminus f^{-1}([c-\bar{\varepsilon}, c+\bar{\varepsilon}])} = id,$

(2) $\eta(0,\cdot) = id,$

(3) $\eta(1, f^{c+\varepsilon} \setminus N) \subset f^{c-\varepsilon},$

(4) $\forall\, t \in [0,1]$, $\eta(t,\cdot)$ *is a homeomorphism.*

Second Deformation Lemma. *Assume* (PS) *holds for* f *and that* df *is locally Lipschitzian. If there are no critical values in the open interval* (a, b), *then* f^a *is a strong deformation retract of* $(f^b \setminus K_b)$.

For the proofs, cf. [22], [8], [24], [6]. Since $\mathcal{M}$ is a Riemannian manifold, we can consider the flow determined by the gradient of f. The deformations are then constructed along this flow.

Let $a < b$, and suppose that $x_1, \ldots, x_i$ are the only critical points of f in $f^{-1}([a,b])$. Let $C_n(f, x_i)$ denote the n^{th} critical group of x_i, and suppose that all these critical groups (which are vector spaces) are finite dimensional, and that they are trivial for n sufficiently large. Then we can define the Morse polynomial

$$M(t,f^b,f^a) = \sum_{n=0}^{\infty} \left(\sum_{i=1}^{j} \dim C_n(f,x_i)\right) t^n .$$

Moreover, the Poincaré polynomial

$$P(t,f^b,f^a) = \sum_{n=0}^{\infty} \dim H_n(f^b,f^a)\, t^n$$

is also well defined and we have that

$$M(t,f^b,f^a) = P(t,f^b,f^a) + (1+t)\, Q(t), \tag{2.5}$$

where Q is a polynomial with nonnegative integer coefficients (cf. [22]).

As a consequence of (2.5), if for every $i \in \{1, \dots, j\}$ we have that $\sum_{n=0}^{\infty} \dim C_n(f,x_i) \leq 1$, then

$$j \geq \sum_{n=0}^{\infty} \dim H_n(f^b,f^a). \tag{2.6}$$

This is the case if all the critical points $x_1, \dots, x_j$ are nondegenerate. Indeed, in that case one has

$$C_n(f,x_i) = \delta_{n,m_i} \mathbb{R}$$

where m_i is the Morse index of the point x_i.

3. An abstract multiplicity result

In this section we expose some results of Chang [6].

Definition. *Let* X *be a topological space and* $A \subset X$. *Consider two non-zero singular homology classes*

$$z_1 \in H_m(X,A), \quad z_2 \in H_{n+m}(X,A).$$

We set $z_1 \prec z_2$ *whenever* $n > 0$ *and there exists* $\omega \in H^n(X)$ *such that*

$$z_1 = z_2 \cap \omega$$

(the cap product is as in (2.2), *with* $B = \emptyset$*).*

Theorem 1. *Assume* (PS) *holds for* f. *Let* a < b *be two real numbers such that* f *has only a finite number of isolated critical points in* $f^{-1}[a,b]$. *If there are* k *non-zero singular homology classes* $z_1 \in H_{n_1}(f^b, f^a)$, ..., $z_k \in H_{n_k}(f^b, f^a)$ *with* $z_1 \prec z_2 \prec \ldots \prec z_k$, *then* f *has at least* k *distinct critical values.*

Proof. Define the following quantities :

$$c_i = \inf_{\eta \in z_i} \sup_{x \in |\eta|} f(x) \qquad (i = 1, 2, \ldots, k).$$

The Minimax Principle tells us that whenever c_i is finite and the family of sets z_i is invariant under homeomorphisms, c_i is a critical value of f. Since $z_i \in H_{n_i}(f^b, f^a)$, for all $\eta \in z_i$ we have that $|\eta| \subset f^b$. Hence $c_i \leq b$. Moreover, since z_i is supposed to be non-zero in $H_{n_i}(f^b, f^a)$, this means that for any $\eta \in z_i$ we have that $|\eta|$ has non empty intersection with $f^b \setminus f^a$. Hence $c_i \geq a$. So c_i is finite; z_i is invariant under homeomorphisms, as we said in section 2.

So the c_i's are critical values of f. We want to prove now that we have

$$c_1 < c_2 < \ldots < c_k.$$

Let us concentrate on c_1 and c_2, the reasoning being the same for the others. We know, since $z_1 \prec z_2$ that there exists $\omega \in H^{n_2 - n_1}(f^b)$, $n_2 - n_1 > 0$ such that $z_1 = z_2 \cap \omega$. This means that for any $\eta_2 \in z_2$ and any $c \in \omega$ we can define $\eta_1 = \eta_2 \cap c \in z_1$, and by (2.4) we have that $|\eta_1| \subset |\eta_2|$. Hence

$$\forall \eta_2 \in z_2 \, \exists \, \eta_1 \in z_1 : \sup_{x \in |\eta_1|} f(x) \leq \sup_{x \in |\eta_2|} f(x).$$

This immediately implies $c_1 \leq c_2$.

Suppose by contradiction that $c_1 = c_2$, and denote by c this common value. Then for every $\varepsilon > 0$ there must exist a $\eta_2 \in z_2$ such that $|\eta_2| \subset f^{c+\varepsilon}$. Since K_c is a set of isolated critical points, we choose two contractible neighborhoods of K_c in $f^b \setminus f^a$: $N \subset N'$. We can then write $\eta_2 = \eta'_2 + \eta''_2$, where $|\eta'_2| \subset N'$ and $|\eta''_2| \subset f^{c+\varepsilon} \setminus N$.
We can consider η'_2 as a cycle of f^b relative to $f^b \setminus N$. Hence $[\eta'_2] \in H_{n_2}(f^b, f^b \setminus N)$ and $[\eta'_2] \cap \omega \in H_{n_1}(f^b, f^b \setminus N)$. The cap product does not change if we shrink N', and hence η'_2, to a point. Since $\omega \in H^{n_2 - n_1}(f^b)$ and $n_2 - n_1 > 0$, there exists $c \in \omega$ which, applied to any chain having support in N', gives 0. In particular, by the definition of the cap product, $\eta'_2 \cap c = 0$. Set $\eta_1 = \eta_2 \cap c \in z_1$; then

$$\eta_1 = \eta_2 \cap c = \eta'_2 \cap c + \eta''_2 \cap c = \eta''_2 \cap c.$$

Hence $|\eta_1| \subset |\eta''_2 \cap c| \subset |\eta''_2| \subset f^{c+\varepsilon} \setminus N$.

Consider now the homeomorphism $\tau : f^{c+\epsilon} \setminus N \to f^{c-\epsilon}$ given by the First Deformation Lemma. We have that $\tau(|\eta_1|) \subset f^{c-\epsilon}$. But $\tau \circ \eta_1 \in z_1$ because of the invariance under homeomorphisms, and this contradicts the definition of c_1 (= c).

Given a compact manifold $\mathcal{V}$, let us define cuplength ($\mathcal{V}$) as the greatest natural number ℓ such that

$$(\forall\, i \in \{1, \ldots, \ell\})\ (\exists\, k_i > 0)\ (\exists\, \omega_i \in H^{k_i}(\mathcal{V})) : \omega_1 \cup \ldots \cup \omega_\ell \neq 0.$$

It can be seen that for a compact manifold such a number indeed exists.

Theorem 2. *Let* L *be a bounded self-adjoint operator with a bounded inverse, defined on a Hilbert space* H. *Suppose that the negative space determined by* L *is finite dimensional. Let* $\mathcal{V}$ *be a* C^2*-compact manifold without boundary. Let* $g \in C^2(H \times \mathcal{V}, \mathbb{R})$ *be a function having bounded and compact differential* dg. *Then the function*

$$f(x, v) = \frac{1}{2}(Lx, x) + g(x, v)$$

has at least [cuplength ($\mathcal{V}$) + 1] *critical points.*

Moreover, if all the critical points of f *are non degenerate, there are at least* $\sum_{n=0}^{\infty} \dim H_n(\mathcal{V})$ *of them.*

<u>Proof.</u> 1) In order to apply the previous theorem, let us verify that (PS) holds for f on $H \times \mathcal{V}$. Take a sequence (x_n, v_n) in $H \times \mathcal{V}$ such that $(f(x_n, v_n))$ is bounded and

$$df(x_n,v_n) = Lx_n + dg(x_n,v_n) \to 0. \tag{3.1}$$

Since $(dg(x_n,v_n))$ is bounded by hypothesis, it follows that (Lx_n) is also bounded. We deduce that, L having a bounded inverse, (x_n) is bounded, and since $\mathcal{V}$ is compact, the sequence (x_n,v_n) is bounded. Since dg is supposed to be compact and $\mathcal{V}$ is compact, there exists a subsequence (x_{n_k},v_{n_k}) such that $(dg(x_{n_k},v_{n_k}))$ and (v_{n_k}) are convergent. From (3.1) and the boundedness of L^{-1}, it then follows that (x_{n_k},v_{n_k}) itself is convergent.

2) Let $H = H^+ \oplus H^-$, where H^+ and H^- are the invariant subspaces corresponding to the positive and negative spectrum of L, respectively. Accordingly, every $x \in H$ can be written as $x = x^+ + x^-$, with $x^+ \in H^+$ and $x^- \in H^-$. Let $\gamma = \dim H^-$. Set $\varepsilon_\pm = \inf\{\|Lx^\pm\| : \|x^\pm\| = 1\}$. These are positive numbers. If $\bar{m}$ is such that $\|dg(x,v)\| \leq \bar{m}$ $\forall\,(x,v) \in H \times \mathcal{V}$, set $R_+ = (\bar{m} + 1) / \varepsilon_+$. From now on it will be convenient to work on the manifold $\mathcal{M} = (H^+ \cap B_{R_+}) \times H^- \times \mathcal{V}$.

In order to be sure that the Deformation Lemmas can be used, we have to check that $-df$ points inward to $\mathcal{M}$ on each point of the boundary $\partial\mathcal{M}$, i.e. for every (x, v) such that $\|x^+\| = R_+$. Indeed in such a case we have

$$\begin{aligned}(-df(x,v), x^+) &= -(Lx^+,x^+) - (dg(x,v), x^+)\\ &\le -\varepsilon_+ \|x^+\|^2 + \bar{m}\|x^+\|\\ &= -R_+(\varepsilon_+R_+ - \bar{m}) = -R_+ < 0.\end{aligned}$$

Since (on $\mathcal{M}$)

$$f(x,v) \le \frac{1}{2}\|L\| R_+^2 - \frac{1}{2}\varepsilon_-\|x^-\|^2 + \bar{m}(R_+ + \|x^-\| + \|v\|)$$

and since $\mathcal{V}$ is compact, we have that

$$f(x, v) \to -\infty \text{ as } \|x^-\| \to \infty \text{ uniformly in } x^+ \text{ and } v. \tag{3.2}$$

It is not restrictive to suppose there exist only a finite number of critical points, which are isolated and contained in $f^b \setminus f^a$, for certain fixed $a < b$.

From (3.2) it follows that there exists an $R_1 > 0$ such that

$$(H^+ \cap B_{R_+}) \times (H^- \setminus B_{R_1}) \times \mathcal{V} \subset f^a.$$

Since

$$f(x,v) \ge \frac{1}{2}\varepsilon_+ R_+^2 - \frac{1}{2}\|L\|\|x^-\|^2 - \bar{m}(R_+ + \|x^-\| + \|v\|)$$

this implies f is bounded below on $(H^+ \cap B_{R_+}) \times (H^- \cap B_{R_1}) \times \mathcal{V}$, and hence there exists an $a' < a$ such that

$$f^{a'} \subset (H^+ \cap B_{R_+}) \times (H^- \setminus B_{R_1}) \times \mathcal{V}$$

and, again, there exists an $R_2 > R_1$ such that

$$(H^+ \cap B_{R_+}) \times (H^- \setminus B_{R_2}) \times \mathcal{V} \subset f^{a'}.$$

The Second Deformation Lemma gives us a strong deformation retraction $\tau_1 : f^a \to f^{a'}$. Moreover a strong deformation retraction

$$\tau_2 : (H^+ \cap B_{R_+}) \times (H^- \setminus B_{R_1}) \times \mathcal{V} \to (H^+ \cap B_{R_-}) \times (H^- \setminus B_{R_2}) \times \mathcal{V}$$

can be constructed by hand (see [6]). So $\tau = \tau_2 \cdot \tau_1$ is a strong deformation retraction from f^a to $(H^+ \cap B_{R_+}) \times (H^- \setminus B_{R_2}) \times \mathcal{V}$, and we have

$$
\begin{aligned}
H_n(f^b, f^a) &\approx H_n(\mathcal{M}, f^a) && \text{(deformation)}\\
&\approx H_n(\mathcal{M}, (H^+ \cap B_{R_+}) \times (H^- \setminus B_{R_2}) \times \mathcal{V}) && \text{(deformation)}\\
&\approx H_n(H^- \times \mathcal{V}, (H^- \setminus B_{R_2}) \times \mathcal{V}) && \text{(Künneth)}\\
&\approx \oplus_{p+q=n} [H_p(H^-, H^- \setminus B_{R_2}) \otimes H_q(\mathcal{V})] && \text{(Künneth)}\\
&\approx \oplus_{p+q=n} [H_p(B_{R_2}, \partial B_{R_2}) \otimes H_q(\mathcal{V})] && \text{(deformation)}\\
&\approx H_{n-\gamma}(\mathcal{V}).
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
H^n(f^b) &\approx H^n(\mathcal{M})\\
&\approx H^n((H^+ \cap B_{R_+}) \times H^- \times \mathcal{V})\\
&\approx \oplus_{p+q=n} [H^p((H^+ \cap B_{R_+}) \times H^-) \otimes H^q(\mathcal{V})]\\
&\approx H^n(\mathcal{V}).
\end{aligned}
$$

Let ℓ = cuplength $(\mathcal{V})$. Then, according to the above isomorphisms, for every $i \in \{1, \ldots, \ell\}$ there exists $\omega_i \in H^{k_i}(f^b)$, $k_i > 0$ such that $\omega_1 \cup \ldots \cup \omega_\ell \neq 0$. Hence there exists $z_1 \in H_{k_1 + \ldots + k_\ell + \gamma}(f^b, f^a)$ such that

$$(z_1, \omega_1 \cup \ldots \cup \omega_\ell) \neq 0.$$

Then we can define recursively $z_{j+1} \in H_{k_{j+1} + \ldots + k_\ell + \gamma}(f^b, f^a)$ by

$$z_{j+1} = z_j \cap \omega_j,$$

$j = 1, \ldots, \ell$. We thus obtain $(\ell+1)$ non-zero homology classes such that $z_{\ell+1} \prec z_\ell \prec \ldots \prec z_1$. Theorem 1 then proves the first part of the theorem.

As for the second part, it is immediate from (2.6) and the above isomorphisms.

4. Periodic solutions of second order systems

In this section we will apply the abstract multiplicity results of section 3 to the following second order system.

$$
\begin{aligned}
&(M(t)u')' + Au + D_uF(t,u) = h(t)\\
&u(0) - u(T) = u'(0) - u'(T) = 0
\end{aligned}
\qquad (4.1)
$$

We assume $T > 0$, $F = [0,T] \times \mathbb{R}^n \to \mathbb{R}$ is continuous, bounded and such that

(i) $F(t+T,u) = F(t,u)$ for all $(t,u) \in [0,T] \times \mathbb{R}^n$.

(ii) D_uF exists, is continuous and bounded, and $D_{uu}F$ exists and is continuous.

Moreover, $S(\mathbb{R}^n,\mathbb{R}^n)$ being the space of symmetric real $(n \times n)$-matrices, we have $A \in S(\mathbb{R}^n,\mathbb{R}^n)$, $M : [0,T] \to S(\mathbb{R}^n,\mathbb{R}^n)$ is continuous and such that, for some $\mu > 0$ and all $(t,v) \in [0,T] \times \mathbb{R}^n$,

$$(M(t)v \mid v) \geq \mu \, |v|^2, \tag{4.2}$$

and $h \in L^1(0,T; \mathbb{R}^n)$.

Using Schauder's fixed point theorem, one can prove the existence of at least one solution to (4.1) whenever the "linearized" problem

$$\begin{gathered} (M(t)u')' + Au = 0 \\ u(0) - u(T) = u'(0) - u'(T) = 0 \end{gathered} \tag{4.3}$$

has only the zero solution.

We will consider the situation described by the following assumptions, where N(A) denotes the null-space of A.

(A1) $N(A) = \text{span}\,\{a_1, \ldots, a_m\}$, $1 \leq m \leq n$, *and problem* (4.3) *has as solutions only the elements of* N(A).

(A2) *For every* $v \in N(A)$,

$$\int_0^T (h(t) \mid v)\, dt = 0.$$

(A3) *There are positive numbers* $T_1, \ldots, T_m$ *such that*

$$F(t, u + T_j a_j) = F(t,u)$$

for every $(t,u) \in [0,T] \times \mathbb{R}^n$ *and* $j \in \{1, \ldots, m\}$.

Theorem 3. *Under the above assumptions, problem* (4.1) *has at least* $(m+1)$ *geometrically distinct solutions. If moreover all the solutions of* (4.1) *are nondegenerate, then there are at least* 2^m *of them.*

Remark. Theorem 3 generalizes previous results by Mawhin [19] where A was supposed to be semi-negative definite.

Proof. Let us consider the Hilbert space

$$H_T^1 = \{u \in H^1(0,T; \mathbb{R}^n) : u(0) = u(T)\}$$

equipped with the inner product

$$\langle u \mid v \rangle = \int_0^T [(M(t)u'(t) \mid v'(t)) + (u(t) \mid v(t))]\, dt.$$

The corresponding norm $\|u\| = \langle u \mid u \rangle^{1/2}$ is by (4.2) equivalent to the classical norm of $H^1(0,T; \mathbb{R}^n)$.

Let us define the operator $L : H_T^1 \to H_T^1$ such that

$$\langle Lu \mid v \rangle = \int_0^T [(M(t)u'(t) \mid v'(t)) - (Au(t) \mid v(t))]\, dt.$$

It can easily be seen that L is a self-adjoint operator on H_T^1. Because of the compact imbedding of H_T^1 into $C([0,T], \mathbb{R}^n)$, $(I - L)^{-1}$ is compact. This implies that, writing $H_T^1 = H^- \oplus H^o \oplus H^+$, where H^-, H^o and H^+ are the invariant subspaces corresponding to the negative, zero and positive spectrum of L, respectively, the space H^- is finite dimensional. Moreover, by (A1), $H^o = N(A)$.

Let us consider the space $H = H^- \oplus H^+$. Then L can be considered as a bounded self-adjoint operator on H with a bounded inverse.

Let $T^m = \mathbb{R}^m/\mathbb{Z}^m$ be the m-fold torus, and define on $H \times T^m$ the following functional

$$g(u,(v_1, \ldots, v_m)) = \int_0^T (-F(t, u(t) + \sum_{i=1}^m v_i a_i) + (h(t) \mid u(t))]\, dt.$$

By (ii), g is of class C^2. Because of (A2), (A3) and classical arguments, the critical points of the C^2-functional f defined by

$$f(u,v) = \frac{1}{2}(Lu,u) + g(u,v)$$

correspond to geometrically distinct solutions of (4.1). It is easy to see that dg is bounded and compact, because of the compact imbedding of H_T^1 into the space of continuous functions. So all the assumptions of Theorem 2 are satisfied, and the result follows from the following well known facts :

$$\text{cuplength}\,(T^m) = m + 1 \qquad (4.4)$$

$$\dim H_n\,(T^m) = \binom{m}{n}. \qquad (4.5)$$

5. Periodic solutions of Hamiltonian systems

In this section we will give a multiplicity result for the following Hamiltonian system

$$\begin{aligned} J\dot{u} + Au + D_uH(t,u) &= h(t) \\ u(0) &= u(T) \end{aligned} \tag{5.1}$$

We assume $T > 0$, $H = [0,T] \times \mathbb{R}^{2n} \to \mathbb{R}$ is continuous, bounded and such that

(i) $H(t + T, u) = H(t,u)$ for all $(t,u) \in [0,T] \times \mathbb{R}^{2n}$;
(ii) D_uH and $D_{uu}H$ exist, they are continuous and bounded.

Moreover, A is a symmetric real $(2n \times 2n)$-matrix with null-space $N(A)$, $h \in L^2(0,T; \mathbb{R}^{2n})$ and $J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$ is the standard symplectic matrix.

It is not difficult to see, by Schauder's fixed point theorem, that if the "linearized" problem

$$\begin{aligned} J\dot{u} + Au &= 0 \\ u(0) &= u(T) \end{aligned} \tag{5.2}$$

has only the zero solution, then (5.1) has at least one solution.

We will consider the situation described by the following assumptions.

(A1) $N(A) = \text{span}\,\{a_1, \ldots, a_m\}$, $1 \le m \le 2n$, *and problem* (5.2) *has as solutions only the elements of* $N(A)$.

(A2) *For every* $v \in N(A)$,

$$\int_0^T (h(t) \mid v)\, dt = 0.$$

(A3) *There are positive numbers* $T_1, \ldots, T_m$ *such that*

$$H(t,u + T_j a_j) = H(t,u)$$

for every $(t,u) \in [0,T] \times \mathbb{R}^{2n}$ *and* $j \in \{1, \ldots, m\}$.

Theorem 4. *Under the above assumptions, problem* (5.1) *has at least* $(m + 1)$ *geometrically distinct solutions.*

If moreover all the solutions of (5.1) *are nondegenerate, then there are at least* 2^m *of them.*

Remark. Theorem 4 generalizes previous results obtained by Conley and Zehnder [9, 10] and Chang [6]. They all consider the case $A = 0$ and hence $N(A) = \mathbb{R}^{2n}$. In [6] and [9] a conjecture of Arnold was proved (cf. [3], [4]).

Proof. We define a self-adjoint operator L on the Hilbert space $X = L^2(0,T; \mathbb{R}^{2n})$:

$$D(L) = \{u \in H^1(0,T; \mathbb{R}^{2n}) : u(0) = u(T)\}, \qquad Lu = J\dot{u} + Au.$$

It is well known that L is self-adjoint, has closed range and a discrete spectrum $\sigma(L) = \{\dots < \lambda_{-1} < \lambda_0 < \lambda_1 < \dots\}$ unbounded from below and from above, made of eigenvalues of finite multiplicity which do not accumulate at any finite point.

We define the operator $N : X \to X$ by

$$(Nu)(t) = -D_u H(t,u(t)) + h(t).$$

By (ii), N is Lipschitz continuous and possesses a symmetric Gateaux derivative given by

$$[N'(u)\eta](t) = -D_{uu}H(t,u(t))\eta(t) \tag{5.3}$$

for every $\eta \in X$.

Let α be the Lipshitz constant of N. By the properties of $\sigma(L)$ we can choose $\alpha' \geq \alpha$ and $\varepsilon > 0$ such that $[-(\alpha' + \varepsilon), -\alpha'] \cap \sigma(L)$ and $[\alpha', \alpha' + \varepsilon] \cap \sigma(L)$ are both empty.

By the above definitions of the operators L and N, it is clear that problem (5.1) is equivalent to the equation

$$Lu = Nu. \tag{5.4}$$

In order to be able to apply the abstract results of section 3 we need a reduction to a finite dimensional equation. To this aim, let us consider $\{E_\lambda : \lambda \in \mathbb{R}\}$, the spectral resolution of L, and define the following orthogonal projector in X :

$$P = \int_{-(\alpha'+\varepsilon)}^{\alpha'+\varepsilon} dE_\lambda.$$

For any $u \in D(L)$, we will write $u = v + w$, where $v = Pu$ and $w = (I - P)u$. Equation (5.4) is then equivalent to the following system.

$$Lw = (I - P)\, N\,(v + w) \tag{5.5}$$
$$Lv = P\, N\,(v + w). \tag{5.6}$$

Remark that, if V denotes the range of P and W the range of $(I - P)$, then V is finite dimensional, and for every $u \in D(L)$, Pu can be expressed as a finite sum of terms in the spaces $\ker(\lambda I - L) \subset D(L)$, with $\lambda \in [-(\alpha' + \varepsilon), \alpha' + \varepsilon]$. Hence, if we pose $E = C([0,T], \mathbb{R}^{2n})$, we have that $V \subset D(L) \hookrightarrow E$. From (5.3) it follows moreover that $N'|_E$ is continuous.

We will now prove the following

Claim. *For each fixed* $v \in V$ *there exists a unique* $w \in W$ *which solves* (5.5). *Further,* $w \in E$ *and the map* ξ *associating* w *to* v *is of class* C^1 *from* V *to* E. *Moreover, for every* $v \in V$ *and* $j \in \{1, \ldots, m\}$,

$$\xi(v + T_j a_j) = \xi(v). \tag{5.7}$$

First of all, notice that, taking $\tau \in\,]-(\alpha' + \varepsilon), \alpha' + \varepsilon[\, \setminus \sigma(L)$, (5.5) becomes equivalent to the following fixed point problem:

$$w = (L - \tau I)^{-1} (I - P) [N(v + w) - \tau w] := T_v(w).$$

We want to show that, if τ is appropriately chosen, the map T_v is a contraction, for all v. Since

$$(L - \tau I)^{-1}(I - P) = \int_{-\infty}^{-(\alpha'+\varepsilon)} (\lambda - \tau)^{-1}\, d E_\lambda + \int_{\alpha'+\varepsilon}^{+\infty} (\lambda - \tau)^{-1}\, d E_\lambda,$$

we have that

$$|(L - \tau I)^{-1}(I - P)| \le (\alpha' + \varepsilon - |\tau|)^{-1}.$$

Hence, since N is Lipshitzian of constant α,

$$|T_v(w) - T_v(\tilde{w})| \le (\alpha' + \varepsilon - |\tau|)^{-1}(\alpha + |\tau|)\, |w - \tilde{w}|.$$

This shows that T_v is a contraction whenever $|\tau| < \frac{1}{2}(\alpha' + \varepsilon - \alpha)$, and such a choice is always possible because of the structure of $\sigma(L)$. Then T_v has a unique fixed point w, and we set

$$\xi(v) = w.$$

Hence we have that, for all $v \in V$,

$$\xi(v) = (L - \tau I)^{-1}(I - P) [N(v + \xi(v)) - \tau\xi(v)] \tag{5.8}$$

It can be shown (see [16]) that ξ is Lipshitz continuous from V to W. Moreover, for every $v \in H$,

$$[(\tau - L)^{-1}v](t) = \exp[t J(A - \tau)]u_0 + \int_0^t \exp[(t - s)J(A - \tau)]\, Jv(s)\, ds$$

where

$$u_0 = [I_{2n} - \exp[T J(A - \tau)]]^{-1} \int_0^T \exp[(T - s)J(A - \tau)] Jv(s)\, ds.$$

This implies

$$(\tau I - L)^{-1} \in \mathcal{L}(X,E).$$

Hence from (5.8) we have that ξ is continuous from V to E. Now consider the function

$$\varphi : V \times E \to E$$
$$\varphi(v,w) = w - T_v(w).$$

Recalling the fact that $N'|_E$ is continuous, since T_v is also a contraction as a map from E to E, we have that the implicit function theorem can be applied to φ. As a consequence, we have that ξ is of class C^1 from V to E. Finally, (5.7) holds since, by (A3), equation (5.5) does not change if we substitute v with $v + T_i a_i$. The Claim is then proved.

By the Claim proved above, we have that equation (5.4) is reduced to equation (5.6), with $w = \xi(v)$, i.e. to

$$Lv = PN(v + \xi(v)) \tag{5.9}$$

where v varies in the finite dimensional space V. By the spectral decomposition of L we can write $V = V^- \oplus V^o \oplus V^+$, where V^-, V^o and V^+ are the invariant subspaces of V corresponding to the negative, zero and positive spectrum of L. By (A1), $V^o = N(A)$.Let us consider the Hilbert space $H = V^- \oplus V^+$. Then L can be considered as a bounded self-adjoint operator on H with a bounded inverse.

Let $T^m = \mathbb{R}^m/\mathbb{Z}^m$ be the m-fold torus, and define on $H \times T^m$ the following map

$$g(v, (\tau_1,\ldots,\tau_m)) = \int_0^T [-H(t,v(t) + \xi(v)(t) + \sum_{i=1}^m \tau_i a_i) + (h(t), v(t) + \xi(v)(t))]\, dt.$$

Since $\xi \in C^1(H,E)$, $H \hookrightarrow E$ and $N'|_E$ is continuous, g is of class C^2. Set

$$f(v,\tau) = \frac{1}{2}(Lv,v) + g(v,\tau).$$

By (A2), (A3) and classical arguments, the critical points of the functional f correspond to geometrically distinct solutions of (5.9).

It is easy to see that dg is bounded and continuous. Since g is defined on a finite dimensional space, this implies dg is also compact. So all the assumptions of Theorem 2 are satisfied, and the result follows from (4.4) and (4.5).

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