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Periodic Solutions of Nonlinear Differential Equations with Double Resonance (*).

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Summary. – *We prove the existence of periodic solutions of a second order nonlinear ordinary differential equation whose nonlinearity is at resonance with two successive eigenvalues of the associated linear operator and satisfies some Landesman-Lazer type conditions at both of them.*

1. – Introduction.

Consider the nonlinear periodic boundary value problem

$$x'' + g(t, x) = 0 \tag{1}$$

$$x(0) - x(T) = x'(0) - x'(T) = 0 \tag{2}$$

where $g\colon [0, T]\times \boldsymbol{R} \to \boldsymbol{R}$ satisfies Carathéodory conditions and grows at most linearly. More precisely, letting $I = [0, T]$, we assume the existence of two functions $a, b \in L^\infty(I)$ such that

$$a(t) \leqslant \liminf_{|x|\to\infty} \frac{g(t, x)}{x} \leqslant \limsup_{|x|\to\infty} \frac{g(t, x)}{x} \leqslant b(t)\,, \tag{3}$$

uniformly for a.e. $t \in I$. We will suppose that $a(t) < b(t)$ on a subset of I of positive measure. The following property will be imposed on a, b:

(A) *Taken any function $p \in L^\infty(I)$ such that $a(t) \leqslant p(t) \leqslant b(t)$ a.e. on I, the inequalities being strict on sets of positive measure, the problem*

$$x'' + p(t)x = 0 \tag{4}$$

$$x(0) - x(T) = x'(0) - x'(T) = 0 \tag{2}$$

has only the trivial solution.

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If the trivial solution is also the only solution of (4) (2) for $p \equiv a$ and for $p \equiv b$, the problem (1)(2) can be called nonresonant. Such problems have been extensively studied: let us mention, among many others, the works of LASOTA and OPIAL [9] (where a property analogous to (A) does already appear), MAWHIN and WARD [13], MAWHIN [11], HABETS and METZEN [5].

In this paper, we are interested mainly in the case where the problem (4)(2) has nontrivial solutions for $p \equiv a$ and for $p \equiv b$; this will be called a double resonance situation. This occurs, for instance, when, for some $n \in \boldsymbol{N}$, we have $a(t) \equiv n^2(2\pi/T)^2$ and $b(t) \equiv (n+1)^2(2\pi/T)^2$. For such problems, we will assume that a pair of so-called « Landesman-Lazer conditions » is satisfied. Such conditions, introduced by LANDESMAN and LAZER [8], appear in many studies of resonance problems: see, for instance, the papers of BRÉZIS and NIRENBERG [1], DE FIGUEIREDO [2], IANNACCI and NKASHAMA [6] and references therein. However, these papers always consider situations in which (4) (2) is admitted to have nontrivial solutions for $p \equiv a$ or for $p \equiv b$, but not for both. For different type of conditions, we refer to DING [3] and OMARI and ZANOLIN [15].

The proofs of our results are based on coincidence degree arguments (see MAWHIN [12] for the basic theory). The paper is organized as follows: in section 2, we associate to the functions a, b a positive semidefinite quadratic form which will play a major role in the proof of our main result. In section 3, preliminary lemmas are presented, which will be useful to produce a priori estimates for components of the solutions of (1) (2). The main theorem is then proved is section 4, whereas section 5 is devoted to the more specific case when 0 is the first eigenvalue of the operator $L\colon x \to -x'' - a(\cdot)x$. Finally, in section 6, we describe the resonant case to the left of the first eigenvalue.

2. – Property (A) and positive definiteness of a quadratic form.

To the functions a, b appearing in (3), we associate the linear operators $L_a, L_b\colon \operatorname{Dom} L_a = \operatorname{Dom} L_b \subset L^2(I) \to L^2(I)$ defined by

$$\operatorname{Dom} L_a = \operatorname{Dom} L_b = \{x \in H^2(I)\colon x \text{ verifies } (2)\},$$

$$L_a x = -x'' - a(\cdot)x,$$

$$L_b x = -x'' - b(\cdot)x,$$

where

$$H^2(I) = \{x\colon I \to \boldsymbol{R}\colon x \text{ and } x' \text{ are absolutely continuous on } I \text{ and } x'' \in L^2(I)\}.$$

It is well known (see, for instance, EASTHAM [4]) that L_a, L_b are self-adjoint operators with discrete spectrum whose eigenvalues, denoted respectively by α_i and β_i, are such that

$$\alpha_1 < \alpha_2 \leqslant \alpha_3 < \alpha_4 \leqslant \alpha_5 < \ldots \quad \text{and } \alpha_n \to \infty \text{ as } n \to \infty ,$$

$$\beta_1 < \beta_2 \leqslant \beta_3 < \beta_4 \leqslant \beta_5 < \ldots \quad \text{and } \beta_n \to \infty \text{ as } n \to \infty ,$$

any double eigenvalue appearing twice. Moreover, a comparison argument shows that

$$\beta_i \leqslant \alpha_i \quad \text{for } i = 1, 2, \ldots .$$

Using the spectra of L_a and L_b, we will relate property (A) to the positive semi-definiteness of a quadratic form, defined on the space

$$H^1_T = \{x \in H^1(I) : x(0) = x(T)\} ,$$

where

$$H^1(I) = \{x : I \to \mathbf{R} : x \text{ is absolutely continuous on } I \text{ and } x' \in L^2(I)\} .$$

PROPOSITION 1. – *The following conditions are equivalent:*

(i) *the functions a, b satisfy property (A);*

(ii) *either $\beta_1 \geqslant 0$ or, for some $\bar{n} \geqslant 1$, one has $\alpha_{\bar{n}} \leqslant 0 \leqslant \beta_{\bar{n}+1}$;*

(iii) *the space $L^2(I)$ can be decomposed into a direct sum $H^- \oplus H^+$, where either*

$$H^- = \{0\} , \quad H^+ = L^2(I) ,$$

in which case, for any $x \in H^1_T \setminus \operatorname{Ker} L_b$, we have

$$\int_I (x'^2 - bx^2) > 0$$

or, for some $\bar{n} \geqslant 1$,

$$H^- = \bigoplus_{j=1,\ldots,\bar{n}} \operatorname{Ker}(L_a - \alpha_j I) , \qquad H^+ = \overline{\bigoplus_{j \geqslant \bar{n}+1} \operatorname{Ker}(L_b - \beta_j I)} , \tag{5}$$

in which case, for any $x = \bar{x} + \tilde{x} \in H^1_T$, with $\bar{x} \in H^-$, $\tilde{x} \in H^+$, we have

$$B_{a,b}(x) := \int_I (\tilde{x}'^2 - b\tilde{x}^2) - \int_I (\bar{x}'^2 - a\bar{x}^2) \geqslant 0 ,$$

the inequality being strict when $x \notin \operatorname{Ker} L_a \oplus \operatorname{Ker} L_b$.

PROOF. – *a*) (i) $\Rightarrow$ (ii). Suppose (ii) is not true. Let n be the largest integer such that $\beta_n < 0$. Then, $\beta_{n+1} \geqslant 0$ and $\alpha_n > 0$. Defining $p_\lambda(t) = (1-\lambda)a(t) + \lambda b(t)$

and using the continuity of the eigenvalues with respect to λ, we conclude that there exists $\bar{\lambda} \in]0, 1[$ such that the n-th eigenvalue of $L_{p_{\bar{\lambda}}}$ (defined as L_a and L_b) is equal to 0. This contradicts (i).

b) (ii) $\Rightarrow$ (iii). In case $\beta_1 \geqslant 0$, take $H^- = \{0\}$, $H^+ = L^2(I)$. Then, for any $x \in (H^1_T \cap H^2(I)) \setminus \operatorname{Ker} L_b$, one has

$$\int_I (x'^2 - bx^2) = (L_b x, x)_{L^2} \geqslant \sigma \|x\|^2_{L^2}$$

for some $\sigma > 0$. Since $H^2(I)$ is dense in $H^1(I)$, it follows that, for any $x \in H^1_T \setminus \operatorname{Ker} L_b$,

$$\int_I (x'^2 - bx^2) \geqslant \sigma \|x\|^2_{L^2} > 0 \; . \tag{7}$$

In case $\alpha_{\bar{n}} \leqslant 0 \leqslant \beta_{\bar{n}+1}$, take H^-, H^+ as in (5). Since the dimension of H^- is finite and equal to the codimension of H^+, we will have $L^2(I) = H^- \oplus H^+$ if we can prove that $H^- \cap H^+ = \{0\}$ (see LAZER [10]).

Take $x \in H^- \cap H^+$; we then have

$$\alpha_{\bar{n}} \|x\|^2_{L^2} \geqslant (L_a x, x)^2_{L^2} = \int_I (x'^2 - ax^2) \geqslant \int_I (x'^2 - bx^2) = (L_b x, x) \geqslant \beta_{\bar{n}+1} \|x\|^2_{L^2} \, . \tag{8}$$

If $x \neq 0$, it then follows that $\alpha_{\bar{n}} = \beta_{\bar{n}+1} = 0$ and also that $(L_a x, x) = (L_b x, x) = 0$.

On H^-, the quadratic form $(L_a x, x)$ vanishes only if $x \in \operatorname{Ker} L_a$; similarly on $H^+ \cap H^2(I)$, the quadratic form $(L_b x, x)$ vanishes only if $x \in \operatorname{Ker} L_b$; since $x \in H^- \cap H^+$, we conclude that $x \in \operatorname{Ker} L_a \cap \operatorname{Ker} L_b$.

But, it is easy to show that $\operatorname{Ker} L_a \cap \operatorname{Ker} L_b = \{0\}$, because $a(t) < b(t)$ on a subset of positive measure. Hence, we have proved that $H^- \cap H^+ = \{0\}$, from which follows that $L^2(I) = H^- \oplus H^+$. Moreover, if $x \in H^1_T \cap H^2(I)$ is decomposed into $x = \bar{x} + \tilde{x}$ with $\bar{x} \in H^-$, $\tilde{x} \in H^+$, we have

$$B_{a,b}(x) = (L_b \tilde{x}, \tilde{x}) - (L_a \bar{x}, \bar{x}) \geqslant \beta_{\bar{n}+1} \|\tilde{x}\|^2_{L^2} - \alpha_{\bar{n}} \|\bar{x}\|^2_{L^2} \, ,$$

and

$$B_{a,b}(x) \geqslant \sigma \|x\|^2_{L^2} \tag{9}$$

for some $\sigma > 0$, unless $\tilde{x} \in \operatorname{Ker} L_b$ and $\bar{x} \in \operatorname{Ker} L_a$. By a density argument, (9) still holds for any $x \in H^1_T$, unless $\tilde{x} \in \operatorname{Ker} L_b$ and $\bar{x} \in \operatorname{Ker} L_a$.

c) (iii) $\Rightarrow$ (i). Take p as in property (A) and let x be a solution of (4) (2). Multiplying (4) par $(\bar{x} - \tilde{x})$ and integrating over I, one gets

$$0 = \int_I [\tilde{x}'^2 - \bar{x}'^2 + p(\bar{x}^2 - \tilde{x}^2)] \geqslant B_{a,b}(x) \, ,$$

the last inequality being strict unless $x \equiv 0$ (the functions $\bar{x}$ and $\tilde{x}$ cannot vanish simultaneously on a set of positive measure if x is a nontrivial solution of (4)). Condition (iii) then implies that $x \equiv 0$. ∎

3. – A priori estimates.

The main result of this paper is prepared by the lemmas given below. One of the purposes of this section is to provide a priori estimates for components of any solution to (1) (2). We want to distinguish between the component in $\operatorname{Ker} L_a$, the component in $\operatorname{Ker} L_b$ and a complementary component. Therefore, we introduce the subspace H^* such that $L^2(I) = \operatorname{Ker} L_a \oplus \operatorname{Ker} L_b \oplus H^*$ and, for $x \in L^2(I)$, we will write $x = x^a + x^b + x^*$ with $x^a \in \operatorname{Ker} L_a$, $x^b \in \operatorname{Ker} L_b$, $x^* \in H^*$.

LEMMA 1. – *Assume that a, b satisfy property (A). Then there exists a $\delta > 0$ such that, for every $x \in H^1_T$,*

$$B_{a,b}(x) \geqslant \delta \|x^*\|^2_{H^1}. \tag{10}$$

PROOF. – Notice first that $B_{a,b}(x) = B_{a,b}(x^*)$. Hence, if (10) does not hold, there exists a sequence (x_n) in H^* such that $\|x_n\|_{H^1} = 1$ and $B_{a,b}(x_n) \to 0$ for $n \to \infty$. Taking a subsequence, we can suppose that (x_n) converges weakly to some x in $H^1_T \cap H^*$. Then, using the decomposition $x_n = \bar{x}_n + \tilde{x}_n$ of Proposition 1, we assert that $(\bar{x}_n)$ converges strongly to $\bar{x}$ in $H^1(I)$, since all norms are equivalent on a finite dimensional space, and $(\tilde{x}_n)$ converges uniformly to $\tilde{x}$, since $C_0(I)$ is compactly imbedded in $H^1(I)$. Consequently,

$$\int_I (\tilde{x}_n')^2 \to \int_I \left(b\tilde{x}^2 + (\bar{x}')^2 - a\bar{x}^2\right) \qquad \text{for } n \to \infty \tag{11}$$

and, by the weak lower semi-continuity of the L^2-norm, we obtain $B_{a,b}(x) \leqslant 0$. By Proposition 1, this implies that $B_{a,b}(x) = 0$ and that $x \in \operatorname{Ker} L_a \oplus \operatorname{Ker} L_b$. As we also have $x \in H^*$, we conclude that $x = 0$ and, from (11), that $\int_I (\tilde{x}_n')^2 \to 0$ for $n \to \infty$. This, combined with the uniform convergence of $(\tilde{x}_n)$ and the strong convergence in $H^1(I)$ of $(\bar{x}_n)$, shows that $\|x_n\|_{H^1} \to 0$ as $n \to \infty$, contradicting the fact that $\|x_n\|_{H^1} = 1$. ∎

The next lemma shows that, if $\operatorname{Ker} L_a = \{0\}$ (resp. $\operatorname{Ker} L_b = \{0\}$) property (A) is stable with respect to small perturbations of a (resp. b).

LEMMA 2. – *Let the functions a, b satisfy property (A). If $\operatorname{Ker} L_a = \{0\}$, there exists $\varepsilon > 0$ such that the functions $a - \varepsilon$, b still satisfy property (A). Symmetrically, if $\operatorname{Ker} L_b = \{0\}$, there is an $\varepsilon > 0$ such that the functions $a, b + \varepsilon$ satisfy (A).*

PROOF. – By Proposition 1, condition (ii) holds. If $\operatorname{Ker} L_a = \{0\}$, we have either $\alpha_1 > 0$ or $\alpha_{\bar{n}} < 0 \leqslant \beta_{\bar{n}+1}$. If $\alpha_1 > 0$, then $\beta_1 \geqslant 0$ by (ii) in Proposition 1. Hence, property (A) holds for any functions $\tilde{a}, b$ with $\tilde{a} \leqslant b$.

If $\alpha_{\bar{n}} < 0 \leqslant \beta_{\bar{n}+1}$, we deduce from (9) that $B_{a-\varepsilon, b}(x) \geqslant 0$, for any $0 < \varepsilon \leqslant |\alpha_{\bar{n}}|$ the inequality being strict, except when $x \in \operatorname{Ker} L_{a-\varepsilon} \oplus \operatorname{Ker} L_b$. The conclusion then results again from Proposition 1. ■

In the sequel, it will be useful to decompose the function g in (1) as the sum of a « pseudo-linear » function and a bounded function. For this purpose, we introduce the following hypothesis:

(H) *If* $\operatorname{Ker} L_a \neq \{0\}$, *there exists a function* $H \in L^2(I)$ *such that, for every* $x \in \mathbf{R}$,

$$\operatorname{sgn} x[g(t, x) - a(t)x] \geqslant -H(t) \tag{12}$$

If $\operatorname{Ker} L_b \neq \{0\}$, *there exists a function* $K \in L^2(I)$ *such that, for every* $x \in \mathbf{R}$,

$$\operatorname{sgn} x[b(t)x - g(t, x)] \geqslant -K(t)\,. \tag{13}$$

LEMMA 3. – *Let the Carathéodory function g verify* (3), *hypothesis* (H) *and be such that, for all $R > 0$, there exists $h_R \in L^2(I)$ such that*

$$|g(t, x)| \leqslant h_R(t) \qquad \text{for } |x| \leqslant R\,. \tag{14}$$

Assume that the functions a, b satisfy property (A). Then, there exist functions $\bar{a}, \bar{b} \in L^\infty(I)$, with $\bar{a}(t) \leqslant a(t)$, $b(t) \leqslant \bar{b}(t)$ a.e. on I, $\bar{a}, \bar{b}$ still satisfying (A), (H) and we can write

$$g(t, x) = x\gamma(t, x) + h(t, x) \tag{15}$$

where

$$\bar{a}(t) \leqslant \gamma(t, x) \leqslant \bar{b}(t) \tag{16}$$

for a.e. $t \in I$, all $x \in \mathbf{R}$, and $h(\cdot, \cdot)$ is a function satisfying Carathéodory conditions and such that

$$|h(t, x)| \leqslant \hat{h}(t) \tag{17}$$

for a.e. $t \in I$, where $\hat{h} \in L^2(I)$.

PROOF. – By (3) and (14), given $\varepsilon > 0$, we can find $R > 0$ such that, for $x \geqslant 0$,

$$g(t, x) \geqslant (a - \varepsilon)x - h_R(t) - |a - \varepsilon| R\,;$$

a similar inequality holding for $x \leq 0$. By Lemma 2, if $\operatorname{Ker} L_a = \{0\}$, choosing $\varepsilon > 0$ small enough, the functions $a - \varepsilon, b$ satisfy property (A). In this case, we take $\bar{a} \equiv a - \varepsilon$; we notice that (12) holds with a replaced by $\bar{a}$ and H replaced by $h_R + |a - \varepsilon| R$.

If $\operatorname{Ker} L_a \neq \{0\}$, we take $\bar{a} \equiv a$. The function $\bar{b}$ is defined in a similar way. The functions $\bar{a}, \bar{b}$ satisfy property (A) and we can now use, in all cases, the inequalities (12), (13) if we replace therein a by $\bar{a}$, b by $\bar{b}$. We introduce the function $\delta\colon \mathbf{R}^3 \to \mathbf{R}$ defined by

$$\delta(u, x, v) = \begin{cases} \min\{u, v\} & \text{if } x < \min\{u, v\} \\ \max\{u, v\} & \text{if } x > \max\{u, v\} \\ x & \text{otherwise}. \end{cases}$$

Let us then define $\gamma(t, x)$ and $h(t, x)$ by

$$\gamma(t, x) = \begin{cases} \dfrac{1}{x}\, \delta(\bar{a}(t)x, g(t, x), \bar{b}(t)x) & \text{for } x \neq 0, \\ \bar{a}(t) & \text{for } x = 0, \end{cases}$$

$$h(t, x) = g(t, x) - x\gamma(t, x);$$

the function $\gamma(t, \cdot)$ need not be continuous at 0, but the function $x \to x\gamma(t, x)$ will be (for a.e. $t \in I$); so will be the function $h(t, \cdot)$.

It is clear from the definition of $\gamma(t, x)$ that (16) holds. On the other hand, (17) will result from (12) (13) (a, b having been replaced by $\bar{a}, \bar{b}$) with $\hat{h}(t) = \max\{|H(t)|, |K(t)|\}$. ∎

In the sequel, we can assume, without loss of generality, that $\bar{a} \equiv a$, $\bar{b} \equiv b$. Using Lemma 3, we can obtain an estimate of the component in H^* of any solution of (1) (2).

LEMMA 4. – *Assume that g, a, b satisfy the hypotheses of Lemma 3. Then, there exists a constant $C_1 > 0$ such that, for every solution x of (1) (2), we have*

$$\|x^*\|_{H^1}^2 \leq C_1 \|x\|_{L^2}, \tag{18}$$

x^* being the component of x in H^*.

PROOF. – We decompose g as in (15); multiplying (1) by $(\bar{x} - \tilde{x})$, where $\bar{x}, \tilde{x}$ are as in Proposition 1, and integrating, we get

$$0 = \int_I [(\tilde{x}')^2 - (\bar{x}')^2 + (\bar{x}^2 - \tilde{x}^2)\gamma(t, x) + (\bar{x} - \tilde{x})h(t, x)] =$$

$$= B_{a,b}(x) + \int_I \big[\bar{x}^2(\gamma(t, x) - a(t)) + \tilde{x}^2(b(t) - \gamma(t, x)) + (\bar{x} - \tilde{x})h(t, x)\big].$$

By the remark preceding this lemma, we can assume, without loss of generality, that $a(t)\leq\gamma(t,x)\leq b(t)$. Hence, by Lemma 1 and inequality (17), we have

$$0\geq\delta\|x^*\|^2_{H^1}-C\|x\|_{L^2}\|\hat{h}\|_{L^2},$$

for some constant C. The conclusion then follows immediately. ■

Consider, as in Lemma 1, the decomposition $L^2(I)=\operatorname{Ker}L_a\oplus\operatorname{Ker}L_b\oplus H^*$; as above, for $x\in L^2(I)$, we write $x=x^a+x^b+x^*$. The next lemma will be useful to estimate x^b (or x^a) when x is a solution of (1) (2).

LEMMA 5. – *Let a, b satisfy property (A) and (s_n) be a sequence in $L^2(I)$ converging weakly to a, and such that $a(t)\leq s_n(t)\leq b(t)$ a.e. on I. Let x_n be a solution of*

$$x_n''+s_n(t)x_n+h(t,x_n)=0\,, \tag{19}$$

$$x_n(0)-x_n(T)=x_n'(0)-x_n'(T)=0\,, \tag{2}$$

where h is a Carathéodory function satisfying (17). *Then, there exist constants C_2, C_3 such that, for any $n\in\mathbf{N}$,*

$$\|x_n^b\|^2\leq C_2\|x_n\|_{L^2}+C_3\,. \tag{20}$$

REMARK. – Any norm can be used for x_n^b, since $\operatorname{Ker}L_b$ is finite dimensional.

PROOF. – As in Lemma 4, we multiply (19) by $(\bar{x}_n-\tilde{x}_n)$ and integrate, which yields

$$0=B_{a,b}(x_n)+\int_I\big[\bar{x}_n^2\big(s_n(t)-a(t)\big)+\tilde{x}_n^2\big(b(t)-s_n(t)\big)+(\bar{x}_n-\tilde{x}_n)h(t,x_n)\big]\,dt\,.$$

Since $B_{a,b}(x_n)\geq0$ and $s_n(t)\geq a(t)$ a.e. on I, it follows that

$$\int_I\tilde{x}_n^2\big(b(t)-s_n(t)\big)\,dt\leq C\|x_n\|_{L^2}\|\hat{h}\|_{L^2},$$

for some constant C. Isolating the component of $\tilde{x}_n$ in $\operatorname{Ker}L_b$, we obtain

$$\int_I(x_n^b)^2\big(b(t)-s_n(t)\big)\,dt\leq C\|x_n\|_{L^2}\|\hat{h}\|_{L^2}+\|b-a\|_{L^\infty}\int_I\big[|\tilde{x}_n-x_n^b|^2+2|x_n^b||\tilde{x}_n-x_n^b|\big]\leq$$
$$\leq C\|x_n\|_{L^2}\|\hat{h}\|_{L^2}+\|b-a\|_{L^\infty}\big[\|\tilde{x}_n-x_n^b\|^2_{L^2}+2\|x_n^b\|_{L^2}\|\tilde{x}_n-x_n^b\|_{L^2}\big]\,. \tag{21}$$

We claim that there exists $\eta>0$ such that, for n sufficiently large, we have

$$\int_I(x_n^b)^2\big(b(t)-s_n(t)\big)\,dt\geq\eta\|x_n^b\|^2 \tag{22}$$

Indeed, if this would not be the case, we could find a subsequence, which will still be denoted by (x_n), such that

$$\int_I \left(\frac{x_n^b}{\|x_n^b\|}\right)^2 (b(t) - s_n(t))\, dt \to 0 \qquad \text{as } n \to \infty . \tag{23}$$

Since $\operatorname{Ker} L_b$ is finite dimensional, we can assume, without loss of generality, that $x_n^b/\|x_n^b\|$ converges uniformly towards a non-zero function $y \in \operatorname{Ker} L_b$. By (23), it follows that

$$\int_I y^2(t)(b(t) - a(t))\, dt = 0 ,$$

Since $a(t) < b(t)$ on a subset of positive measure, we conclude that y vanishes on such a subset. But since y belongs to $\operatorname{Ker} L_b$, this would imply $y \equiv 0$, leading to a contradiction. Consequently, we deduce from (21) and (22) that, for n sufficiently large,

$$\eta \|x_n^b\|^2 \leqslant C \|x_n\|_{L^2} \|\hat{h}\|_{L^2} + \|b - a\|_{L^\infty} [\|\tilde{x}_n - x_n^b\|_{L^2}^2 + 2 \|x_n^b\|_{L^2} \|\tilde{x}_n - x_n^b\|_{L^2}] .$$

Since $\tilde{x}_n - \bar{x}_n^b$ is a component of x_n^*, the result follows from Lemma 4 and easy computations. ∎

A result similar to Lemma 5 can of course be proven when the sequence (s_n) converges weakly to b, with $a(t) \leqslant s_n(t) \leqslant b(t)$ a.e. on I. In that case, we obtain a relation of the form

$$\|x_n^a\|^2 \leqslant C_2 \|x_n\|_{L^2} + C_3 .$$

4. – The main result.

As explained above, we want to prove the existence of solutions for the problem (1) (2) in particular when a double resonance does occur, i.e. when the problem (4) (2) has non trivial solutions for $p \equiv a$ and for $p \equiv b$. For that purpose, we introduce the following « Landesman-Lazer conditions ».

(LL) *For every* $u \in \operatorname{Ker} L_a \setminus \{0\}$,

$$0 < \int_{u>0} \liminf_{x \to +\infty} [g(t, x) - a(t)x] u(t)\, dt + \int_{u<0} \limsup_{x \to -\infty} [g(t, x) - a(t)x] u(t)\, dt$$

For every $v \in \operatorname{Ker} L_b \setminus \{0\}$,

$$0 < \int_{v>0} \liminf_{x \to +\infty} [b(t)x - g(t, x)] v(t)\, dt + \int_{v<0} \limsup_{x \to -\infty} [b(t)x - g(t, x)] v(t)\, dt .$$

THEOREM 1. – *Let the function g verify* (3), *hypothesis* (H) *and the Landesman-Lazer condition* (LL). *Moreover, assume that, for all $R > 0$, there exists $h_R \in L^2(I)$ such that*

$$|g(t, x)| \leqslant h_R(t) \quad \textit{for } |x| \leqslant \boldsymbol{R} .$$

Assume that the functions a, b satisfy property (A). *Then, problem* (1) (2) *has at least one solution.*

PROOF. – By classical arguments from the theory of the coincidence topological degree (see Mawhin [12]), the result will be proven if we can find an a priori bound for the solutions of the problems

$$x'' + \lambda p(t)x + (1-\lambda)g(t, x) = 0 \tag{24}$$

$$x(0) - x(T) = x'(0) - x'(T) = 0 \tag{2}$$

where $\lambda \in]0, 1[$ and $p(t) = (a(t) + b(t))/2$ (notice that, with this definition of p, the problem (4) (2) has only the trivial solution, by property (A)).

As the function

$$g_\lambda(t, x) = \lambda p(t)x + (1-\lambda)g(t, x)$$

verifies the same hypotheses as the function g, with the same functions a, b in the inequalities corresponding to (3), the estimate (18) of Lemma 4 will hold for the solutions of (24) (2), with a constant C_1 independent of λ.

By contradiction, suppose that there exists sequences (x_n), (λ_n) with $\|x_n\|_{H^1} \to \infty$, such that x_n is a solution of (24) (2) with $\lambda = \lambda_n$. Set $u_n = x_n/\|x_n\|_{H^1}$; u_n then satisfies the equation

$$u''_n + [\lambda_n p(t) + (1-\lambda_n)\gamma(t, x_n)]u_n + (1-\lambda_n)\frac{h(t, x_n)}{\|x_n\|_{H^1}} = 0 , \tag{25}$$

where we have used the decomposition (15). It is easily shown that the sequence (u_n) is bounded in $H^2(I)$; therefore, passing to a subsequence, we can assume that (u_n) converges weakly in $H^2(I)$ and strongly in $C^1(I)$ to a certain map u. Moreover, letting

$$s_n(t) = \lambda_n p(t) + (1-\lambda_n)\gamma(t, x_n(t)) ,$$

we can suppose that (s_n) converges weakly in $L^2(I)$ to some function s and, by the weak closure of the set $\{\sigma \in L^2(I)\colon a(t) \leqslant \sigma(t) \leqslant b(t) \text{ a.e. on } I\}$, we have that $a(t) \leqslant s(t) \leqslant b(t)$ a.e. on I. Hence, passing to the weak limit in (25), we obtain

$$u'' + s(t)u = 0$$

$$u(0) - u(T) = u'(0) - u'(T) = 0$$

and, by property (A), we conclude that either $s \equiv a$, $s \equiv b$, or $u \equiv 0$. But this last possibility is excluded since $\|u_n\|_{H^1} = 1$ and u_n converges strongly to u in $C^1(I)$. So, we must have either $s \equiv a$ and $\operatorname{Ker} L_a \neq \{0\}$ or $s \equiv b$ and $\operatorname{Ker} L_b \neq \{0\}$; let us consider the first case, the other one being treated in a similar way. Multiplying (25) by $v_n = x_n^a / \|x_n\|_{H^1}$ and integrating we get, using the fact that $v_n'' = -a(t)v_n$,

$$\lambda_n \int_I (p(t) - a(t))\, u_n v_n \, dt + (1 - \lambda_n) \int_I \left[(\gamma(t, x_n) - a(t))\, u_n v_n + \frac{h(t, x_n)}{\|x_n\|_{H^1}} v_n \right] dt = 0 . \tag{26}$$

Since s_n converges weakly to a, we can apply Lemma 5; by the estimate (20) of Lemma 5 and the estimate (18) of Lemma 4, it is easy to show that $v_n \to u$ as $n \to \infty$ and, consequently,

$$\lim_{n\to\infty} \int_I (p(t) - a(t))\, u_n v_n \, dt = \int_I (p(t) - a(t))\, u^2 \, dt .$$

So, for n large enough, the first term of the sum in (26) is positive and, therefore, we have the inequality

$$\int_I \left[(\gamma(t, x_n) - a(t))\, u_n v_n + \frac{h(t, x_n)}{\|x_n\|_{H^1}} v_n \right] dt < 0 ,$$

which implies

$$\limsup_{n\to\infty} \int_I (g(t, x_n) - a(t) x_n)\, v_n \, dt \leqslant 0 \tag{27}$$

We want to apply Fatou's Lemma; for that purpose, we need to find a function $\gamma \in L^1(I)$ such that

$$\big(g(t, x_n(t)) - a(t) x_n(t)\big)\, v_n(t) \geqslant \gamma(t) \qquad \text{a.e. on } I .$$

This function will be obtained in two steps. First, since (v_n) is uniformly bounded and since h satisfies (17), there exists a function $\gamma_1 \in L^1(I)$ such that

$$h(t, x_n(t))\, v_n(t) \geqslant \gamma_1(t) \qquad \text{a.e. on } I . \tag{28}$$

Secondly, using the relations

$$2 x_n x_n^a \geqslant -(x_n - x_n^a)^2 = -(x_n^* + x_n^b)^2 \geqslant -2[(x_n^*)^2 + (x_n^b)^2]$$

and the fact that $\gamma(t, x_n(t)) \geqslant a(t)$ a.e. on I, we obtain

$$\big(\gamma(t, x_n(t)) - a(t)\big)x_n(t)v_n(t) \geqslant -\frac{1}{\|x_n\|_{H^1}}\big(\gamma(t, x_n(t)) - a(t)\big)\big[(x_n^*)^2 + (x_n^b)^2\big] .$$

Using the estimates of Lemmas 4 and 5, it is then an easy matter to find a function $\gamma_2 \in L^1(I)$ such that

$$(29) \qquad \big(\gamma(t, x_n(t)) - a(t)\big)x_n(t)v_n(t) \geqslant \gamma_2(t) \qquad \text{a.e. on } I .$$

Combining (28) and (29), we can now apply Fatou's Lemma to the integral in (27). The sequence (x_n) converges pointwise to $+\infty$ on the set $\{t \in I: u(t) > 0\}$, while it converges to $-\infty$ on the set $\{t \in I: u(t) < 0\}$. The application of Fatou's Lemma gives

$$\int_{u>0} \liminf_{n\to+\infty} [g(t, x_n) - a(t)x_n]u(t)\,dt + \int_{u<0} \limsup_{n\to-\infty} [g(t, x_n) - a(t)x_n]u(t)\,dt \leqslant 0 ,$$

contradicting the first of the two Landesman-Lazer conditions in (LL). ■

5. – Resonance at the first two eigenvalues.

When $\alpha_1 = 0$, the first part of condition (LL) can be replaced by a different hypothesis, using the fact that $\operatorname{Ker} L_a$ is of dimension 1 and consists of functions of constant sign.

THEOREM 2. – *Assume that α_1, the first eigenvalue of L_a, is* 0. *Let* $\operatorname{Ker} L_a = \{k\varphi: k \in \mathbf{R}\}$, *where φ is a positive function. Let the function g verify the same hypotheses as in Theorem* 1, *except that the first part of condition* (LL) *is replaced by the following assumption: there exists functions $h^+, h^- \in L^2(I)$ and a number $d > 0$ such that*

$$(30) \qquad g(t, x) - a(t)x \geqslant h^+(t) \quad \text{for } x \geqslant d , \quad \text{a.e. } t \in I$$

$$(31) \qquad g(t, x) - a(t)x \leqslant h^-(t) \quad \text{for } x \leqslant -d , \quad \text{a.e. } t \in I$$

$$(32) \qquad \int_I h^+(t)\varphi(t)\,dt \geqslant 0 ,$$

$$(33) \qquad \int_I h^-(t)\varphi(t)\,dt \leqslant 0 .$$

Then, problem (1) (2) *has at least one solution.*

PROOF. – The proof begins as the proof of Theorem 1. Defining u_n and s_n in the same way, it is shown that (u_n) converges strongly in $C^1(I)$ to some $u \in \operatorname{Ker} L_a \cup$

$\cup \operatorname{Ker} L_b$ and that (s_n) converges weakly in $L^2(I)$ to some function s. If $a(t) < s(t)$ on a subset of positive measure, the proof is unchanged with respect to that of Theorem 1. The only case that needs to be treated here is the case where $s \equiv a$. In that case, multiply (25) by $\|x_n\|_{H^1} u$ and integrate over I. Since $u''(t) = -a(t)u$, this yields

$$\lambda_n \int_I (p(t) - a(t)) x_n(t) u(t)\, dt + (1 - \lambda_n) \int_I [g(t, x_n) - a(t) x_n] u(t)\, dt = 0 . \tag{34}$$

Since $\operatorname{Ker} L_a \setminus \{0\}$ consists of functions of constant sign, let us assume, for instance, that $u(t) > 0$, $\forall t \in I$, the other case being treated in a similar way. We then have, for n sufficiently large, $u_n(t) > 0$ and $x_n(t) \geqslant d$, $\forall t \in I$. Consequently, using (30), (32), we have, for n sufficiently large,

$$\int_I [g(t, x_n) - a(t) x_n] u(t)\, dt \geqslant \int_I h^+(t) u(t)\, dt \geqslant 0 .$$

On the other hand, since $p(t) > a(t)$ on a subset of positive measure, we have

$$\lim_{n \to \infty} \int_I (p(t) - a(t))\, u_n(t) u(t)\, dt = \int_I (p(t) - a(t))\, u^2(t)\, dt > 0 .$$

Hence, the first term of the sum in (34) is positive for n large, while the second is non negative, leading to a contradiction. ∎

REMARKS. – 1) If $h^+(t) = -e(t) + \Gamma$, $h^-(t) \equiv -e(t) + \gamma$, the conditions (30), (31) become

$$g(t, x) - a(t)x + e(t) \geqslant \Gamma \quad \text{for } x \geqslant d , \quad \text{a.e. } t \in I ,$$

$$g(t, x) - a(t)x + e(t) \leqslant \gamma \quad \text{for } x \leqslant -d , \quad \text{a.e. } t \in I ,$$

and (32) (33) write

$$\gamma \leqslant \frac{1}{T} \int_I e(t) \varphi(t)\, dt \leqslant \Gamma .$$

In the particular case where $a(t) \equiv 0$, this corresponds to a result of Mawhin and Ward [14], $\operatorname{Ker} L_a$ consisting then of constant functions.

2) When $\gamma = \Gamma = 0$, the above assumptions become

$$x[g(t, x) - a(t)x + e(t)] \geqslant 0 \quad \text{for } |x| \geqslant d , \quad \text{a.e. } t \in I , \tag{35}$$

$$\int_I e(t) \varphi(t)\, dt = 0 . \tag{36}$$

Similar conditions have been proposed by IANNACCI, NKASHAMA and WARD [7] for elliptic problems. Notice that this situation also covers the linear resonant case, when $g(t, x) = a(t)x - e(t)$.

3) A counterexample of IANNACCI and NKASHAMA [6] shows that conditions (35) (36) are not sufficient to guarantee the existence of solutions when 0 is an eigenvalue of L_a other than the first one.

4) Conditions (30) (31) (32) (33) could be replaced by the following conditions: there exist functions $k^+, k^- \in L^2(I)$ and a number $d > 0$ such that

$$x[g(t, x) - a(t)x] \geqslant k^+(t) \quad \text{for } x \geqslant d\,, \quad \text{a.e. } t \in I\,,$$

$$x[g(t, x) - a(t)x] \geqslant k^-(t) \quad \text{for } x \leqslant -d\,, \quad \text{a.e. } t \in I\,,$$

$$\int_I k^+(t)\,dt > 0\,,$$

$$\int_I k^-(t)\,dt > 0\,.$$

6. – Resonance to the left of the first eigenvalue.

For the sake of completeness, we consider in this section a case of one-sided resonance, assuming only the existence of a function $b \in L^\infty(I)$ such that

$$\lim_{|x| \to \infty} \frac{g(t, x)}{x} \leqslant b(t)\,, \tag{37}$$

uniformly for a.e. $t \in I$.

Let us introduce the following assumption, which is the analogous, in the setting of the above one-sided condition (37), of property (A).

(A') *Taken any function $p \in L^\infty(I)$ such that $p(t) \leqslant b(t)$, a.e. on I, the inequality being strict on a set of positive measure, the problem*

$$x'' + p(t)x = 0$$

$$x(0) - x(T) = x'(0) - x'(T) = 0$$

has only the trivial solution.

Notice that g is no longer required to grow at most linearly. As in Proposition 1 and Lemma 1, one can prove:

PROPOSITION 2. – *The following conditions are equivalent:*

(i) *the function* b *satisfy property* (A');

(ii) $\beta_1 \geqslant 0$;

(iii) *for any* $x \in H^1_T$, *one has*

$$B_b(x) := \int_I (x'^2 - bx^2) \geqslant 0 \, ,$$

the inequality being strict when $x \notin \operatorname{Ker} L_b$.

LEMMA 6. – *Assume that* b *satisfies property* (A'). *Let* H^* *be such that* $L^2(I) = \operatorname{Ker} L_b \oplus H^*$. *Then there exists a* $\delta > 0$ *such that, for every* $x = x^b + x^* \in H^1_T$, *with* $x^b \in \operatorname{Ker} L_b$, $x^* \in H^*$, *one has*

$$B_b(x) \geqslant \delta \|x^*\|^2_{H^1} \, . \tag{38}$$

We will also assume that the second part of condition (H) holds. For convenience, we rewrite it under the following form:

(H') *If* $\operatorname{Ker} L_b \neq \{0\}$, *there exist functions* $k^+, k^- \in L^2(I)$ *and a number* $d \geqslant 0$ *such that*

$$b(t)x - g(t,x) \geqslant k^+(t) \quad \text{for } x \geqslant d \, , \quad \text{a.e. } t \in I \, , \tag{39}$$

$$b(t)x - g(t,x) \leqslant k^-(t) \quad \text{for } x \leqslant -d \, , \quad \text{a.e. } t \in I \, . \tag{40}$$

A slight modification of the proof of Lemma 3 gives the following (see [2, Lemma 2]).

LEMMA 7. – *Let the Carathéodory function* g *verify* (37), *hypothesis* (H') *and be such that, for all* $R > 0$, *there exists* $h_R \in L^2(I)$ *such that*

$$|g(t,x)| \leqslant h_R(t) \quad \text{for } |x| \leqslant R \, .$$

Assume that the function b *satisfies property* (A'). *Then, there exists* $\bar{b} \in L^\infty(I)$, *with* $b(t) \leqslant \bar{b}(t)$ *a.e. on* I, $\bar{b}$ *still satisfying* (A'), (H') *and we can write*

$$g(t,x) = x\gamma(t,x) + h(t,x)$$

where

$$\gamma(t,x) \leqslant \bar{b}(t)$$

for a.e. $t \in I$, all $x \in \boldsymbol{R}$, and $h(\cdot,\cdot)$ is a function satisfying Carathéodory conditions and such that

$$|h(t, x)| \leqslant \hat{h}(t)$$

for a.e. $t \in I$, where $\hat{h} \in L^2(I)$.

We are now able to prove the analogous of Theorems 1 and 2.

THEOREM 3. – *Let the function g verify* (37), *hypotheses (H') and the Landesman-Lazer condition*

(LL′) *For every $\psi \in \operatorname{Ker} L_b \setminus \{0\}$ such that $\psi(t) > 0$ a.e. on I,*

$$\int_I \limsup_{x \to -\infty} [b(t)x - g(t, x)]\psi(t)\, dt < 0 < \int_I \liminf_{x \to +\infty} [b(t)x - g(t, x)]\psi(t)\, dt .$$

Moreover, assume that, for all $R > 0$, there exists $h_R \in L^2(I)$ such that

$$|g(t, x)| \leqslant h_R(t) \qquad \text{for } |x| \leqslant R .$$

If b satisfies property (A'), then problem (1) (2) *has at least one solution.*

THEOREM 4. – *Assume that β_1, the first eigenvalue of L_b, is* 0. *Let* $\operatorname{Ker} L_b = \{k\psi : k \in \boldsymbol{R}\}$, *where ψ is a positive function. Let the function g verify the same hypotheses as in Theorem* 3, *except that the condition* (LL′) *is replaced by*

$$\int_I k^-(t)\, \psi(t)\, dt \leqslant 0 \leqslant \int_I k^+(t)\, \psi(t)\, dt . \tag{41}$$

Then, problem (1) (2) *has at least one solution.*

PROOFS. – The proofs start in the same way as in the one of Theorem 1, with the only differences that one choses $p(t) = b(t) - 1$ and, by Lemma 7, where we assume without loss of generality that $b \equiv \bar{b}$,

$$\gamma(t, x) \leqslant b(t)$$

for all $x \in \boldsymbol{R}$, a.e. $t \in I$.

Once we arrive at (25), multiplying by $(-u_n)$ and integrating one has

$$\int_I (u_n'^2 - s_n(t) u_n^2) = \int_I (1 - \lambda_n) \frac{h(t, x_n)}{\|x_n\|_{H^1}}\, u_n ,$$

which converges to zero as $n \to \infty$. This implies, by (38), that $u_n^* \xrightarrow{H^1} 0$. Passing to a subsequence, Ker L_b being finite dimensional, we can assume $u_n^b \to u \in \operatorname{Ker} L_b$, and hence $u_n \xrightarrow{H^1} u \neq 0$. Multiplying (25) by $\|x_n\|_{H^1} u$ and integrating, by the choice of p, one obtains

$$\lambda_n \int_I x_n(t)\, u(t)\, dt + (1 - \lambda_n) \int_I \big(b(t) x_n - g(t, x_n)\big)\, u(t)\, dt = 0 \tag{42}$$

Since $u \in \operatorname{Ker} L_b \setminus \{0\}$ and β_1 is the first eigenvalue, u has constant sign. Let us suppose $u(t) < 0$ a.e. in I, the other case being treated similarly. By the strong converge $u_n \to u$, we have that, for n sufficiently large, $x_n(t) < -d$ for a.e. $t \in I$. By assumption (H'), a contradiction is obtained from (42) and (41), proving Theorem 4. On the other hand, from (42) one has

$$\limsup_{n \to \infty} \int_I \big(b(t) x_n - g(t, x_n)\big)\, u(t)\, dt \leqslant 0 \,. \tag{43}$$

By assumption (H'), the Fatou's Lemma can be applied to (43), and this leads to a contradiction with (LL$'$), thus proving Theorem 3.

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