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Iterative and Variational Methods for the Solvability of Some Semilinear Equations in Hilbert Spaces

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1. INTRODUCTION

In 1972, using an interesting algebraic lemma, Lazer [22] showed that if there exist symmetric matrices $\mathbb{A}$ and $\mathbb{B}$ with eigenvalues $\alpha_1 \leqslant \cdots \leqslant \alpha_n$ and $\beta_1 \leqslant \cdots \leqslant \beta_n$, respectively, such that

$$\mathbb{A} \leqslant G''(u) \leqslant \mathbb{B} \tag{1}$$

for all $u \in \mathbb{R}^n$ (where $\mathbb{C} \leqslant \mathbb{D}$ for $n \times n$-matrices means $\mathbb{D} - \mathbb{C}$ semi-positive definite), and such that

$$\bigcup_{i=1}^{n} [\alpha_i, \beta_i] \cap \{k^2 : k \in \mathbb{N}\} = \varnothing, \tag{2}$$

then, for each $h \in L^2(0, 2\pi; \mathbb{R}^n)$ the periodic problem

$$u''(t) + G'(u(t)) = h(t), \qquad u(0) - u(2\pi) = u'(0) - u'(2\pi) = 0,$$

has at most one solution. Here, $G: \mathbb{R}^n \to \mathbb{R}$ is a C^2-mapping, G' is its gradient, and G'' its Hessian matrix. One year later, Ahmad [1] used an involved argument based upon the method of continuation to prove that the same conditions imply existence. A unique proof for existence and uniqueness was given in 1980 by Brown and Lin [13], based upon a global implicit function theorem, and in 1981 by Perov [35] based upon a minimax argument. Existence under weaker conditions was also obtained by Ward [41], Bates [7], Ahmad and Salazar [2], Tersian [38], Amaral and Pera [3], Habets and Nkashama [20] using degree arguments.

The corresponding question for the periodic-Dirichlet problem for a

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system of semilinear wave equations (with $\square = D_t^2 - D_x^2$ denoting the one-dimensional Dalembertian)

$$\begin{aligned} \square u(t, x) + G'(u(t, x)) &= h(t, x) \\ u(0, x) - u(2\pi, x) = u_t(0, x) - u_t(2\pi, x) &= 0, \qquad x \in [0, \pi] \\ u(t, 0) = u(t, \pi) &= 0, \qquad t \in [0, 2\pi], \end{aligned} \tag{3}$$

was first considered by Bates and Castro [8] who studied in 1979 the existence and uniqueness of weak solutions for (3) when G satisfies (1) and the following condition, corresponding to (2),

$$\bigcup_{i=1}^{n} [\alpha_i, \beta_i] \cap \{k^2 - j^2 : k \in \mathbb{N}^*, j \in \mathbb{N}\} = \varnothing \tag{4}$$

holds. By a combination of a Galerkin method and a minimax theorem, they obtained the existence and uniqueness when h is continuous and $D_t h \in L^2[(0, 2\pi) \times (0, \pi)]$. This unnatural smoothness condition was removed by Mawhin [28] in 1981 by combining the Galerkin method with a global implicit function theorem and Minty's trick of monotone operators theory. The limit process in [28] used the compactness of the right inverse of $\square$ with the periodic-Dirichlet conditions. But Amann [5] showed in 1982 that there was enough monotonicity in some equivalent formulation of the equation to obtain an existence and uniqueness theorem for abstract semilinear equations in $\mathbb{H}^n$ of the form

$$Lu = N(u), \tag{5}$$

with $\mathbb{H}$ a real Hilbert space, $L: D(L) \subset \mathbb{H}^n \to \mathbb{H}^n$ self-adjoint, $N: \mathbb{H}^n \to \mathbb{H}^n$ a Gateaux-differentiable gradient operator such that

$$A \leqslant N'(u) \leqslant B, \tag{6}$$

where $A: u \mapsto (\mathbb{A}u)(\cdot)$, $B: u \mapsto (\mathbb{B}u)(\cdot)$ correspond in $\mathbb{H}^n$ to the multiplication by the real symmetric $(n \times n)$-matrices $\mathbb{A}$ and $\mathbb{B}$, when the abstract formulation of (2) or (4), namely

$$\bigcup_{i=1}^{n} [\alpha_i, \beta_i] \cap \sigma(L) = \varnothing, \tag{7}$$

holds, with $\sigma(L)$ the spectrum of L, together with the supplementary conditions (satisfied in the above examples), that L commutes with every constant multiplication operator and has a pure point spectrum in

$[\alpha_1, \beta_n]$. Amann used a combination of Galerkin method and an existence theorem for monotone operators. Similar abstract results were proved later when (6) is replaced by the more general condition

$$\langle A(u-v), u-v\rangle \leqslant \langle N(u)-N(v), u-v\rangle \leqslant \langle B(u-v), u-v\rangle, \qquad (8)$$

by Tersian [39], using α-convex functionals and monotone operators, and by Milojević [34] using pseudo-A-proper mappings. Notice that those results and some similar abstract theorems in [5, 9] involve some a priori decomposition of the underlying Hilbert space in a direct sum which only follows from Lazer's lemma when one of the direct summands has finite-dimension. This is not the case in semilinear wave problems and explains the use of the Galerkin method. Finally, in 1984, Dancer [14] weakened, in the above abstract result, the supplementary condition to the commutativity of L with $\mathbb{A}$ and $\mathbb{B}$. His proof is based upon a result of Browder on normally solvable nonlinear operators and a delicate analysis of various spectral decompositions associated to L, A, and B.

The special case of this result when $\mathbb{A} = pI$ and $\mathbb{B} = qI$, for which (6) becomes

$$pI \leqslant N'(u) \leqslant qI, \qquad (9)$$

and (7) becomes

$$[p, q] \cap \sigma(L) = \varnothing, \qquad (10)$$

had already been proved in 1976 by Mawhin [26] with the Banach fixed point theorem only, providing a simple and unifying approach for a series of contributions initiated by Dolph [15] in his seminal paper of 1949 on Hammerstein equations and Dirichlet problems (see also [36, 16, 17] for simplifications of the proof and extensions of the results), and by Loud [24] and Lazer and Sanchez [23] in the case of periodic boundary conditions (see [25] for a survey of these results). When N is a gradient operator verifying the following extension of conditions (9)

$$p\,\|u-v\|^2 \leqslant \langle N(u)-N(v), u-v\rangle \leqslant q\,\|u-v\|^2, \qquad (11)$$

existence and uniqueness for (5) under condition (10) was proved by Brézis and Nirenberg [11] in 1978 and Amann [4] in 1979 using distinct combinations of a Lyapunov–Schmidt-type reduction and monotone operators. A simple proof based upon the Banach fixed point theorem was given in 1981 by Mawhin [27] together with an example showing that the assumption that N is a gradient operator could not be avoided in this theorem.

A natural question to raise is then the possibility of proving the existence and uniqueness of the solution for (5) under condition (8), and assumption (7), using only the Banach fixed point theorem. This is done in Theorem 1 in the more general situation of Eq. (5) with $L: D(L) \subset H \to H$ self-adjoint in the real Hilbert space H, $N: H \to H$ a gradient operator such that condition (8) holds for some continuous self-adjoint operators A, $B: H \to H$ such that $B - A$ is positive, and with condition (7) replaced by the invertibility of the operator $L - (1 - \mu) A - \mu B$ for each $\mu \in [0, 1]$. Notice that this condition can be written

$$[0, 1] \cap \sigma_{B-A}(L - A) = \varnothing, \tag{12}$$

where $\sigma_{B-A}(L - A)$ is the spectrum of $L - A$ with weight $B - A$, and that (12) is in turn equivalent to the condition

$$[0, 1] \cap \sigma(\hat{L}_{A,B}) = \varnothing, \tag{13}$$

where $\hat{L}_{A,B} = (B - A)^{-1/2}(L - A)(B - A)^{-1/2}$ (see, e.g., [21]). The very simple idea of the proof consists in reducing (5) to the equivalent form

$$\hat{L}_{A,B}(v) = \hat{N}_{A,B}(v), \tag{14}$$

where $\hat{L}_{A,B}$ is given above and $\hat{N}_{A,B}(v) = (B - A)^{-1/2}(N - A)(B - A)^{-1/2}(v)$ are easily shown by (11) and (13) to satisfy conditions (9) and (10) with $p = 0$ and $q = 1$. Notice that with respect to the abstract theorems in [5, 14, 34, 40], we replace the assumption of strong positive definiteness of $A - L$ and $L - B$ on direct summands of H respectively by the more natural condition (12) whose application to concrete systems will not require any commutativity assumption between L, A, and B like in the above papers (see our Section 4). Notice also that for all the concrete examples given at the beginning of this Introduction, condition (12) is easily shown to be equivalent to the corresponding condition (7). This equivalence is closely connected to the commutativity properties of the differential operators d^2/dt^2 and $\Box$. Indeed, for a general linear operator L and arbitrary matrices $\mathbb{A}$ and $\mathbb{B}$, the conditions (7) and (12) are independent, as shown by the following simple example with $H = \mathbb{R}^2$ and $L(x_1, x_2) = (x_2, x_1)$. If

$$\mathbb{A} = \mathbb{B} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix},$$

then, for every $\lambda \in \mathbb{R}$,

$$\sigma(L) \cap \sigma(\lambda \mathbb{A} + (1 - \lambda)\, \mathbb{B}) = \varnothing,$$

but $L - \lambda A - (1-\lambda) B = L - \lambda \mathbb{A} - (1-\lambda) \mathbb{B}$ is not invertible. On the other hand, if

$$\mathbb{A} = \mathbb{B} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

then, for every $\lambda \in \mathbb{R}$,

$$\sigma(L) \cap \sigma(\lambda \mathbb{A} + (1-\lambda) \mathbb{B}) = \{1\} \neq \varnothing,$$

but $L - \lambda A - (1-\lambda) B$ is invertible.

The same reduction to (14) is applied in Theorems 2, 3, and 4 to prove the existence conclusion for (5) in situations where conditions (11) and (12) are weakened, but some restrictions are made upon the potential of N. Such situations generalize some consequences of the dual least action principle (see, e.g., [30, 33]) due to Willem [42, 43], Mawhin and Willem [32], and Mawhin [29], to which they are reduced by the above mentioned trick. Notice that Proposition 1 also extends results of Smiley [37] and Bennaoum and Mawhin [10] which cover situations where N need not be a gradient operator.

An application is then given in Theorem 5 to the periodic-Dirichlet problem on $]0, 2\pi[\times (]0, \pi[)^n$ for a system of semilinear wave equations of the form

$$\Box u - V'(t, x, u) = h(t, x)$$

when

$$\begin{aligned} (\mathbb{A}(t, x)(u - v), u - v) &\leqslant (V'(t, x, u) - V'(t, x, v), u - v) \\ &\leqslant (\mathbb{B}(t, x)(u - v), u - v) \end{aligned}$$

for all t, x, v and some measurable bounded matrix functions $\mathbb{A}$ and $\mathbb{B}$ satisfying a vector nonresonance condition of the type introduced in [3] for systems of ordinary differential equations. Further applications will be given in another paper.

2. Existence and Uniqueness by an Iterative Method

Let H be a Hilbert space with scalar product $\langle \cdot, \cdot \rangle$ and corresponding norm $\|\cdot\|$. We consider a linear normal operator $L: D(L) \subset H \to H$ and a possibly nonlinear operator $N: H \to H$. We are interested in finding solutions of the equation

$$Lx = Nx. \tag{15}$$

Let $S: H \to H$ be a given continuous linear selfadjoint operator which is positive and invertible. We will denote its inverse by S^{-1}, and by $S^{1/2}$ and $S^{-1/2}$ the square root operators of S and S^{-1}, respectively (cf. [21] for details).

DEFINITION 1. Setting $L_S := S^{-1/2}LS^{-1/2}$ and $N_S := S^{-1/2}NS^{-1/2}$, the spectrum $\sigma_S(L)$ of L with weight S is the (usual) spectrum of L_S, i.e., $\sigma_S(L) := \{\lambda \in \mathbb{C} : L - \lambda S$ has no bounded inverse$\}$. An eigenvalue of L_S will be called an eigenvalue of L with weight S.

Our first existence result is the following one.

PROPOSITION 1. *Assume that the following conditions hold.*

(i) $\langle Nx - Ny, x - y \rangle \geqslant \langle S^{-1}(Nx - Ny), Nx - Ny \rangle$ *for all* $x, y \in H$;

(ii) $\sigma_S(L) \cap \{\lambda \in \mathbb{C} : |\lambda - (1/2)| < 1/2\} = \varnothing$;

(iii) *there exists* $R > 0$ *such that, if* $\|S^{1/2}x\| = R$, *then* $\langle Nx, x \rangle \geqslant \langle S^{-1}Nx, Nx \rangle$.

Then Eq. (15) *has at least a solution. If we replace conditions* (ii) *and* (iii) *by*

(iv) $\sigma_S(L) \cap \{\lambda \in \mathbb{C} : |\lambda - (1/2)| \leqslant 1/2\} = \varnothing$,

then $L - N$ *is a bijection from* $D(L)$ *to* H. *In particular, Eq.* (15) *has a unique solution which moreover can be obtained, from any* $x_0 \in D(L)$, *by the iteration process defined by*

$$Lx_{k+1} - (1/2)\, Sx_{k+1} = Nx_k - (1/2)\ Sx_k.$$

Proof. By the change of variable $u = S^{1/2}x$ and the invertibility of $L_S - (1/2)\, I$ which follows from condition (ii), Eq. (15) can be written in the fixed point form

$$u = [L_S - (1/2)\, I]^{-1}[N_S u - (1/2)\, u] := Tu.$$

The assumptions (i), (ii), and (iii) become, respectively:

(j) $\|N_S u - (1/2)\, u - N_S v + (1/2)\, v\| \leqslant (1/2)\|u - v\|$;

(jj) $\|(L_S - (1/2)\, I)^{-1}\| \leqslant 2$;

(jjj) $\|N_S u - (1/2)\, u\| \leqslant (1/2)\|u\|$ for $\|u\| = R$.

The operator T is therefore nonexpansive and such that $\|Tu\| \leqslant R$ whenever $\|u\| = R$. Hence, T has a fixed point (see, e.g., [12]). In case condition (iv) holds, one has a strict inequality in (jj) and the operator T comes out to be a contractive mapping. The conclusion follows then from the Banach fixed point theorem. ∎

Remark 1. Notice that the conditions (i)–(iv) in Proposition 1 are symmetrical, in the sense that they hold as well if N is replaced by $S-N$ or L is replaced by $S-L$. For example, the inequality in (i) can be written equivalently as

$$\begin{aligned} &\langle (S-N)\,x-(S-N)\,y, x-y\rangle \\ &\qquad \geqslant \langle S^{-1}((S-N)\,x-(S-N)\,y), (S-N)\,x-(S-N)\,y\rangle. \end{aligned}$$

Remark 2. If the operator L is self-adjoint, the assumptions (ii) and (iv) can be written, respectively,

(ii′) $\sigma_S(L) \cap \,]0, 1[\, = \varnothing$;

(iv′) $\sigma_S(L) \cap [0, 1] = \varnothing$.

When N is of gradient type, condition (i) of Proposition 1 can be characterized in a simple way.

LEMMA 1. *Assume that N is a continuous gradient operator. Then condition* (i) *of Proposition* 1 *is equivalent to the following one*:

(i′) *both N and $S-N$ are monotone.*

Proof. If (i) holds, N is monotone since S^{-1} is positive semi-definite, and the same is true for $S-N$, by Remark 1. So (i) implies (i′). Assume now that (i′) holds. It is easy to see that both N_S and $I-N_S$ are monotone. It follows then from [27, Lemma 1], that condition (j) in the proof of Proposition 1 holds, and hence its equivalent formulation (i) holds too. ∎

As a consequence of Proposition 1, Lemma 1, and Remark 2, we have the following two existence and uniqueness results.

THEOREM 1. *Assume that L is selfadjoint and that N is a continuous gradient operator. Let A, B: $H \to H$ be two continuous, linear, and selfadjoint operators such that the following conditions hold.*

(i) *$N-A$ and $B-N$ are monotone*;

(ii) *$L-(1-\lambda)\,A-\lambda B$ has a bounded inverse for every $\lambda \in [0, 1]$.*

Then $L-N$ is a bijection from $D(L)$ to H. In particular, Eq. (15) *has a unique solution which moreover can be obtained, from any $x_0 \in H$, by the iteration process defined by*

$$Lx_{k+1} - (1/2)(A+B)\,x_{k+1} = Nx_k - (1/2)(A+B)\,x_k.$$

Proof. By condition (i), the operator $B-A$ is nonnegative. We need to define a nonnegative and invertible operator S. To this end, as the set of operators with bounded inverse is open, it follows from condition (ii) that there exists $\varepsilon > 0$ such that $L-(1-\lambda)(A-\varepsilon I)-\lambda(B+\varepsilon I)$ has a bounded inverse for every $\lambda \in [0, 1]$. Define $S = B - A + 2\varepsilon I$, and set $\tilde{L} = L - (A-\varepsilon I)$, $\tilde{N} = N-(A-\varepsilon I)$. Clearly, both $\tilde{N}$ and $S-\tilde{N}$ are monotone. Moreover, $\sigma_S(\tilde{L}) \cap [0, 1] = \varnothing$, and the bijectivity follows from Proposition 1 via Lemma 1 and Remark 2, with $\tilde{L}$ and $\tilde{N}$ instead of L and N, respectively, and by the fact that the assumptions are invariant if we replace Nx by $Nx-f$ for any $f \in H$. For the iterative method, it suffices to check that the convergent iterative process

$$\tilde{L}u_{k+1} - (1/2)\, Su_{k+1} = \tilde{N}u_k - (1/2)\, Su_k,$$

is equivalent to the one given in the assertion. ▮

COROLLARY 1. *Assume that L is selfadjoint and that N is Gateaux-differentiable and such that, for every $x \in H$, $N'(x)$ is a symmetric operator. Let A, B: $H \to H$ be two continuous, linear, and selfadjoint operators such that the following conditions hold.*

(i) *$A \leqslant N'(x) \leqslant B$ for each $x \in H$;*

(ii) *$L-(1-\lambda)\,A-\lambda B$ has a bounded inverse for every $\lambda \in [0, 1]$.*

Then the conclusion of Theorem 1 holds.

Remark 3. Corollary 1 is a generalization of a result by Mawhin [26], while Theorem 1 generalizes a result of Amann [4] (see also [27] which simplifies [4] and completes it by an iterative process). Both results hold in the case where L is normal provided that the assumption (ii) is replaced by the following one:

(ii′) $L-(1-\lambda)\,A-\lambda B$ has a bounded inverse for every $\lambda \in \{\mu \in \mathbb{C} : |\mu-(1/2)| \leqslant 1/2\}$.

3. EXISTENCE RESULTS BY A VARIATIONAL METHOD

In this section, we will assume that the linear operator L introduced in Section 2 is a selfadjoint operator with closed range $R(L)$, and that the nonlinear part N is the gradient of a differentiable function η: $H \to \mathbb{R}$. Our aim is to study some situation where the hypotheses of Theorem 1 are appropriately weakened.

The following result generalizes Theorem 2 in [42].

Remark 4. In what follows, we use the convention that $\inf \varnothing = +\infty$.

PROPOSITION 2. *Assume that η is convex and that the following conditions are satisfied.*

(i) *There exist positive constants δ, d such that condition*

$$-d+\frac{\delta}{2}\langle Sx, x\rangle \leqslant \eta(x) \leqslant \frac{1-\delta}{2}\langle Sx, x\rangle + d$$

holds for all $x \in H$;

(ii) $\sigma_S(L) \cap]0, 1[= \varnothing$;

(iii) $\sigma_S \cap]0, l[$ *consists of isolated eigenvalues with weight S of finite multiplicity, where*

$$l := \inf\{k > 0 : kS - N \textit{ is monotone}\}.$$

Then $L - N$ is onto H.

Proof. The assumptions being invariant for the substitution of Nx into $Nx - z$, it suffices to prove that Eq. (15) has a solution. It is easy to see that N_S is the gradient of the convex function η_S: $H \to \mathbb{R}$ defined by

$$\eta_S(y) = \eta(S^{-1/2}y).$$

Let us denote by η^*_S the Fenchel transform of η_S, namely

$$\eta^*_S(u) = \sup_{x \in H} [\langle u, x\rangle - \eta_S(x)],$$

and by K_S: $R(L_S) \to R(L_S)$ the continuous right inverse of L_S. Consider the functional ϕ defined on $R(L_S)$ by

$$\phi(u) = \eta^*_S(u) - (1/2)\langle K_S u, u\rangle.$$

By the left inequality in (i), ϕ takes values in $\mathbb{R}$. We will show as in [42] that ϕ is coercive and weakly lower semicontinuous. As it is well known, this will imply that ϕ has a minimum reached at some $u \in R(L_S) = R(S^{-1/2}L)$, which implies then (see, e.g., [42]) that there exists a solution x of (15) such that $u = S^{-1/2}Lx$.

From assumptions (i) and (ii), one gets the inequalities

(j)* $\eta^*_S(u) \geqslant (1/2(1-\delta))\|u\|^2 - d$;

(jj)* $\langle K_S u, u\rangle \leqslant \|u\|^2$.

This implies immediately that the functional ϕ is coercive, and it remains to prove that ϕ is weakly lower semicontinuous. For this, let us first suppose that $l < +\infty$. Then both N and $lS - N$ are monotone, and we can apply Lemma 2, with S replaced by lS, to get the analogous form of (j) in the proof of Proposition 1, i.e.,

$$\left\| N_S u - N_S v - \frac{l}{2}(u-v) \right\| \leqslant \frac{l}{2} \|u-v\|.$$

In particular, N_S is Lipschitzian with Lipschitz constant l. By Corollary 10 of [6] we obtain that the functional

$$\phi_0(u) = \eta_S^*(u) - \frac{1}{2l} \|u\|^2$$

is convex and continuous. Let $\{P_\lambda : \lambda \in \mathbb{R}\}$ denote the spectral resolution of L_S, and consider the following projectors:

$$Q_1 = \int_{]-\infty, 0]} dP_\lambda, \qquad Q_2 = \int_{]0, l[} dP_\lambda, \qquad Q_3 = \int_{[l, +\infty[} dP_\lambda.$$

Then the functionals

$$\phi_1(u) = -\frac{1}{2}\langle K_S Q_1 u, Q_1 u\rangle, \qquad \phi_3(u) = \frac{1}{2l}\|u\|^2 - \frac{1}{2}\langle K_S Q_3 u, Q_3 u\rangle$$

are convex and continuous, while, because of (iii), the functional

$$\phi_2(u) = -\tfrac{1}{2}\langle K_S Q_2 u, Q_2 u\rangle$$

is weakly continuous, $K_S Q_2$ being a compact operator. Hence, all the functionals ϕ_i, $i = 0$, 1, 2, 3, are weakly lower semicontinuous, and the same is true for $\phi = \phi_0 + \phi_1 + \phi_2 + \phi_3$. This proves the result when $l < +\infty$. If $l = +\infty$, the proof is easier, since Q_3 does not appear. Then one simply has $\phi_0 = \eta_S^*(u)$, ϕ_1 and ϕ_2 as above, and $\phi_3 \equiv 0$. ∎

As a consequence of Proposition 2, we have the following result.

THEOREM 2. *Let A, B: $H \to H$ be two continuous, linear, and selfadjoint operators such that the following conditions hold.*

(i) *$N - A$ and $B - N$ are monotone;*

(ii) *$L - (1-\lambda) A - \lambda B$ has a continuous inverse for every $\lambda \in]0, 1[$;*

(iii) *There exist two continuous, linear, and selfadjoint operators A_1, B_1: $H \to H$, with $A \leqslant A_1 \leqslant B_1 \leqslant B$, such that for some $d \geqslant 0$ and all $x \in H$, one has*

$$-d(\|x\| + 1) + \tfrac{1}{2}\langle A_1 x, x\rangle \leqslant \eta(x) \leqslant \tfrac{1}{2}\langle B_1 x, x\rangle + d(\|x\| + 1),$$

with moreover $A < A_1$ *in case* $L - A$ *is not invertible, and* $B_1 < B$ *in case* $L - B$ *is not invertible.*

Then $L - N$ *is onto* H.

Proof. If either $L - A$ or $L - B$ is not invertible, it follows from the assumption (iii) that $A < B$. Otherwise, we have seen in the proof of Theorem 1 that one can perturb a little bit the operators without changing the setting of the theorem. We can therefore assume without loss of generality that $A < A_1 \leqslant B_1 < B$. At this point, set $S = B - A$, $L' = L - A$, $N' = N - A$. It is not difficult to show that the assumptions of Proposition 2 hold true, and the result follows. ∎

Remark 5. The assumptions of Theorem 2 generalize those of Theorem 1. Indeed, when both $L - A$ and $L - B$ are invertible, it is sufficient to take $A_1 = A$ and $B_1 = B$.

Whenever assumption (iii) of Proposition 2 fails, we can still, following Willem [43] prove a density result for the range of $L - N$.

THEOREM 3. *Assume that* η *is convex and that the following conditions are satisfied.*

(i) *There exist positive constants* δ, d *such that*

$$-d + \frac{\delta}{2}\langle Sx, x\rangle \leqslant \eta(x) \leqslant \frac{1-\delta}{2}\langle Sx, x\rangle + d,$$

for all $x \in H$;

(ii) $\sigma_S(L) \cap]0, 1[= \varnothing$;

(iii) $l < +\infty$, *i.e., there exists* $k \geqslant 0$ *such that* $kS - N$ *is monotone.*

Then the range of $L - N$ *is dense in* H.

Proof. We consider the functional ϕ introduced in the proof of Proposition 2. One can again show that ϕ is coercive but we cannot assert that ϕ is weakly lower semicontinuous. On the other hand, it was shown in the proof of Proposition 2 that the assumption (iii) above implies that N_S is Lipschitz continuous with Lipschitz constant l.

By the remark at the beginning of the proof of Proposition 2, it will be sufficient to show that $0 \in cl[R(L - N)]$. Let $\varepsilon > 0$ be fixed and let us set $\gamma = \varepsilon / l \|S^{1/2}\|$. By a theorem of Ekeland [18] (see also [33]), there exists $v \in R(L_S)$ such that, for every $h \in R(L_S)$ and $t > 0$, one has

$$\phi(v) \leqslant \phi(v + th) + \gamma t \|h\|,$$

i.e.,

$$\langle K_S v, h\rangle \leqslant \frac{1}{t}(\eta_S^*(v+th)-\eta_S^*(v))+\gamma\|h\|-\frac{t}{2}\langle K_S h, h\rangle.$$

Then, since

$$\delta^+\eta_S^*(v,h) := \limsup_{t\to 0^+}\frac{1}{t}(\eta_S^*(v+th)-\eta_S^*(v)) \leqslant \eta_S^*(v+h)-\eta_S^*(v),$$

and $\delta^+\eta_S^*(v,\cdot)+\gamma\|\cdot\|$ is positively homogeneous and subadditive, the Hahn–Banach theorem implies the existence of $w\in\ker L_S$ such that, for every $h\in H$, one has

$$\langle w+K_S v, h\rangle \leqslant \eta_S^*(v+h)-\eta_S^*(v)+\gamma\|h\|.$$

Using the geometrical version of the Hahn–Banach theorem, we can now find $f\in H$ such that, for every $h\in H$, one has

$$-\gamma\|h\| \leqslant \langle f, h\rangle \leqslant \eta_S^*(v+h)-\eta_S^*(v)-\langle w+K_S v, h\rangle.$$

This implies that $\|f\|\leqslant\gamma$, and that $w+K_S v+f\in\partial\eta_S^*(v)$, the subdifferential of η_S^* in v. By duality, $v=N_S(w+K_S v+f)$. Setting $x=S^{-1/2}(w+K_S v)$, we have

$$\begin{aligned}\|Lx-Nx\| &= \|S^{1/2}(v-N_S(w+K_S v))\| \\ &\leqslant \|S^{1/2}\|\;\|N_S(w+K_S v+f)-N_S(w+K_S v)\| \\ &\leqslant l\|S^{1/2}\|\;\|f\|\leqslant\varepsilon. \blacksquare\end{aligned}$$

Finally, let us consider, in analogy with [29, 32], a resonance case where condition (i) is weakened and η is coercive on $\ker L$.

THEOREM 4. *Assume that* $\ker L\neq\{0\}$ *and that the following conditions hold.*

(i) *There exist positive constants* δ, d *such that*

$$\eta(x)\leqslant\frac{1-\delta}{2}\langle Sx, x\rangle+d$$

for all $x\in H$.

(ii) $\sigma_S(L) \cap]0, 1[= \varnothing$.

(iii) $\sigma_S(L) \cap]0, +\infty[$ *consists of isolated eigenvalues with weight* S *of* L *having finite multiplicities.*

(iv) $\eta(x) \to +\infty$ *as* $\|x\| \to \infty$, $x \in \ker L$.

Then Eq. (15) *has a solution* x *such that* $S^{-1/2}Lx$ *minimizes the functional*

$$\phi(u) = \eta_S^*(u) - \tfrac{1}{2}\langle K_S u, u\rangle.$$

We need not give the details of the proof since it follows from [29], using the reduction made in the proof of Proposition 2.

4. Nonuniform Nonresonance Conditions for Systems of Semilinear Wave Equations

Let $\Omega =]0, 2\pi[\times (]0, \pi[)^n$ in $\mathbb{R} \times \mathbb{R}^n$, $n \geqslant 1$ an integer, $h \in L^2(\Omega, \mathbb{R}^n)$, $V\colon \Omega \times \mathbb{R}^N \to \mathbb{R}$, $((t, x), u) \mapsto V(t, x, u)$ be a Caratheodory function such that $V' = D_u V$ exists and is a Caratheodory function, $N \geqslant 1$ being an integer. We are interested in the existence and uniqueness of weak solutions for the system of semilinear wave equations (with $\square = D_t^2 - \sum_{j=1}^n D_{x_j}^2$)

$$\square u - V'(t, x, u) = h(t, x) \tag{16}$$

with periodic-Dirichlet boundary conditions on Ω, i.e., in the existence and uniqueness of $u \in L^2(\Omega, \mathbb{R}^N)$ such that

$$\int_\Omega [(u(t, x), \square v(t, x)) - (V'(t, x, u(t, x)) - h(t, x), v(t, x))]\, dt\, dx = 0$$

for all $v \in C^2(\bar{\Omega}, \mathbb{R}^N)$ such that

$$v(0, x) - v(2\pi, x) = D_t v(0, x) - D_t v(2\pi, x) = 0, \qquad x \in [0, \pi]^n,$$
$$v(t, x) = 0,\ t \in [0, 2\pi], \qquad x \in bdry([0, \pi]^n).$$

If $V'(t, x, u) = \mathbb{D}u$, with $\mathbb{D}$ an $(N \times N)$-symmetric real matrix, those weak solutions are the $u \in L^2(\Omega, \mathbb{R}^N)$ whose Fourier series

$$u(t, x) \approx \sum_{j=-\infty}^{\infty} \sum_{k_1=1}^{\infty} \cdots \sum_{k_n=1}^{\infty} \exp(ijt) \sin(k_1 x) \cdots \sin(k_n x)\, u_{j, k_1, \ldots, k_n}$$

formally satisfies (16), where $u_{-j, k_1, \ldots, k_n} = \bar{u}_{j, k_1, \ldots, k_n}$.

Consequently, the periodic-Dirichlet problem on Ω for the system

$$\Box u - \mathbb{D}u = 0$$

will have a nontrivial weak solution if and only if $\mathbb{D}$ has an eigenvalue of the form $\sum_{l=1}^{n} k_l^2 - j^2$ for some $j \in \mathbb{Z}$, $k_l \in \mathbb{N}^*$, $1 \leqslant l \leqslant n$, the solutions being linear combinations of the functions

$$\exp(ijt)\sin(k_1 x)\cdots\sin(k_n x)\, c$$

for all $j \in \mathbb{Z}$, $k_l \in \mathbb{N}^*$, $1 \leqslant l \leqslant n$, and all eigenvectors c of $\mathbb{D}$ corresponding to the eigenvalue $\sum_{l=1}^{n} k_l^2 - j^2$. Of course, infinitely many of those eigenvalues have infinite multiplicity when $n \geqslant 2$. If $\mathbb{D}$ has no eigenvalue of the type above, it is easy to check that the nonhomogeneous system

$$\Box u - \mathbb{D}u = h(t, x)$$

with the periodic-Dirichlet conditions on Ω has a unique weak solution for each $h \in L^2(\Omega, \mathbb{R}^N)$, and that the spectrum of the abstract realization in $H := L^2(\Omega, \mathbb{R}^N)$ of $\Box - \mathbb{D}$ with the periodic-Dirichlet boundary conditions on Ω is made of eigenvalues only. In particular, if L denotes the abstract realization in H of $\Box$ with the periodic-Dirichlet boundary conditions, then L is a selfadjoint operator in H, with a discrete spectrum given by

$$\sigma(L) = \left\{ \sum_{l=1}^{n} k_l^2 - j^2 : k_j \in \mathbb{N}^*, j \in \mathbb{Z} \right\}.$$

Thus $\sigma(L)$ is discrete and unbounded from below and from above. We may denote it by $\sigma(L) = \{\lambda_p : p \in \mathbb{Z}\}$ with $\lambda_p < \lambda_{p+1}$, $p \in \mathbb{Z}$.

Let us introduce the following assumptions.

(H_1) There exist two constant symmetric $(N \times N)$-matrices $\mathbb{A}_0$ and $\mathbb{B}_0$ with respective eigenvalues

$$\lambda_{n_m}\ (1 \leqslant m \leqslant N) \qquad \text{and} \qquad \lambda_{n_m+1}\ (1 \leqslant m \leqslant N)$$

being elements of the spectrum of L (with the indices n_l nondecreasing in l), and there exist two measurable bounded mappings $\mathbb{A}$, $\mathbb{B}: \Omega \to \mathscr{S}(\mathbb{R}^N, \mathbb{R}^N)$, where $\mathscr{S}(\mathbb{R}^N, \mathbb{R}^N)$ denotes the set of symmetric $(N \times N)$-matrices, such that the inequalities

$$\begin{aligned}(\mathbb{A}_0(u-v), u-v) &\leqslant (\mathbb{A}(t,x)(u-v), u-v)\\ &\leqslant (V'(t,x,u) - V'(t,x,v), u-v)\\ &\leqslant (\mathbb{B}(t,x)(u-v), u-v)\\ &\leqslant (\mathbb{B}_0(u-v), u-v) \qquad (17)\end{aligned}$$

hold for a.e. $(t, x) \in \Omega$ and all $u, v, \in \mathbb{R}^N$, (u, v) denoting the inner product of u and v in $\mathbb{R}^N$.

(H_2) Let

$$\mathbb{A}(\cdot, \cdot)\, u(\cdot, \cdot) \neq \mathbb{A}_0 u(\cdot, \cdot)$$

for each $u \in N(L - A_0)\backslash\{0\}$, and

$$\mathbb{B}(\cdot, \cdot)\, u(\cdot, \cdot) \neq \mathbb{B}_0 u(\cdot, \cdot)$$

for each $u \in N(L - B_0)\backslash\{0\}$, where

$$A_0, B_0 : H \to H, \qquad u \mapsto \mathbb{A}_0 u(\cdot, \cdot), \mathbb{B}_0 u(\cdot, \cdot).$$

(H_3) For each $\mu \in [0, 1]$, $L - (1 - \mu)\, A_0 - \mu B_0$ has closed range.

It follows from inequalities (17) and classical results that the mapping N defined by

$$(Nu)(t, x) = V'(t, x, u(t, x))$$

is a continuous gradient mapping from H into itself, so that our problem is equivalent to solving the abstract equation (15) in $D(L) \subset H$. We want to apply Theorem 1 with

$$A, B\colon H \to H, \qquad u \mapsto \mathbb{A}(\cdot, \cdot)\, u(\cdot, \cdot), \mathbb{B}(\cdot, \cdot)\, u(\cdot, \cdot),$$

and it is clear from (17) that assumption (i) is satisfied. We need a few lemmas to verify condition (ii). Let $\mathbb{S}_0 = \mathbb{B}_0 - \mathbb{A}_0$, $S_0 = B_0 - A_0$, so that S_0 is a positive definite invertible selfadjoint operator on H.

LEMMA 2. *Assume that* (H_1) *holds. Then the weighted eigenvalue problem*

$$Lu - A_0 u - \mu S_0 u = 0 \tag{18}$$

has a nontrivial solution if and only if $\mu \in \sigma_{S_0}(L - A_0)$, *where*

$$\sigma_{S_0}(L - A_0) = \left\{\sigma_m^{-1}\left(\sum_{l=1}^{n} k_l^2 - j^2 - \lambda_{n_m}\right) : j \in \mathbb{Z}, k_l \in \mathbb{N}^*, 1 \leqslant m \leqslant N\right\}, \tag{19}$$

where the σ_m *are the eigenvalues of* $\mathbb{S}_0$. *Moreover, if* ϕ *and* ψ *are eigenfunctions corresponding to two distinct eigenvalues* μ *and* v, *then*

$$\langle S_0 \phi, \psi \rangle := \int_\Omega (\mathbb{S}_0 \phi(t, x), \psi(t, x))\, dt\, dx = 0. \tag{20}$$

Finally,

$$\sigma_{S_0}(L - A_0) \cap]0, 1[= \varnothing.$$

Proof. By the discussion at the beginning of this section, Eq. (18) will have a nontrivial solution if and only if

$$\det\left(\left(\sum_{l=1}^{n} k_l^2 - j^2\right) I - \mathbb{A}_0 - \mu \mathbb{S}_0\right) = 0 \tag{21}$$

for some $j \in \mathbb{Z}$, $k_l \in \mathbb{N}^*$. Now, $\mathbb{A}_0$ and $\mathbb{S}_0$ can be diagonalized simultanously by an unitary matrix, so that (21) is equivalent to

$$\det\left(\left(\sum_{l=1}^{n} k_l^2 - j^2\right) I - \operatorname{diag}(\lambda_{n_m}) - \mu \operatorname{diag}(\sigma_m)\right) = 0,$$

i.e., to

$$\mu = \sigma_m^{-1}\left(\sum_{l=1}^{n} k_l^2 - j^2 - \lambda_{n_m}\right), \qquad j \in \mathbb{Z}, k_l \in \mathbb{N}^*, 1 \leqslant m \leqslant N, \tag{22}$$

and the first part of the lemma is proved. Notice also that if μ is not one of the numbers in (22), then $(L - A_0 - \mu S_0)^{-1}$ exists and is continuous, and hence the set in (22) is nothing but $\sigma_{S_0}(L - A_0)$. Now, if

$$L\phi - A_0\phi - \mu S_0 \phi = 0 = L\psi - A_0 \psi - v S_0 \psi,$$

with $\mu \neq v$, then

$$\begin{aligned} &\langle L\phi, \psi \rangle - \langle A_0\phi, \psi \rangle - \mu \langle S_0\phi, \psi \rangle \\ &\quad = \langle L\psi, \phi \rangle - \langle A_0\psi, \phi \rangle - v \langle S_0\psi, \phi \rangle = 0, \end{aligned}$$

and, using the symmetry of the operators, we get

$$\langle S_0\phi, \psi \rangle = \langle S_0\psi, \phi \rangle = 0.$$

Now, Eq. (18) can be written

$$Lu - D(\mu)\, u = 0$$

with $D(\mu)\colon u \mapsto \mathbb{D}_0(\mu)\, u(\cdot, \cdot)$ and

$$\mathbb{A}_0 < \mathbb{D}_0(\mu) = (1 - \mu)\, \mathbb{A}_0 + \mu \mathbb{B}_{0p} < \mathbb{B}_0,$$

for $\mu \in]0, 1[$. Consequently, if $\delta_m(\mu)$ are the eigenvalues of $\mathbb{D}_0(\mu)$ ordered increasingly with respect to k, then we have

$$\lambda_{n_m} < \delta_m(\mu) < \lambda_{n_m+1}, \qquad 1 \leqslant m \leqslant N, \tag{23}$$

which shows that the $\delta_m(\mu)$ cannot be of the form λ_j for some $j \in \mathbb{Z}$, and hence, by the discussion at the beginning of the section, (18) has only the trivial solution for each $\mu \in]0, 1[$, and the proof of the lemma is complete. ▮

Remark 6. Notice that by assumption (H_1), 0 and 1 are eigenvalues of $L - A_0$ with weight S_0.

Let us write $H = H_0 \oplus H_1$, where H_0 is the closure of the vector space spanned by the $S_0^{1/2}\phi$ for the eigenfunctions ϕ of the problem (equivalent to (18))

$$S_0^{-1/2}(L - A_0)\, S_0^{-1/2} v - \mu v = 0, \tag{24}$$

corresponding to the eigenvalues smaller or equal to 0, and H_1 is obtained in the same way from the eigenfunctions associated to the eigenvalues larger or equal to 1. Let us write, for $u \in H$, $u = u_0 + u_1$ with $u_0 \in H_0$, $u_1 \in H_1$, and let us define on $D(L)$ the quadratic form $Q_{A,B}$ by

$$Q_{A,B}(u) = \langle (L - B)\, u_1, u_1 \rangle - \langle (L - A)\, u_0, u_0 \rangle. \tag{25}$$

LEMMA 3. *Assume that* (H_1) *holds and that* $Q_{A,B}(u) > 0$ *for all* $u \neq O$ *in* $D(L)$. *Then the problem*

$$Lu - (1 - \mu)\, Au - \mu Bu = 0 \tag{26}$$

has only the trivial solution for each $\mu \in [0, 1]$.

Proof. Assume that $u \neq 0$ is a solution of (26) for some $\mu \in [0, 1]$. Then, letting $u = u_0 + u_1$ like above, we get, from the symmetry of the operators,

$$\begin{aligned} 0 &= \langle Lu - (1 - \mu)\, Au - \mu Bu, u_1 - u_0 \rangle \\ &= \langle Lu_1 + Lu_0 - (1 - \mu)(Au_1 + Au_0) - \mu(Bu_1 + Bu_0), u_1 - u_0 \rangle \\ &= \langle [L - (1 - \mu)\, A - \mu B]\, u_1, u_1 \rangle - \langle [L - (1 - \mu)\, A - \mu B]\, u_0, u_0 \rangle \\ &\geqslant \langle (L - B)\, u_1, u_1 \rangle - \langle (L - A)\, u_0, u_0 \rangle = Q_{A,B}(u) > 0, \end{aligned}$$

a contradiction. ▮

LEMMA 4. *Assume that* (H_1) *and* (H_2) *hold. Then, for each* $u \in D(L)\setminus\{0\}$, *one has* $Q_{A,B}(u) > 0$.

Proof. Let $u_1 = w_1 + v_1$ with w_1 in the eigenspace associated to the eigenvalue 1 of $L - A_0$ with weight S_0 and v_1 orthogonal (in the sense of (20)) to w_1. Then, using the symmetry, Fourier series, S_0-orthogonality, and the fact that $(L - A_0) w_1 = S_0 w_1$, i.e., $Lw_1 = B_0 w_1$, we have

$$\begin{aligned}\langle (L-B) u_1, u_1 \rangle &\geqslant \langle (L-B_0) u_1, u_1 \rangle = \langle (L-B_0) v_1, w_1 + v_1 \rangle \\ &= \langle (L-B_0) v_1, v_1 \rangle = \langle (L-A_0) v_1, v_1 \rangle - \langle S_0 v_1, v_1 \rangle \\ &\geqslant (\mu_2 - 1) \langle S_0 v_1, v_1 \rangle \geqslant 0,\end{aligned}$$

where μ_2 is the smallest eigenvalue of $L - A_0$ with weight S_0 strictly greater than 1. Similarly, writing $u_0 = w_0 + v_0$ with w_0 in the eigenspace associated to the eigenvalue 0 of $L - A_0$ with weight S_0 and v_0 orthogonal (in the sense of (20)) to w_0, we get

$$\begin{aligned}-\langle (L-A) u_0, u_0 \rangle &\geqslant -\langle (L-A_0) u_0, u_0 \rangle \\ &= -\langle (L-A_0) v_0, v_0 \rangle \geqslant -\mu_{-1} \langle S_0 v_0, v_0 \rangle \geqslant 0,\end{aligned}$$

where μ_{-1} is the largest negative eigenvalue of $L - A_0$ with weight S_0. Thus, $Q_{A,B}(u) \geqslant 0$ for all $u \in D(L)$ and if $Q_{A,B}(u) = 0$, then necessarily $v_1 = 0$, $v_0 = O$, i.e., $u_1 = w_1$, $u_0 = w_0$, and

$$\langle (L-B) w_1, w_1 \rangle = \langle (L-A) w_0, w_0 \rangle = 0,$$

i.e.,

$$\langle (B_0 - B) w_1, w_1 \rangle = \langle (A - A_0) w_0, w_0 \rangle = 0.$$

Those quadratic forms being nonnegative definite, this implies that they reach their minimum at w_1 and w_0, so that

$$(B_0 - B) w_1 = (A - A_0) w_0 = 0,$$

and hence $w_1 = w_0 = 0$ by assumption (H_2). Thus $Q_{A,B}(u) > 0$ for $u \in D(L)\setminus\{0\}$, and the proof is complete.

By Lemmas 2 to 4, condition (ii) of Theorem 1 is satisfied and we have proved the following existence and uniqueness result.

THEOREM 5. *If conditions* (H_1), (H_2) *and* (H_3) *hold, the periodic-Dirichlet problem for the system* (16) *has, for each* $h \in L^2(\Omega, \mathbb{R}^N)$ *a unique* *weak solution u which can be obtained as the limit of the successive approximations given by $u_0 \in H$ arbitrary and*

$$\Box u_{k+1} - \tfrac{1}{2}(\mathbb{A}(t,x) + \mathbb{B}(t,x))\, u_{k+1}$$
$$= V'(t, x, u_k) - \tfrac{1}{2}(\mathbb{A}(t,x) + \mathbb{B}(t,x))\, u_k - h(t,x), \qquad k \in \mathbb{N}.$$

Remarks. 1. Conditions (H_1) and (H_2) were essentially introduced in [3] for periodic solutions of systems of ordinary differential equations. They constitute vector extensions of the nonuniform nonresonance conditions introduced in [31].

2. This example shows that conditions of type (ii) of Theorem 1 are at least as easy to check than the other formulations like (2), (4), or (7).

3. The methodology used in Lemmas 2 to 4 was introduced for scalar two-point boundary value problems in [19].

4. The reader will easily state generalizations of conditions (H_1)–(H_3) which provide, via Theorem 3, density of the range of $L-N$ and, through Theorem 2, imply that $L-N$ is onto.

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