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Periodic Solutions of Second Order Ordinary Differential Equations.

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Sunto. – *Si studia l'esistenza di soluzioni periodiche per l'equazione differenziale:*

$$-x'' = g(t, x(t))$$

quando la nonlinearità g interagisce con i primi due autovalori dell'operatore differenziale.

1. – Introduction.

In this paper we study the existence of periodic solutions of second order differential equations of the form

$$-x''(t) = g(t, x(t)),$$

where the nonlinearity g interacts, roughly speaking, with the first two eigenvalues of the differential operator. More precisely we impose at the first eigenvalue a resonance condition of the type introduced by Ahmad, Lazer and Paul [2], and a nonuniform nonresonance condition at the second eigenvalue.

We prove the existence of solutions using a variational approach. Our geometrical setting is the one of the Saddle Point Theorem of Rabinowitz [12]. It is obtained by splitting naturally the considered space into the eigenspace corresponding to the first eigenvalue and its orthogonal space.

The main technical problem lies in the verification of the Palais-Smale condition. This is done in the line of an argument introduced in [11] in order to find an a priori bound for the solutions of a Liénard type equation, and later developed in [3, 4].

In order to obtain the Palais-Smale condition, we have to require $g(t, x)$ to be bounded below for x positive and bounded above for x negative.

Our theorem then gives a partial answer to a problem raised by Mawhin [8]. However, the problem of eliminating the boundedness assumptions on g remains still open.

2. – The main result.

We consider the following one dimensional periodic problem.

$$(1) \qquad \begin{cases} x''(t) + g(t, x(t)) = 0 \,, \\ x(0) - x(T) = x'(0) - x'(T) = 0 \,. \end{cases}$$

Being $T > 0$, we set $I = [0, T]$. The function $g\colon I \times \mathbf{R} \to \mathbf{R}$ is of Carathéodory type, i.e.

(*a*) $g(\cdot, x)$ is measurable for every $x \in \mathbf{R}$,

(*b*) $g(t, \cdot)$ is continuous for a.e. $t \in I$,

and

(*c*) for every $R > 0$ there exists a $k_R \in L^1(I; \mathbf{R})$ such that

$$|g(t, x)| \leq k_R(t)$$

for all $|x| \leq R$ and a.e. $t \in I$.

Statements concerning the variable t will always be intended to be true a.e.

We define the function G as follows:

$$G(t, x) = \int_0^x g(t, s)\, ds \,.$$

THEOREM 1. – *Assume the following conditions.*

(i) *There exists* $K \in L^1(I; \mathbf{R})$ *such that*

$$g(t, x) \geq -K(t) \qquad \text{for } x \geq 0 \,,$$

$$g(t, x) \leq K(t) \qquad \text{for } x \leq 0 \,.$$

(ii) $\lim\limits_{|x| \to \infty} \int_I G(t, x)\, dt = +\infty.$

(iii) $\limsup\limits_{x \to \pm\infty} g(t, x)/x \leq \Gamma_\pm(t) \leq (2\pi/T)^2$

uniformly in t, where $\Gamma_\pm \in L^1(I)$ *and the set*

$$S = \{t \in [0, T] \colon \Gamma_+(t) < (2\pi/T)^2 \text{ and } \Gamma_-(t) < (2\pi/T)^2\}$$

has positive measure.

Then problem (1) *has a solution.*

REMARK 1. – Condition (i) generalizes some assumptions which are frequently made in studying a problem like (1). It is in particular satisfied if one of the following holds.

(i)′ $\exists \varrho > 0\colon\ g(t, x)x \geqslant 0\ \ (|x| \geqslant \varrho)$.

(i)″ $\exists k \in L^1(I; \boldsymbol{R})\colon\ |g(t, x)| \leqslant k(t)\ \ (x \in \boldsymbol{R})$.

(i)‴ $g(t, \cdot)$ is increasing.

In the following, we will denote by H^1_T the Hilbert space of absolutely continuous functions $x\colon I \to \boldsymbol{R}$ such that $x(0) = x(T)$ and whose derivatives are square integrable, with the usual norm

$$\|x\| = \Big\{ \int_I [|x(t)|^2 + |x'(t)|^2]\, dt \Big\}^{1/2}.$$

Define the functional $f\colon H^1_T \to \boldsymbol{R}$, associated to problem (1), by

$$f(x) = \int_I [(1/2)|x'(t)|^2 - G(t, x(t))]\, dt.$$

It is well known that solving problem (1) is equivalent to finding a critical point of the functional f.

REMARK 2. – Condition (ii) imposes the functional f to be coercive on the space of constant functions, i.e. on the eigenspace associated to the first eigenvalue of the operator $(-x'')$ under periodic boundary conditions. An assumption of this type was first introduced by Ahmad, Lazer and Paul in [2].

Condition (iii) means, roughly speaking, that the nonlinearity stays asymptotically below the second eigenvalue. Assumptions of this type are usually called «nonuniform nonresonance conditions», and were first introduced by Mawhin and Ward [9, 7].

PROOF OF THEOREM 1. – We will use the Saddle Point Theorem of Rabinowitz (cf. [13]). Let us write $H^1_T = H_0 \oplus H_1$, where H_0 is the space of constant functions, and H_1 is the space of functions having mean value zero. First of all, we will show that there exists an $R > 0$ such that, setting

$$S_0 = \{x \in H_0\colon \|x\| = R\}$$

one has

$$\max_{S_0} f < \inf_{H_1} f . \tag{2}$$

Indeed, proceeding as in [7, Lemma 2.2], from (iii) it is possible to find a $\delta > 0$ such that, for every $x_1 \in H_1$,

$$\int_I [|x_1'(t)|^2 - \Gamma_+(t)|x_1^+(t)|^2 - \Gamma_-(t)|x_1^-(t)|^2]\, dt \geq 2\delta \|x_1\|^2 ,$$

where, as usual, $x^+ = \max\{x, 0\}$ and $x^- = \max\{-x, 0\}$.

Moreover, (iii) implies that there is a $K_\delta \in L^1(I; \mathbf{R})$ such that

$$G(t, x) \leq (1/2)(\Gamma_+(t) + \delta) x^2 + K_\delta(t)$$

for every $x \geq 0$, and

$$G(t, x) \leq (1/2)(\Gamma_-(t) + \delta) x^2 + K_\delta(t)$$

for every $x \leq 0$.

Then, for $x_1 \in H_1$, we have

$$f(x_1) = \int_I (1/2)|x_1'(t)|^2\, dt - \int_{\{x_1 \geq 0\}} G(t, x_1(t))\, dt - \int_{\{x_1 < 0\}} G(t, x_1(t))\, dt \geq$$

$$\geq (1/2) \int_I [|x_1'(t)|^2 - (\Gamma_+(t) + \delta)|x_1^+(t)|^2 - (\Gamma_-(t) + \delta)|x_1^-(t)|^2 - 2K_\delta(t)]\, dt \geq$$

$$\geq (1/2)[2\delta \|x_1\|^2 - \delta \|x_1\|_{L^2}^2] - \|K_\delta\|_{L^1} .$$

This implies

$$\inf_{H_1} f \geq - \|K_\delta\|_{L^1} .$$

On the other hand, if $x_0 \in H_0$,

$$f(x_0) = - \int_I G(t, x_0)\, dt .$$

By (ii), there exists an $R > 0$ such that, if $\|x_0\| = R$, then $f(x_0) < - \|K_\delta\|_{L^1} - 1$. So (2) is satisfied.

To conclude the proof, we need to show that the functional f satisfies the Palais-Smale condition. By means of [13], it will be sufficient to show that, if (x_n) is a sequence in H such that $f(x_n)$ is bounded and $f'(x_n) \to 0$, then $\|x_n\|$ has a bounded subsequence.

First of all we show that there is a subsequence of (x_n) which is eigher uniformly bounded from above or from below. In fact, if this is not true, there must exist a subsequence, still denoted by (x_n), such that

(3) $$m_n = \min x_n \to -\infty$$

and

(4) $$M_n = \max x_n \to +\infty .$$

Then there surely exist α_n, β_n, γ_n and δ_n in $[0, 2T]$ such that, extending by T-periodicity x_n over $[0, 2T]$, one has

$$x_n(\alpha_n) = x_n(\beta_n) = 0 ,$$

$$x_n(t) > 0 \qquad \text{for } t \in]\alpha_n, \beta_n[,$$

$$M_n = \max\{x_n(t) \colon t \in [\alpha_n, \beta_n]\} ,$$

and

$$x_n(\gamma_n) = x_n(\delta_n) = 0 ,$$

$$x_n(t) < 0 \qquad \text{for } t \in]\gamma_n, \delta_n[,$$

$$m_n = \min\{x_n(t) \colon t \in [\gamma_n, \delta_n]\} .$$

Using a technical lemma proved in [4], we can say that, for a further subsequence (x_n), either there is a $\varrho_- > 0$ such that

(5) $$\int_{\gamma_n}^{\delta_n} [|x_n'(t)|^2 - \Gamma_-(t)|x_n(t)|^2]\, dt > (\varrho_-/2)\int_{\gamma_n}^{\delta_n} |x_n'(t)|^2\, dt$$

or there is a $\varrho_+ > 0$ such that

(6) $$\int_{\alpha_n}^{\beta_n} [|x_n'(t)|^2 - \Gamma_+(t)|x_n(t)|^2]\, dt > (\varrho_+/2)\int_{\alpha_n}^{\beta_n} |x_n'(t)|^2\, dt$$

for n sufficiently large.

Assume for example that (6) holds. Define the function y_n as follows:

$$y_n(t) = x_n(t) \qquad \text{when } t \in [\alpha_n, \beta_n] ,$$

$$y_n(t) = 0 \qquad \text{otherwise} .$$

It is possible to see that $y_n \in H_T^1$ (cf. [5]).

Since $f'(x_n) \to 0$, for some constant $C_1 > 0$ we have that

$$|\langle f'(x_n), v\rangle| \leqslant C_1\|v\| \tag{7}$$

for every $v \in H_T^1$. Fix $\varepsilon < \varrho_+\pi^2/2T^2$. From (iii) and (c), there exists a $\tilde{k}_\varepsilon \in L^1(I; \mathbf{R})$ such that

$$g(t, x)x \leqslant (\Gamma_+(t) + \varepsilon)x^2 + \tilde{k}_\varepsilon(t)$$

for all $x \geqslant 0$ and a.e. $t \in I$. Hence, by (7), the definition of y_n, and (6), one has

$$\begin{aligned} C_1\|y_n\| \geqslant \langle f'(x_n), y_n\rangle &= \int_{\alpha_n}^{\beta_n} [|x_n'(t)|^2 - g(t, x_n(t))x_n(t)]\,dt \geqslant \\ &\geqslant \int_{\alpha_n}^{\beta_n} [|x_n'(t)|^2 - (\Gamma_+(t) + \varepsilon)|x_n(t)|^2 - \tilde{k}_\varepsilon(t)]\,dt \geqslant \\ &\geqslant (\varrho_+/2)\int_{\alpha_n}^{\beta_n} |x_n'(t)|^2\,dt - \varepsilon\left(\frac{\beta_n - \alpha_n}{\pi}\right)^2 \int_{\alpha_n}^{\beta_n} |x_n'(t)|^2\,dt - \|\tilde{k}_\varepsilon\|_{L^1} \geqslant \\ &\geqslant C_2\int_{\alpha_n}^{\beta_n} |x_n'(t)|^2\,dt - \|\tilde{k}_\varepsilon\|_{L^1}, \end{aligned}$$

where C_2 is a positive constant. It follows from the Poincaré inequality that

$$\int_{\alpha_n}^{\beta_n} |x_n'(t)|^2\,dt \leqslant C_3,$$

for a certain constant C_3. But this implies

$$M_n \leqslant (TC_3)^{1/2},$$

which is in contradiction with (4). Analogously, if (5) holds, we get a contradiction with (3).

We then proved that (x_n) is uniformly bounded either from above of from below. Let us suppose for example that (x_n) is uniformly bounded from above by a constant M. Taking $v \equiv 1$ in (7) we get

$$\left|\int_I g(t, x_n(t))\,dt\right| \leqslant C_1\sqrt{T}. \tag{8}$$

On the other hand, since $x_n(t) \leq M$ for every n and a.e. $t \in I$, one has, by (i), the compactness of $[0, M]$ and (c), that

$$\int_{\{g\geq 0\}} g(t, x_n(t))\, dt = \int_{\{g\geq 0\}\cap\{0\leq x_n\leq M\}} g + \int_{\{g\geq 0\}\cap\{x_n<0\}} g \leq C_4 . \tag{9}$$

Combining (8) and (9) one easily gets

$$\int_I |g(t, x_n(t))|\, dt \leq C_5 . \tag{10}$$

The same conclusion (10) can be obtained when supposing (x_n) uniformly bounded from below.

Writing $x_n = x_0^n + x_1^n$, with $x_0^n \in H_0$ and $x_1^n \in H_1$, and recalling the Sobolev inequality

$$\|x_1^n\|_{L^\infty} \leq \sqrt{(T/12)}\, \|x_n'\|_{L^2} ,$$

from (7) and (10) we get

$$C_1 \|x_1^n\| \geq \langle f'(x_n), x_1^n\rangle =$$

$$= \int_I [|x_n'(t)|^2 - g(t, x_n(t))\, x_1^n(t)]\, dt \geq \|x_n'\|_{L^2}^2 - C_5 \sqrt{(T/12)}\, \|x_n'\|_{L^2} .$$

From the Wirtinger inequality we can conclude that

$$\|x_n'\|_{L^2} \leq C_6 . \tag{11}$$

Suppose by contradiction that $\|x_n\|$ is not bounded. By (11), this implies that there is a subsequence, still denoted (x_n), such that either

$$M_n = \max x_n \to -\infty \tag{12}$$

or

$$m_n \triangleq \min x_n \to +\infty . \tag{13}$$

In case (12) holds, by the definition of G, hypothesis (i) and (11), for n large we have:

$$\int_I G(t, x_n(t))\, dt = \int_I \Big[G(t, M_n) + \int_{M_n}^{x_n(t)} g(t, s)\, ds\Big]\, dt \geq \tag{14}$$

$$\geq \int_I G(t, M_n)\, dt - \int_I (M_n - x_n(t))\, K(t)\, dt \geq \int_I G(t, M_n)\, dt - C_7 ,$$

which tends to $+\infty$ by hypotheses (ii).

Recall now that $f(x_n)$ is supposed to be bounded, i.e. there is a constant C_8 such that

$$\left| \int_I [(1/2)|x_n'(t)|^2 - G(t, x_n(t))]\, dt \right| < C_8 .$$

This is in contradiction with (11) and (14). An analogous contradiction is obtained when (13) holds. The Theorem of Rabinowitz can thus be applied, to achieve the proof.

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