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Periodic Solutions of Asymptotically Positively Homogeneous Differential Equations

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1. INTRODUCTION

First studies of periodic solutions for a differential equation

$$\ddot{x} + c\dot{x} + g(x) = e(t),$$

where g is asymptotically linear in some sense, are due to W. S. Loud [20] and A. C. Lazer [16]. This was the starting point of a vast literature on the Liénard equation

$$\ddot{x} + f(x)\dot{x} + g(t, x) = e(t) \tag{1}$$

and its special case, the Duffing equation

$$\ddot{x} + c\dot{x} + g(t, x) = e(t). \tag{2}$$

One can mention for example the papers by R. Reissig [31], M. Martelli [21], J. Mawhin and J. R. Ward [27], J. Mawhin [25], C. Fabry [6], and the literature therein. In these papers, the asymptotic behaviour of the nonlinearity g is controlled through inequalities such as

$$a(t) \leqslant \liminf_{|x| \to \infty} \frac{g(t, x)}{x} \leqslant \limsup_{|x| \to \infty} \frac{g(t, x)}{x} \leqslant b(t). \tag{3}$$

These tend to keep away the quotient $g(t, x)/x$ from the spectrum of the linear operator $Lx = -\ddot{x}$ as $|x| \to \infty$. Closely related results can be found in J. Mawhin [23], J. Mawhin and J. R. Ward [28], P. Omari and F. Zanolin [30]. Similar results for systems have been worked out in A. C. Lazer and D. A. Sanchez [17], P. Habets and M. N. Nkashama [12], for a Rayleigh equation in R. Reissig [32], and for third order equations in G. Villari [33], O. C. Ezeilo and M. N. Nkashama [5]. See also the references therein.

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A major generalization was considered in E. N. Dancer [2, 3] and S. Fučik [9, 10]. There, existence of solutions for the equation

$$\ddot{x} + g(x) = e(t) \tag{4}$$

is investigated when the function g is asymptotically positively homogeneous, i.e.,

$$\lim_{x \to +\infty} \frac{g(x)}{x} = \mu, \qquad \lim_{x \to -\infty} \frac{g(x)}{x} = \nu. \tag{5}$$

Noticing that the quotient $g(x)/x$ could vary from one eigenvalue of L as $x \to -\infty$ to the next one as $x \to +\infty$, or even could cross eigenvalues of L, S. Fučik called the function g a "jumping nonlinearity." These authors considered the positively homogeneous equation

$$\ddot{x} + \mu x_+ - \nu x_- = 0, \tag{6}$$

where $x_+ = \max(x, 0)$ and $x_- = \max(-x, 0)$ and introduced the set K, known as Fučik spectrum, of points $(\mu, \nu) \in \mathbb{R}^2$ such that (6) has a non-zero periodic solution. Basically they proved that in $(\mu, \nu) \notin K$ and g satisfies (5), Eq. (4) has a periodic solution. Later, condition (5) has been generalized for a Duffing equation (2) using assumptions of the type (3). In P. Habets and G. Metzen [11], the asymptotic values of the quotient $g(t, x)/x$ are controlled by the inequalities

$$a(t) \leqslant \liminf_{x \to +\infty} \frac{g(t, x)}{x} \leqslant \limsup_{x \to +\infty} \frac{g(t, x)}{x} \leqslant b(t),$$

$$c(t) \leqslant \liminf_{x \to -\infty} \frac{g(t, x)}{x} \leqslant \limsup_{x \to -\infty} \frac{g(t, x)}{x} \leqslant d(t),$$

together with a condition called property P. This property replaces the assumption $(\mu, \nu) \notin K$ by imposing that zero is the only periodic solution of the positively homogeneous equation

$$\ddot{x} + c\dot{x} + p(t)\, x_+ - q(t)\, x_- = 0,$$

whenever $a(t) \leqslant p(t) \leqslant b(t)$, $c(t) \leqslant q(t) \leqslant d(t)$. Such a property P appears already more or less implicitly in A. Lasota and Z. Opial [18] and S. Invernizzi [15]. Recent results along these lines are in P. Drabek and S. Invernizzi [4], R. Iannacci, M. N. Nkashama, P. Omari, and F. Zanolin [14]. In the case of one-sided growth restrictions, see also P. Omari, G. Villari, and F. Zanolin [29] and L. Fernandes and F. Zanolin [7].

A similar phenomenon was observed by A. Fonda and F. Zanolin [8] for the Liénard equation

$$\ddot{x} + f(x)\,\dot{x} + g(x) = e(t). \tag{7}$$

Assuming (5) as well as

$$\lim_{x \to +\infty} f(x) = p, \qquad \lim_{x \to -\infty} f(x) = q,$$

they indicate a set K in the (μ, ν, p, q) space which generalizes the Fučik spectrum and is such that if $(\mu, \nu, p, q) \notin K$, the equation (7) has at least one periodic solution.

The original motivation of our paper was to prove the existence of periodic solutions for (7) using a property P so as to weaken the above conditions on f and g. Our purpose was also to apply these ideas to other problems such as the Rayleigh equation

$$\ddot{x} + f(t, \dot{x}) + g(t, x) = e(t) \tag{8}$$

and the third order equation

$$\dddot{x} + a\ddot{x} + b\dot{x} + g(t, x) = e(t). \tag{9}$$

The paper is organized as follows. In Section 2, we consider a general first order equation in $\mathbb{R}^n$

$$\dot{x} = F(t, x). \tag{10}$$

We describe what we mean by F being asymptotically positively homogeneous and check this property in applications. Section 3 is devoted to property P and the main existence theorem for periodic solutions of (10). In Section 4, we investigate property P for equations in $\mathbb{R}^2$ using phase plane methods. This applies to Liénard and Rayleigh equations. Section 5 studies property P for equations in $\mathbb{R}^3$ using L^2-estimates on the solutions and their derivatives. In Section 6, we deduce some existence theorems for Liénard equation (1), Rayleigh equation (8) and the third order equation (9). These contain and generalize results in P. Drabek and S. Invernizzi [4], P. Habets and G. Metzen [11], A. Fonda and F. Zanolin [8], and O. C. Ezeilo and M. N. Nkashama [5].

2. The Main Problem

2.1. Consider the periodic boundary value problem

$$\begin{aligned} \dot{x} &= F(t, x) \\ x(0) &= x(2\pi), \end{aligned} \tag{11}$$

where $F\colon [0, 2\pi] \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function.

The following assumption expresses the fact that F is asymptotically positively homogeneous.

Assumption H. (i) Let

$$G(t, x, u) = G_0(t, x) + G_*(t, x)u, \qquad (t, x, u) \in [0, 2\pi] \times \mathbb{R}^n \times \mathbb{R}^p,$$

be a continuous function which is positively homogeneous in x, i.e.,

$$\forall (t, x, u) \in [0, 2\pi] \times \mathbb{R}^n \times \mathbb{R}^p, \qquad \forall \lambda > 0, \qquad G(t, \lambda x, u) = \lambda G(t, x, u);$$

(ii) let

$$\alpha\colon [0, 2\pi] \to \mathbb{R}^p \qquad \text{and} \qquad \beta\colon [0, 2\pi] \to \mathbb{R}^p$$

be continuous functions and

(iii) assume that for any $\varepsilon > 0$, there exist $\gamma > 0$ and a continuous function $u(t, x)$ such that for every $(t, x) \in [0, 2\pi] \times \mathbb{R}^n$ one has $u(t, x) \in [\alpha(t) - \varepsilon\mathbf{e}, \beta(t) + \varepsilon\mathbf{e}]$,

where $\mathbf{e} \in \mathbb{R}^p$ is the vector with all components equal to 1, and

$$|G(t, x, u(t, x)) - F(t, x)| \leqslant \gamma.$$

This assumption holds true in several important applications.

2.2. *Application* 1. Consider the system of equations

$$\begin{aligned} \dot{x} &= y - f(t, x), \\ \dot{y} &= e(t) - g(t, x), \end{aligned} \tag{12}$$

where f, g, and e are continuous functions defined for $t \in [0, 2\pi]$, $x \in \mathbb{R}$. Recall that the Liénard equation

$$\ddot{x} + h(x)\dot{x} + g(t, x) = e(t)$$

can be written in such a form.

In this application, we assume the following.

Assumption A1. There exist continuous functions a, b, c, d, p, q, r, s such that the following inequalities hold uniformly in t:

$$a(t) \leqslant \liminf_{x \to +\infty} \frac{f(t, x)}{x} \leqslant \limsup_{x \to +\infty} \frac{f(t, x)}{x} \leqslant b(t),$$

$$c(t) \leqslant \liminf_{x \to -\infty} \frac{f(t, x)}{x} \leqslant \limsup_{x \to -\infty} \frac{f(t, x)}{x} \leqslant d(t),$$

$$p(t) \leqslant \liminf_{x \to +\infty} \frac{g(t, x)}{x} \leqslant \limsup_{x \to +\infty} \frac{g(t, x)}{x} \leqslant q(t),$$

$$r(t) \leqslant \liminf_{x \to -\infty} \frac{g(t, x)}{x} \leqslant \limsup_{x \to -\infty} \frac{g(t, x)}{x} \leqslant s(t).$$

Let us show that the function

$$F(t, x, y) = (y - f(t, x), e(t) - g(t, x))$$

satisfies Assumption H.

We shall first introduce the functions

$$\begin{aligned} \delta(a, x, b) &= a, && \text{if} \quad x \leqslant a, \\ &= x, && \text{if} \quad x \in (a, b), \\ &= b, && \text{if} \quad x \geqslant b, \end{aligned}$$

and

$$\begin{aligned} \varphi(x) &= 0, && \text{if} \quad x \in [0, 1], \\ &= x - 1, && \text{if} \quad x \in (1, 2], \\ &= 1, && \text{if} \quad x > 2. \end{aligned}$$

With these notations and for any $\varepsilon > 0$, we write (12) as

$$\begin{aligned} \dot{x} &= y - u_1(t, x) x_+ + u_2(t, x) x_- + h_1(t, x), \\ \dot{y} &= -u_3(t, x) x_+ + u_4(t, x) x_- + h_2(t, x), \end{aligned}$$

where

$$x_+ = \max(x, 0), \qquad x_- = \max(-x, 0),$$

$$u_1(t, x) = \delta\left(a(t) - \varepsilon, \frac{f(t, x)}{x} \varphi(|x|), b(t) + \varepsilon\right),$$

$$u_2(t,x)=\delta\left(c(t)-\varepsilon, \frac{f(t,x)}{x}\varphi(|x|), d(t)+\varepsilon\right),$$

$$u_3(t,x)=\delta\left(p(t)-\varepsilon, \frac{g(t,x)}{x}\varphi(|x|), q(t)+\varepsilon\right),$$

$$u_4(t,x)=\delta\left(r(t)-\varepsilon, \frac{g(t,x)}{x}\varphi(|x|), s(t)+\varepsilon\right).$$

Notice that we can choose R large enough, so that if $x \geqslant R$ one has

$$a(t)-\varepsilon \leqslant \frac{f(t,x)}{x} \leqslant b(t)+\varepsilon,$$

$$p(t)-\varepsilon \leqslant \frac{g(t,x)}{x} \leqslant q(t)+\varepsilon.$$

Similarly, if $x \leqslant -R$, one has

$$c(t)-\varepsilon \leqslant \frac{f(t,x)}{x} \leqslant d(t)+\varepsilon,$$

$$r(t)-\varepsilon \leqslant \frac{g(t,x)}{x} \leqslant s(t)+\varepsilon.$$

If we define

$$u=(u_1, u_2, u_3, u_4),$$
$$G(x,y,u)=(y-u_1x_+ + u_2x_-, -u_3x_+ + u_4x_-),$$
$$\alpha(t)=(a(t), c(t), p(t), r(t)),$$
$$\beta(t)=(b(t), d(t), q(t), s(t)),$$

it is clear that

$$\alpha(t)-\varepsilon\mathbf{e} \leqslant u(t,x) \leqslant \beta(t)+\varepsilon\mathbf{e}$$

and that the function

$$h(t,x)=F(t,x,y)-G(x,y,u(t,x))$$
$$=(-f+u_1x_+ - u_2x_-, e-g+u_3x_+ - u_4x_-)$$

is bounded as it is continuous with compact support.

2.3. *Application* 2. The Rayleigh equation

$$\ddot{x}+f(t,\dot{x})+g(t,x)=e(t)$$

can be written in vector form

$$\dot{x} = y, \qquad \dot{y} = e(t) - g(t, x) - f(t, y). \tag{13}$$

As in Application 1, we assume that the functions f, g, and e are continuous functions defined for $t \in [0, 2\pi]$, $x \in \mathbb{R}$ and $y \in \mathbb{R}$. We also assume that Assumption A1 holds.

It is then easy to see that the function

$$F(t, x, y) = (y, e(t) - g(t, x) - f(t, y))$$

verifies Assumption H with

$$G(x, y, u) = (y, -u_1 y_+ + u_2 y_- - u_3 x_+ + u_4 x_-),$$

$$u(t, x, y) = (u_1(t, y), u_2(t, y), u_3(t, x), u_4(t, x)),$$

where the functions u_i, α, and β are defined as in Application 1.

2.4. *Application* 3. The third order equation

$$\dddot{x} + a\ddot{x} + b\dot{x} + g(t, x) = e(t, x, \dot{x}, \ddot{x})$$

can be written as

$$\dot{x} = y, \qquad \dot{y} = z, \qquad \dot{z} = e(t, x, y, z) - g(t, x) - by - az. \tag{14}$$

We assume that the functions g and e are continuous functions defined for $t \in [0, 2\pi]$, $x \in \mathbb{R}$, $y \in \mathbb{R}$, $z \in \mathbb{R}$, and that the following condition holds.

Assumption A3. There exist continuous functions p, q, r, s such that the following inequalities hold uniformly in t

$$p(t) \leqslant \liminf_{x \to +\infty} \frac{g(t, x)}{x} \leqslant \limsup_{x \to +\infty} \frac{g(t, x)}{x} \leqslant q(t),$$

$$r(t) \leqslant \liminf_{x \to -\infty} \frac{g(t, x)}{x} \leqslant \limsup_{x \to -\infty} \frac{g(t, x)}{x} \leqslant s(t),$$

and there exist $\delta_0 > 0$, $\Delta_0 > 0$ such that for any $t \in [0, 2\pi]$, $(x, y, z) \in \mathbb{R}^3$, one has

$$|e(t, x, y, z)| \leqslant \delta_0 + \Delta_0(|x| + |y| + |z|).$$

Let us prove that the function

$$F(t, x, y, z) = (y, z, e(t, x, y, z) - g(t, x) - by - az)$$

verifies Assumption H.

For any $\varepsilon > 0$, we write (14) as

$$\dot{x} = y, \qquad \dot{y} = z,$$
$$\dot{z} = u_1(t, x, y, z)(|x| + |y| + |z|) - u_2(t, x)x_+ + u_3(t, x)x_- - by - az + h(t, x, y, z),$$

where

$$u_1(t, x, y, z) = e(t, x, y, z)\,\varphi\left(\frac{\varepsilon}{\delta_0}(|x| + |y| + |z|)\right)\Big/(|x| + |y| + |z|),$$

$$u_2(t, x) = \delta\left(p(t) - \varepsilon, \frac{g(t, x)}{x}\varphi(|x|), q(t) + \varepsilon\right),$$

$$u_3(t, x) = \delta\left(r(t) - \varepsilon, \frac{g(t, x)}{x}\varphi(|x|), s(t) + \varepsilon\right).$$

If we define

$$G(x, y, z, u) = (y, z, u_1(|x| + |y| + |z|) - u_2 x_+ + u_3 x_- - by - az),$$
$$\alpha(t) = (-\Delta_0, p(t), r(t)),$$
$$\beta(t) = (\Delta_0, q(t), s(t)),$$

it is clear that

$$\alpha(t) - \varepsilon \mathbf{e} \leqslant u(t, x, y, z) \leqslant \beta(t) + \varepsilon \mathbf{e}.$$

Moreover

$$h(t, x, y, z) = F(t, x, y, z) - G(x, y, z, u(t, x, y, z)) = (0, 0, h_1(t, x, y, z) + h_2(t, x))$$

is bounded since the functions

$$h_1(t, x, y, z) = e(t, x, y, z) - u_1(t, x, y, z)(|x| + |y| + |z|)$$
$$h_2(t, x) = -g(t, x) + u_2(t, x)x_+ - u_3(t, x)x_-$$

are continuous functions with compact support.

3. Property P and the Main Theorem

3.1. Definition. Given functions $G(t, x, u)$, $\alpha(t)$, and $\beta(t)$ as in Assumption H, we say that the triplet (G, α, β) has *property* P if for any $u \in L^2$ such that

$$\forall t \in [0, 2\pi], \qquad u(t) \in [\alpha(t), \beta(t)],$$

zero is the only solution of the boundary value problem

$$\dot{x} = G(t, x, u(t))$$
$$x(0) = x(2\pi).$$

3.2. In order to prove our main theorem, we need the following lemma.

LEMMA 1. *Let $G(t, x, u)$, $\alpha(t)$, and $\beta(t)$ be as in Assumption H. If (G, α, β) has property P, then there exists $\varepsilon > 0$ such that $(G, \alpha - \varepsilon\mathbf{e}, \beta + \varepsilon\mathbf{e})$ has property P.*

Proof. Suppose the contrary is true. Then for any $n \in \mathbb{N}$, there exists $u_n \in L^2$ such that

$$\forall t \in [0, 2\pi], \qquad u_n(t) \in \left[\alpha(t) - \frac{1}{n}\mathbf{e}, \beta(t) + \frac{1}{n}\mathbf{e}\right]$$

and $x_n \in H^1$, $x_n \neq 0$ such that

$$\dot{x}_n = G(t, x_n(t), u_n(t)) \tag{15}$$
$$x_n(0) = x_n(2\pi). \tag{16}$$

The positive homogeneity of G in x allows us to choose x_n such that $\|x_n\|_{H^1} = 1$.

As $C \subset H^1$, the x_n are uniformly bounded in $\|\cdot\|_\infty$. The u_n are also uniformly bounded. Hence from (15) it follows that the x_n are equicontinuous. Going to a subsequence, we can then suppose $x_n \xrightarrow{C} x$.

Likewise, since u_n is a bounded sequence in L^2 we can suppose $u_n \xrightarrow{L^2} u$, for some $u \in L^2$. It follows that

$$G(\cdot, x_n, u_n) \xrightarrow{L^2} G(\cdot, x, u).$$

Indeed, for any $\varphi \in L^2$ we have

$$\int_0^{2\pi} [G(t, x_n(t), u_n(t)) - G(t, x(t), u(t))]\, \varphi(t)\, dt$$
$$= \int_0^{2\pi} [G_0(t, x_n(t)) - G_0(t, x(t))]\, \varphi(t)\, dt + \int_0^{2\pi} [G_*(t, x_n(t))$$
$$- G_*(t, x(t))]\, u_n(t)\, \varphi(t)\, dt + \int_0^{2\pi} G_*(t, x(t))(u_n(t) - u(t))\, \varphi(t)\, dt.$$

From Lebesgue dominated convergence theorem, the two first terms go to zero. As $u_n \xrightarrow{L^2} u$, the same holds true for the third one.

Taking the weak limit of (15) in L^2, and the limit in (16) we obtain

$$\begin{aligned} \dot{x}(t) &= G(t, x(t), u(t)) \\ x(0) &= x(2\pi). \end{aligned} \tag{17}$$

As $u_n \xrightarrow{L^2} u$, it is easy to see that for each $i = 1, ..., p$ and almost every $t \in [0, 2\pi]$, one has

$$\alpha_i(t) \leqslant \liminf_{n \to \infty} u_{ni}(t) \leqslant \limsup_{n \to \infty} u_{ni}(t) \leqslant \beta_i(t).$$

Hence changing u on a set of measure zero, we can assume

$$\forall t \in [0, 2\pi], \qquad u(t) \in [\alpha(t), \beta(t)].$$

As (G, α, β) has Property P, we deduce from (17) that $x \equiv 0$.

On the other hand, from the positive homogeneity of G in x, we can find $K > 0$ such that

$$\forall (t, x) \in [0, 2\pi] \times \mathbb{R}^n, \qquad \forall u \in [\alpha(t) - \mathbf{e}, \beta(t) + \mathbf{e}], \qquad |G(t, x, u)| \leqslant K\,|x|.$$

Hence we can write

$$\begin{aligned} 1 &= \|x_n\|_{H^1}^2 = \|x_n\|_{L^2}^2 + \|\dot{x}_n\|_{L^2}^2 \\ &\leqslant 2\pi\, \|x_n\|_\infty^2 + \int_0^{2\pi} G^2(tx_n(t), u_n(t))\, dt \leqslant 2\pi(1+K)^2\, \|x_n\|_\infty^2 \end{aligned}$$

which implies

$$x = \lim x_n \neq 0.$$

This is a contradiction. ∎

3.3. To prove the existence of solutions of (11), we shall apply coincidence degree theory [24]. It is clear that Leray–Schauder's degree [19] could be used at the expense of reformulating the problem as a fixed point problem.

Given functions $F(t, x)$ and $G(t, x, u)$ as in Assumption H, and a continuous function $u_0\colon [0, 2\pi] \to \mathbb{R}^p$, we shall use the following notations:

$$\begin{aligned} &\text{Dom}\, L = \{x \in C^1 \mid x(0) = x(2\pi)\}; \\ &L\ : \text{Dom}\, L \to C,\ x \to x'; \\ &N_1 : C \to C,\ x \to F(\cdot, x); \\ &N_0 : C \to C,\ x \to G(\cdot, x, u_0). \end{aligned} \tag{18}$$

It is clear that N_0 and N_1 are L-compact on bounded subsets of C and that L is a linear Fredholm map of index zero.

THEOREM 1. *Assume*:

(i) *F satisfies Assumption H*;

(ii) *the triplet (G, α, β) has property P*;

(iii) *for some continuous function u_0: $[0, 2\pi] \to \mathbb{R}^p$ such that*

$$\forall t \in [0, 2\pi], \qquad u_0(t) \in [\alpha(t), \beta(t)],$$

one has

$$d(L - N_0, \Omega_0) \neq 0,$$

where N_0 is defined as in (18) *and $\Omega_0 = \{x \in C \mid \|x\|_\infty < 1\}$.*

Then the problem (11) *has at least one solution.*

Proof. We consider the homotopy

$$Lx - \lambda N_1 x - (1-\lambda) N_0 x = 0$$

which corresponds to the boundary value problem

$$\begin{aligned} \dot{x} &= \lambda F(t, x) + (1-\lambda) G(t, x, u_0(t)), \\ x(0) &= x(2\pi). \end{aligned} \tag{19}$$

For any $\lambda \in [0, 1]$, the function

$$\Psi(t, x, \lambda) = \lambda F(t, x) + (1-\lambda) G(t, x, u_0(t))$$

verifies Assumption H with the same functions G, α, β. Indeed, let us fix $\varepsilon > 0$. There exist $\gamma > 0$ and $u(t, x) \in [\alpha(t) - \varepsilon\mathbf{e}, \beta(t) + \varepsilon\mathbf{e}]$ such that the function

$$H(t, x) = F(t, x) - G(t, x, u(t, x))$$

verifies $|H(t, x)| \leqslant \gamma$. Further, we can write

$$\Psi(t, x, \lambda) = G(t, x, \lambda u(t, x) + (1-\lambda) u_0(t)) + \lambda H(t, x)$$

which is such that

$$\lambda u(t, x) + (1-\lambda) u_0(t) \in [\alpha(t) - \varepsilon\mathbf{e}, \beta(t) + \varepsilon\mathbf{e}]$$

and

$$|\lambda H(t, x)| \leqslant \gamma.$$

By Lemma 1, we can choose $\varepsilon > 0$ such that $(G, \alpha - \varepsilon\mathbf{e}, \beta + \varepsilon\mathbf{e})$ has property P. Let us prove that there exists an a priori bound for the solutions of (19). Assume on the contrary that there exist sequences of real numbers (λ_n) in $[0, 1]$ and of functions x_n such that $\forall n \in \mathbb{N}$, $\|x_n\|_\infty \geqslant n$, and

$$\dot{x}_n = \lambda_n F(t, x_n) + (1 - \lambda_n) G(t, x_n, u_0(t)),$$
$$x_n(0) = x_n(2\pi).$$

Let $v_n = x_n / \|x_n\|_\infty$. The functions v_n are solutions of

$$\begin{aligned} \dot{v}_n(t) &= G(t, v_n(t), \lambda_n u(t, x_n(t)) + (1 - \lambda_n) u_0(t)) + \lambda_n \frac{H(t, x_n(t))}{\|x_n\|_\infty} \\ v_n(0) &= v_n(2\pi). \end{aligned} \tag{20}$$

Clearly the $\dot{v}_n$ are uniformly bounded. Hence the v_n are bounded in H^1 and, going to subsequences, we can assume $v_n \xrightarrow{H^1} v \in H^1$, $v_n \xrightarrow{C} v \neq 0$, and $\lambda_n \to \lambda \in [0, 1]$. Likewise, as the functions $u_n(t) = \lambda_n u(t, x_n(t)) + (1 - \lambda_n) u_0(t)$ are uniformly bounded, we can assume $u_n \xrightarrow{L^2} u \in L^2$. Going to the limit in (20) we obtain

$$\dot{v} = G(t, v(t), u(t))$$
$$v(0) = v(2\pi).$$

As in Lemma 1, it is clear that changing u on a set of measure zero, we have

$$\forall t \in [0, 2\pi], \qquad u(t) \in [\alpha(t) - \varepsilon\mathbf{e}, \beta(t) + \varepsilon\mathbf{e}]$$

and, from property P, that $v = 0$, which contradicts $v_n \xrightarrow{C} v$. Hence, there is a constant c such that, for any λ and any soution x of (19),

$$\|x\|_\infty \leqslant c.$$

By invariance of the degree with respect to an homotopy and excision, one has

$$d(L - N_1, \Omega_1) = d(L - N_0, \Omega_1) = d(L - N_0, \Omega_0) \neq 0,$$

where

$$\Omega_1 = \{x \in C \mid \|x\|_\infty < c + 1\}.$$

Hence, there exists $x \in \bar{\Omega}_1$ such that $Lx = N_1 x$; i.e., the problem (11) has at least one solution. ∎

COROLLARY 1. *Assume:*

(i) *F satisfies Assumption H;*

(ii) *the triplet (G, α, β) has property P;*

(iii) *for some continuous function u_0: $[0, 2\pi] \to \mathbb{R}^p$ such that*

$$\forall t \in [0, 2\pi], \qquad u_0(t) \in [\alpha(t), \beta(t)],$$

the function $G(t, x, u_0(t))$ is linear in x.

Then the problem (1) *has at least one solution.*

The proof follows from the observation that property P implies $L - N_0$ is one to one and therefore that $d(L - N_0, \Omega_0) \neq 0$.

4. THE PROPERTY P FOR SECOND ORDER SYSTEMS

4.1. Consider the equation

$$\dot{x} = G(t, x), \tag{21}$$

where the function G: $[0, 2\pi] \times \mathbb{R}^2 \to \mathbb{R}^2$ satisfies Caratheodory conditions and is positively homogeneous in x:

$$\forall (t, x) \in [0, 2\pi] \times \mathbb{R}^2, \qquad \forall \lambda > 0: G(t, \lambda x) = \lambda G(t, x).$$

We will establish some conditions under which the only 2π-periodic solution of (21) is the trivial one. To this end, let us introduce polar coordinates

$$x = (r \cos\theta, r \sin\theta).$$

One computes

$$\begin{aligned} \dot{\theta} &= \mathscr{G}(t, \cos\theta, \sin\theta) \\ &= \cos\theta G_2(t, \cos\theta, \sin\theta) - \sin\theta G_1(t, \cos\theta, \sin\theta). \end{aligned} \tag{22}$$

Consider also comparison systems

$$\dot{x} = A(t, x) \qquad \text{and} \qquad \dot{x} = B(t, x),$$

where the functions A and B are positively homogeneous in x and such that the functions

$$\mathscr{A}(t, x) = x_1 A_2(t, x) - x_2 A_1(t, x), \qquad \mathscr{B}(t, x) = x_1 B_2(t, x) - x_2 B_1(t, x)$$

are continuous. Introducing polar coordinates, we have respectively

$$\dot{\theta} = \mathscr{A}(t, \cos\theta, \sin\theta), \tag{23}$$

$$\dot{\theta} = \mathscr{B}(t, \cos\theta, \sin\theta). \tag{24}$$

PROPOSITION 1. *Assume*

$$\mathscr{A}(t, x) \leqslant \mathscr{G}(t, x) \leqslant \mathscr{B}(t, x) \tag{25}$$

and let θ *be a solution of* (22), φ *be a minimal solution of* (23), *and* ψ *be a maximal solution of* (24), *each of them defined on* $[0, 2\pi]$ *and such that* $\theta(0) = \varphi(0) = \psi(0)$. *Then for any* $t \in [0, 2\pi]$ *one has*

$$\varphi(t) \leqslant \theta(t) \leqslant \psi(t).$$

Proof. See P. Hartman [13, Theorem 4.1, p. 26]. ∎

COROLLARY 2. *Assume* (25) *holds. For any* $\theta_0 \in [0, 2\pi]$, *suppose that the functions* φ, *minimal solution of* (23) *such that* $\varphi(0) = \theta_0$, *and* ψ, *maximal solution of* (24) *such that* $\psi(0) = \theta_0$, *are such that*

$$[\varphi(2\pi) - \theta_0, \psi(2\pi) - \theta_0] \cap (2\pi)\mathbb{Z} = \varnothing.$$

Then Eq. (21) *has no nontrivial* 2π*-periodic solution.*

Suppose now that A and B are independent of t and that for any x

$$\mathscr{B}(x) = x_1 B_2(x) - x_2 B_1(x) < 0. \tag{26}$$

Then we know that $\varphi(t)$ and $\psi(t)$, solutions of (23) and (24), decrease. In this case, let t_φ and t_ψ be the time necessary for φ and ψ to decrease of 2π.

COROLLARY 3. *Assume* (25) *and* (26) *hold and* A, B *are independent of* t; φ *and* ψ *are defined as in Proposition* 1.

If $t_\varphi \geqslant 2\pi/(n+1)$, *then any* 2π*-periodic solution* x *of* (21) *has at most* $2(n+1)$ *zeros in* $[0, 2\pi[$ *and if* $t_\varphi > 2\pi/(n+1)$ *then* x *has less than* $2(n+1)$ *zeros.*

If $t_\psi \leqslant 2\pi/n$, *then* x *has at least* $2n$ *zeros and if* $t_\psi < 2\pi/n$, x *has more than* $2n$ *zeros.*

4.2. In order to compute t_φ and t_ψ in applications, we often have to investigate a comparison system which is piecewise linear

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} J & 1 \\ -K & J-L \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

The equivalent of (23) or (24) reads then

$$\dot{\theta} = -(\sin^2\theta + L\cos\theta\sin\theta + K\cos^2\theta), \tag{27}$$

the solution of which is decreasing if

$$L^2 - 4K < 0.$$

Let $t_1(L, K)$ be the smallest positive time such that (27) has a solution with

$$\theta(0) = \frac{\pi}{2}, \qquad \theta(t_1) = 0.$$

One computes (see P. B. Bailey, L. F. Shampine, and P. E. Waltman [1, p. 36])

$$\begin{aligned} t_1(L, K) &= \int_0^{\pi/2} \frac{d\theta}{\sin^2\theta + L\cos\theta\sin\theta + K\cos^2\theta} \\ &= \frac{1}{\sqrt{K - L^2/4}} \cos^{-1}\frac{L}{2\sqrt{K}}. \end{aligned} \tag{28}$$

It is easy to see also that if we define $t_i(L, K)$ $(i = 1, ..., 4)$ as the time necessary for a soution of (27) to go from $\theta = \pi - i(\pi/2)$ to $\theta = (\pi/2) - i(\pi/2)$, one has

$$t_1(L, K) = t_2(-L, K) = t_3(L, K) = t_4(-L, K).$$

4.3. *Application* 1. Considering (12), we have to investigate property P for the functions

$$\begin{aligned} G(x, y, u) &= (y - u_1 x_+ + u_2 x_-, \; -u_3 x_+ + u_4 x_-), \\ \alpha &= (a, c, p, r), \\ \beta &= (b, d, q, s). \end{aligned} \tag{29}$$

We shall assume for simplicity that the functions α and β are constant. Hence, we must prove that under appropriate conditions on α and β, the system

$$\begin{aligned} \dot{x} &= y - u_1(t)\, x_+ + u_2(t) x_- \\ \dot{y} &= -u_3(t)\, x_+ + u_4(t)\, x_- \end{aligned} \tag{30}$$

has no nontrivial periodic solution if $u \in C$ and

$$\alpha \leqslant u(t) \leqslant \beta.$$

PROPOSITION 2. *Assume* $a \leqslant b$, $c \leqslant d$, $p \leqslant q$, $r \leqslant s$, *and*

$$a^2 - 4p < 0, \qquad b^2 - 4p < 0, \qquad c^2 - 4r < 0, \qquad d^2 - 4r < 0,$$

and that for some $n \in \mathbb{N}$

$$\begin{aligned} \frac{\pi}{n+1} &< \frac{1}{\sqrt{4q-a^2}} \cos^{-1}\left(-\frac{a}{2\sqrt{q}}\right) + \frac{1}{\sqrt{4q-b^2}} \cos^{-1}\left(\frac{b}{2\sqrt{q}}\right) \\ &\quad + \frac{1}{\sqrt{4s-c^2}} \cos^{-1}\left(-\frac{c}{2\sqrt{s}}\right) + \frac{1}{\sqrt{4s-d^2}} \cos^{-1}\left(\frac{d}{2\sqrt{s}}\right) \\ &\leqslant \frac{1}{\sqrt{4p-b^2}} \cos^{-1}\left(-\frac{b}{2\sqrt{p}}\right) + \frac{1}{\sqrt{4p-a^2}} \cos^{-1}\left(\frac{a}{2\sqrt{p}}\right) \\ &\quad + \frac{1}{\sqrt{4r-d^2}} \cos^{-1}\left(-\frac{d}{2\sqrt{r}}\right) + \frac{1}{\sqrt{4r-c^2}} \cos^{-1}\left(\frac{c}{2\sqrt{r}}\right) < \frac{\pi}{n} \end{aligned} \tag{31}$$

then the triplet (G, α, β) *defined in* (29) *has property P.*

Remark. Notice that if $a = b = c = d = 0$, the assumption (31) reduces to

$$\frac{2}{n+1} < \frac{1}{\sqrt{q}} + \frac{1}{\sqrt{s}} \leqslant \frac{1}{\sqrt{p}} + \frac{1}{\sqrt{r}} < \frac{2}{n}, \tag{32}$$

which is the usual condition imposing that the rectangle $[p, q] \times [r, s]$ keeps away from the Fučik's spectrum (see, e.g., [11]).

Proof. Let u be a function such that $\alpha \leqslant u(t) \leqslant \beta$ and let (x, y) be a nontrivial solution of (30). Consider the functions

$$\begin{aligned} A_1(x, y) &= y - ax_+ + dx_-, & &\text{if } y \geqslant 0, \\ &= y - bx_+ + cx_-, & &\text{if } y < 0, \\ A_2(x, y) &= -qx_+ + sx_-, \\ B_1(x, y) &= y - bx_+ + cx_-, & &\text{if } y \geqslant 0, \\ &= y - ax_+ + dx_-, & &\text{if } y < 0, \\ B_2(x, y) &= -px_+ + rx_-. \end{aligned}$$

One easily checks that (25) and (26) hold. Next one computes from (28)

$$
\begin{aligned}
t_\varphi &= t_1(-a, q) + t_1(b, q) + t_1(-c, s) + t_1(d, s) \\
&= \frac{2}{\sqrt{4q-a^2}} \cos^{-1}\left(-\frac{a}{2\sqrt{q}}\right) + \frac{2}{\sqrt{4q-b^2}} \cos^{-1}\left(\frac{b}{2\sqrt{q}}\right) \\
&\quad + \frac{2}{\sqrt{4s-c^2}} \cos^{-1}\left(-\frac{c}{2\sqrt{s}}\right) + \frac{2}{\sqrt{4s-d^2}} \cos^{-1}\left(\frac{d}{2\sqrt{s}}\right) \\
&> \frac{2\pi}{n+1}
\end{aligned}
$$

and

$$
\begin{aligned}
t_\psi &= t_1(-b, p) + t_1(a, p) + t_1(-d, r) + t_1(c, r) \\
&= \frac{2}{\sqrt{4p-b^2}} \cos^{-1}\left(-\frac{b}{2\sqrt{p}}\right) + \frac{2}{\sqrt{4p-a^2}} \cos^{-1}\left(\frac{a}{2\sqrt{p}}\right) \\
&\quad + \frac{2}{\sqrt{4r-d^2}} \cos^{-1}\left(-\frac{d}{2\sqrt{r}}\right) + \frac{2}{\sqrt{4r-c^2}} \cos^{-1}\left(\frac{c}{2\sqrt{r}}\right) \\
&< \frac{2\pi}{n}.
\end{aligned}
$$

From Corollary 3, it follows that the number N_0 of zeros of x on $[0, 2\pi[$ is such that

$$2(n+1) > N_0 > 2n,$$

which is a contradiction. ∎

COROLLARY 4. *Assume $a = b$, $c = d$, $p \leqslant q$, $r \leqslant s$, and*

$$a^2 - 4p < 0, \qquad c^2 - 4r < 0$$

and that, for some $n \in \mathbb{N}$

$$\frac{1}{n+1} < \frac{1}{\sqrt{4q-a^2}} + \frac{1}{\sqrt{4s-c^2}} \leqslant \frac{1}{\sqrt{4p-a^2}} + \frac{1}{\sqrt{4r-c^2}} < \frac{1}{n},$$

then the triplet (G, α, β) defined in (29) has property P.

PROPOSITION 3. *Assume $0 \leqslant a \leqslant b$, $0 \leqslant c \leqslant d$, $p \leqslant q$, $r \leqslant s$, and*

$$b^2 - 4p < 0, \qquad d^2 - 4r < 0.$$

Assume further that

$$\frac{1}{\sqrt{4q-a^2}}+\frac{1}{\sqrt{4s-c^2}}\geqslant 2.$$

Then the triplet (G, α, β) *defined in* (29) *has property P.*

Remark. Similar propositions can be obtained depending on the sign of a, b, c, and d. For instance, if $a\geqslant 0$, $d\leqslant 0$, one can use the condition

$$\frac{1}{\sqrt{4q-a^2}}+\frac{1}{\sqrt{4s-d^2}}\geqslant 2.$$

Proof of Proposition 2. To prove this result, one computes as in Proposition 2

$$t_\varphi>\frac{\pi}{\sqrt{4q-a^2}}+\frac{\pi}{\sqrt{4s-c^2}}\geqslant 2\pi.$$

From Proposition 1, it follows that the time necessary for θ to decrease of 2π is larger than $t_\varphi>2\pi$. Hence, we have no nontrivial periodic solution. ▮

PROPOSITION 4. *Assume* $a=b$, $c=d$, $p\leqslant q$, $r\leqslant s$, *and*

$$a^2-4p<0, \qquad c^2-4r<0.$$

If further

$$\frac{1}{\sqrt{4q-a^2}}+\frac{1}{\sqrt{4s-c^2}}>1,$$

then the triplet (G, α, β) *defined in* (29) *has property P.*

Proof. As above one computes

$$t_\varphi=\frac{2\pi}{\sqrt{4q-a^2}}+\frac{2\pi}{\sqrt{4s-c^2}}>2\pi,$$

and the proof follows. ▮

The following proposition gives a necessary and sufficient condition for the system with constant coefficients

$$\dot{x}=y-ax_++cx_-$$

$$\dot{y}=-px_++rx_-$$

to have only the trivial solution.

Proposition 5. *Assume* $a=b$, $c=d$, $p=q\neq 0$, $r=s\neq 0$. *Then the triplet* (G, α, β) *defined in* (29) *has property* P *if and only if one of the following does not hold*:

(i) $a^2-4p<0$, $c^2-4r<0$;

(ii) $c/\sqrt{r}+a/\sqrt{p}=0$;

(iii) $(1/\sqrt{4p-a^2}+1/\sqrt{4r-c^2})^{-1}\in\mathbb{N}$.

Proof. The proof follows from direct computation of the solutions (see A. Fonda and F. Zanolin [8, Lemma 1]).

4.4. *Application* 2. To investigate periodic solutions of (13) we consider property P for the functions

$$\begin{aligned} G(x, y, u) &= (y, -u_1 y_+ + u_2 y_- - u_3 x_+ + u_4 x_-), \\ \alpha &= (a, c, p, r), \\ \beta &= (b, d, q, s), \end{aligned} \tag{33}$$

and we assume as above that α and β are constant.

Proposition 6. *Assume* $a\leqslant b$, $c\leqslant d$, $p\leqslant q$, $r\leqslant s$, *and*

$$a^2-4p<0, \qquad d^2-4p<0, \qquad c^2-4r<0, \qquad b^2-4r<0.$$

If further

$$\begin{aligned} \frac{\pi}{n+1} &< \frac{1}{\sqrt{4q-b^2}}\cos^{-1}\left(\frac{b}{2\sqrt{q}}\right)+\frac{1}{\sqrt{4q-c^2}}\cos^{-1}\left(-\frac{c}{2\sqrt{q}}\right) \\ &\quad +\frac{1}{\sqrt{4s-d^2}}\cos^{-1}\left(\frac{d}{2\sqrt{s}}\right)+\frac{1}{\sqrt{4s-a^2}}\cos^{-1}\left(-\frac{a}{2\sqrt{s}}\right) \\ &\leqslant \frac{1}{\sqrt{4p-a^2}}\cos^{-1}\left(\frac{a}{2\sqrt{p}}\right)+\frac{1}{\sqrt{4p-d^2}}\cos^{-1}\left(-\frac{d}{2\sqrt{p}}\right) \\ &\quad +\frac{1}{\sqrt{4r-c^2}}\cos^{-1}\left(\frac{c}{2\sqrt{r}}\right)+\frac{1}{\sqrt{4r-b^2}}\cos^{-1}\left(-\frac{b}{2\sqrt{r}}\right)<\frac{\pi}{n} \end{aligned} \tag{34}$$

then the triplet (G, α, β) *defined in* (33) *has property* P.

Remark. As for Proposition 2, we notice that if $a=b=c=d=0$, assumption (34) reduces to (32).

Proof. Let $u\in L^2$ be such that $\alpha\leqslant u(t)\leqslant\beta$ and (x, y) be a nontrivial solution of

$$\dot{x}=y, \qquad \dot{y}=-u_1(t)\,y_+ + u_2(t)\,y_- - u_3(t)\,x_+ + u_4(t)\,x_-.$$

Consider the functions

$$\begin{aligned} A(x, y) &= (y, -by_+ + cy_- - qx), && \text{if } x \geqslant 0, \\ &= (y, -ay_+ + dy_- - sx), && \text{if } x < 0, \\ B(x, y) &= (y, -ay_+ + dy_- - px), && \text{if } x \geqslant 0, \\ &= (y, -by_+ + cy_- - rx), && \text{if } x < 0. \end{aligned}$$

The proof follows then as the proof of Proposition 2. ∎

Statements similar to Corollary 4, Propositions 3 and 4, are easy to obtain. For example, we can write the following.

PROPOSITION 7. *Assume* $a = b = c = d$, $p \leqslant q$, $r \leqslant s$,

$$a^2 - 4p < 0, \qquad a^2 - 4r < 0$$

and

$$\frac{1}{\sqrt{4q - a^2}} + \frac{1}{\sqrt{4s - a^2}} > 1.$$

Then the triplet (G, α, β) *defined in* (33) *has property* P.

The constant coefficient case can also be investigated and needs some more care.

PROPOSITION 8. *Assume* $a = b$, $c = d$, $p = q \neq 0$, $r = s \neq 0$, *and*

$$a + c \neq 0.$$

Then the triplet (G, α, β) *defined in* (33) *has property* P.

Proof. Property P refers to 2π-periodic solutions of the differential equation

$$\dot{x} = y, \qquad \dot{y} = -(ay_+ - cy_-) - (px_+ - rx_-). \tag{35}$$

Assume $a + c < 0$ and let $(x(t), y(t))$ be such a periodic solution. We can assume that for some $t_1 > 0$

$$y(0) = y(t_1) = 0 \qquad \text{and} \qquad \forall t \in (0, t_1),\ y(t) > 0.$$

Notice also that from the positive homogeneity of (35) the functions $k(x(t), y(t))$, $k \in \mathbb{R}_+$, are also periodic solutions. All the solutions of (35) are periodic and the origin is a global center. Next we consider the closed curves defined by the functions

$$\gamma_k\colon [0, 2t_1] \to \mathbb{R}^2, \qquad t \leqslant t_1 \to \gamma_k^+(t) = k(x(t), y(t)), \qquad k \in \mathbb{R}_+.$$
$$t > t_1 \to \gamma_k^-(t) = k(x(2t_1 - t), -y(2t_1 - t)).$$

These curves cover the whole plane and for each t we can define $k(t) \in \mathbb{R}_+$ such that $(x(t), y(t)) \in \gamma_{k(t)}$. Clearly $k(t) = 1$ if $y(t) \geqslant 0$. Moreover $k(t)$ is strictly increasing if $y(t) < 0$. This follows from the fact that along $\gamma_k^-(t)$ the vector field G points outward the regions Γ_k bounded by γ_k. Indeed for such a point one computes

$$\gamma_k'(t) = k(-y(2t_1 - t), -ay(2t_1 - t) - px_+(2t_1 - t) + rx_-(2t_1 - t))$$

and

$$G(\gamma_k(t)) = k(-y(2t_1 - t), cy(2t_1 - t) - px_+(2t_1 - t) + rx_-(2t_1 - t)).$$

Since

$$cy - px_+ + rx_- < -ay - px_+ + rx_-,$$

the vector field G points outward. At last, as $k(2\pi) > k(0)$, we cannot have $(x(2\pi), y(2\pi)) = (x(0), y(0))$ which contradicts the periodicitiy of (x, y). ∎

The above proposition can also be proved by direct computation of the solutions. In fact we can prove more generally the following necessary and sufficient condition for the constant coefficient case to have property P.

PROPOSITION 9. *Assume $a = b$, $c = d$, $p = q \neq 0$, $r = s \neq 0$. Then the triplet (G, α, β) defined in* (33) *has property P if and only if one of the following does not hold*:

(i) $a^2 - 4p < 0$, $a^2 - 4r < 0$;

(ii) $a + c = 0$;

(iii) $\pi[(1/\sqrt{4p - a^2}) \cos^{-1} a/(2\sqrt{p}) + (1/\sqrt{4r - a^2}) \cos^{-1} a/(2\sqrt{r})] \in \mathbb{N}$.

5. PROPERTY P FOR 3d ORDER SYSTEMS

5.1. Consider the equation

$$\dot{x} = y, \qquad \dot{y} = z, \qquad \dot{z} = -u_2(t)\, x_+ + u_3(t)\, x_- - by - az \tag{36}$$

or, which is equivalent,

$$\dddot{x} + a\ddot{x} + b\dot{x} + u_2(t)\, x_+ - u_3(t)\, x_- = 0 \tag{37}$$

together with the conditions $a \in \mathbb{R}$, $b \in \mathbb{R}$,

$$p(t) \leqslant u_2(t) \leqslant q(t),$$
$$r(t) \leqslant u_3(t) \leqslant s(t).$$

PROPOSTITION 10. *Let $a \neq 0$ and assume that for some $n \in \mathbb{N}$*

$$n^2 a \leqslant p(t) = r(t), \qquad q(t) = s(t) \leqslant (n+1)^2 a,$$

both inequalities being strict on a subset of $[0, 2\pi]$ of positive measure.
Then the triplet

$$G(x, y, z, u) = (y, z, -u_2 x_+ + u_3 x_- - by - az), \tag{38}$$
$$\alpha(t) = (p(t), p(t)),$$
$$\beta(t) = (q(t), q(t)),$$

has property P.

Proof. The proof follows from Lemma 1 in O. C. Ezeilo and M. N. Nkashama [5]. ▮

We can extend this result to the somewhat more general equation

$$\dot{x} = y, \qquad \dot{y} = z,$$
$$\dot{z} = u_1(t)(|x| + |y| + |z|) - u_2(t)\, x_+ + u_3(t)\, x_- - by - az.$$

for which we have the following.

COROLLARY 5. *If the triplet (G, α, β) with G defined as in (38) and*

$$\alpha(t) = (p(t), r(t)), \qquad \beta(t) = (q(t), s(t)),$$

has property P, there exists $\varepsilon_0 > 0$ such that the triplet

$$\hat{G}(x, y, z, u) = (y, z, u_1(|x| + |y| + |z|) - u_2 x_+ + u_3 x_- - by - az),$$
$$\hat{\alpha}(t) = (-\varepsilon_0, p(t), r(t)),$$
$$\hat{\beta}(t) = (\varepsilon_0, q(t), s(t))$$

has property P.

Proof. It is clear that the triplet

$$\hat{G}, \qquad \tilde{\alpha}(t) = (0, p(t), r(t)), \qquad \tilde{\beta}(t) = (0, q(t), s(t))$$

has property P. The proof follows then from Lemma 1. ▮

5.2. PROPOSITION 11. *Let $\mu > 0$, $v > 0$, $\rho \geqslant 0$ and define*

$$m = \min(\mu, v), \qquad M = \max(\mu, v), \qquad b_0 = \rho \left(\frac{a}{m-\rho}\right)^{1/2}.$$

Assume $m > \rho$ and one of the following conditions holds:

(i) $a \leqslant 0$;
(ii) $a > M + b_0$;
(iii) $b > 1 - b_0$;
(iv) $m > ab + b_0(a + \sqrt{b + b_0})$;
(v) $M < ab - b_0(a + \sqrt{b + b_0})$.

Then the triplet

$$G(x, y, z, u) = (y, z, -u_2 x_+ + u_3 x_- - by - az),$$
$$\alpha = (\mu - \rho, v - \rho),$$
$$\beta = (\mu + \rho, v + \rho)$$

has property P.

Proof. Let us suppose that x is a nontrivial 2π-periodic solution of (37) and let

$$f(t) = (\mu - u_2(t))\, x_+(t) - (v - u_3(t))\, x_-(t).$$

Equation (37) reads

$$\dddot{x} + a\ddot{x} + b\dot{x} + \mu x_+ - v x_- = f(t). \tag{39}$$

Multiplying (39) by x and integrating gives

$$m\, \|x\|_{L^2}^2 - \|f\|_{L^2} \|x\|_{L^2} \leqslant a\, \|\dot{x}\|_{L^2}^2.$$

We notice further that $\|f\|_{L^2} \leqslant \rho\, \|x\|_{L^2}$, from which follows

$$0 < (m - \rho)\, \|x\|_{L^2}^2 \leqslant a\, \|\dot{x}\|_{L^2}^2, \tag{40}$$

i.e.,

$$a > 0.$$

This contradicts (i).

Multiplying (39) by $\dot{x}$ and integrating, one gets

$$| \ \|\ddot{x}\|_{L^2}^2 - b\ \|\dot{x}\|_{L^2}^2| \leqslant \|f\|_{L^2}\ \|\dot{x}\|_{L^2} \leqslant \rho\ \|x\|_{L^2}\ \|\dot{x}\|_{L^2}$$

and from (40)

$$| \ \|\ddot{x}\|_{L^2}^2 - b\ \|\dot{x}\|_{L^2}^2| \leqslant \rho \left(\frac{a}{m-\rho}\right)^{1/2} \|\dot{x}\|_{L^2}^2 = b_0\ \|\dot{x}\|_{L^2}^2,$$

i.e.,

$$(b - b_0)\ \|\dot{x}\|_{L^2}^2 \leqslant \|\ddot{x}\|_{L^2}^2 \leqslant (b + b_0)\ \|\dot{x}\|_{L^2}^2. \qquad (41)$$

In particular, by Wirtinger inequality,

$$1 \leqslant b + b_0,$$

which contradicts (iii).

Multiplying (39) by $\ddot{x}$ and integrating, one gets

$$m\ \|\dot{x}\|_{L^2}^2 - \|f\|_{L^2}\ \|\ddot{x}\|_{L^2} \leqslant a\ \|\ddot{x}\|_{L^2}^2 \leqslant M\ \|\dot{x}\|_{L^2}^2 + \|f\|_{L^2}\ \|\ddot{x}\|_{L^2}. \qquad (42)$$

From (40) and the Wirtinger inequality we have

$$a \leqslant M + b_0,$$

contradicting (ii).

From (42), (41), and (40), it follows

$$[m - b_0(b + b_0)^{1/2}]\ \|\dot{x}\|_{L^2}^2 \leqslant a\ \|\ddot{x}\|_{L^2}^2 \leqslant a(b + b_0)\ \|\dot{x}\|_{L^2}^2$$

and

$$m - b_0(b + b_0)^{1/2} \leqslant a(b + b_0),$$

which contradicts (iv).

Similarly, it follows from (42), (41), and (40) that

$$a(b - b_0)\ \|\dot{x}\|_{L^2}^2 \leqslant a\ \|\ddot{x}\|_{L^2}^2 \leqslant [M + b_0(b + b_0)^{1/2}]\ \|\dot{x}\|_{L^2}^2$$

and

$$a(b - b_0) \leqslant M + b_0(b + b_0)^{1/2},$$

which contradicts (v). ∎

5.3. Let us consider the constant coefficients case

$$\dddot{x} + a\ddot{x} + b\dot{x} + \mu x_+ - \nu x_- = 0, \tag{43}$$

with $(\mu, \nu) \in \mathbb{R}^2$. We define a subset $U(a, b)$ of $\mathbb{R}^2$ as follows.

(1) If $a = 0$ or $b < 1$, set

$$U(a, b) = \{(\mu, \nu) : \mu \cdot \nu > 0\}.$$

(2) If $a \neq 0$, $b \geqslant 1$, and $n \in \mathbb{N}$ is such that $b \in [n^2, (n+1)^2[$, set

$$U(a, b) = \{(\mu, \nu) \mid \mu \cdot \nu > 0, (\mu - ab)(\nu - ab) > 0\}$$
$$\cup \{(\mu, \nu) \mid \mu, \nu \in]\, an^2, a(n+1)^2[\,\}.$$

PROPOSITION 12. *Assume $(\mu, \nu) \in U(a, b)$. Then the triplet*

$$G(x, y, z, u) = (y, z, -u_2 x_+ + u_3 x_- - by - az)$$
$$\alpha(t) = (\mu, \nu)$$
$$\beta(t) = (\mu, \nu)$$

has property P.

Proof. It is a consequence of Propositions 10 and 11, together with a symmetric formulation of Proposition 11 for the case $\mu < 0$, $\nu < 0$.

Remark. Proposition 12 gives sufficient conditions for a third order system with constant coefficients to have only the trivial solution. In case $\mu = \nu$, it is well known (see [5]) that a necessary and sufficient condition for property P to hold is

$$\mu \neq 0 \qquad \text{and} \qquad \forall n \in \mathbb{N}^* [b \neq n^2 \text{ or } \mu \neq an^2].$$

One can check that such an assumption is equivalent to

$$(\mu, \mu) \in U(a, b).$$

Hence, Proposition 12 generalizes the linear case. Necessary and sufficient conditions in the general case $\mu \neq \nu$ seem to be unknown.

6. Existence of Periodic Solutions in Applications

6.1. Consider the boundary value problem

$$\begin{aligned} \dot{x} &= y - f(t, x), \\ \dot{y} &= e(t) - g(t, x), \\ x(0) = x(2\pi), &\qquad y(0) = y(2\pi). \end{aligned} \tag{44}$$

THEOREM 2. *Assume*:

(i) *the functions f, g, and e are continuous and defined for $t \in [0, 2\pi]$, $x \in \mathbb{R}$*;

(ii) *Assumption* A1 *holds*;

(iii) *the triplet*

$$\begin{aligned} G(x, y, u) &= (y - u_1 x_+ + u_2 x_-, \ -u_3 x_+ + u_4 x_-), \\ \alpha(t) &= (a(t), c(t), p(t), r(t)), \\ \beta(t) &= (b(t), d(t), q(t), s(t)), \end{aligned}$$

has property P;

(iv) *there exists some constants $u^0 \in \mathbb{R}^4$ such that*

$$\alpha(t) \leqslant u^0 \leqslant \beta(t).$$

Then the problem (44) *has at least one solution.*

Proof. We will apply Theorem 1. From Paragraph 2.2 it is clear that Assumption H holds.

Next, using Proposition 5, we can find a path $u^\lambda = (u_1^\lambda, u_2^\lambda, u_3^\lambda, u_4^\lambda)$ in $\mathbb{R}^4$, $\lambda \in [0, 1]$, that links u^0 to a point u^1 such that

$$u_1^1 = u_2^1, \qquad u_3^1 = u_4^1$$

and for any $\lambda \in]0, 1]$, the differential equations

$$\begin{aligned} \dot{x} &= y - u_1^\lambda x_+ + u_2^\lambda x_- \\ \dot{y} &= -u_3^\lambda x_+ + u_4^\lambda x_- \end{aligned} \tag{45}$$

have no nontrivial 2π-periodic solutions. Indeed, in case $u_3^0 \leqslant 0$ or $u_4^0 \leqslant 0$ a path can be found such that condition (i) of Proposition 5 does not hold.

In case $u_3^0 > 0$ and $u_4^0 > 0$ we can choose a path such that, for any $\lambda \in]0, 1]$,

$$\frac{u_2^\lambda}{\sqrt{u_4^\lambda}} + \frac{u_1^\lambda}{\sqrt{u_3^\lambda}} \neq 0$$

(see [8]), so that condition (ii) of Proposition 5 does not hold. As for $\lambda = 1$, (44) reduces to a linear system, it follows that for $\Omega_0 = \{x \in C : \|x\|_\infty < 1\}$ and N_0 defined in (18),

$$d(L - N_0, \Omega_0) \neq 0.$$

The proof follows now from Theorem 1. ∎

Conditions for (iii) to hold are given in Propositions 2, 3, 4, and in Corollary 4. Other methods can be used as in P. Habets and G. Metzen [11]. Theorem 2 generalizes among others results from [11] and A. Fonda and F. Zanolin [8].

6.2. In our second application we consider the boundary value problem

$$\begin{gathered} \dot{x} = y, \qquad \dot{y} = e(t) - g(t, x) - f(t, y), \\ x(0) = x(2\pi), \qquad y(0) = y(2\pi). \end{gathered} \tag{46}$$

THEOREM 3. *Assume*:

(i) *the functions* f, g, *and* e *are continuous and defined for* $t \in [0, 2\pi]$, $x \in \mathbb{R}$, $y \in \mathbb{R}$;

(ii) *Assumption* A1 *holds*;

(iii) *the triplet*

$$\begin{gathered} G(x, y, u) = (y, -u_1 y_+ + u_2 y_- - u_3 x_+ + u_4 x_-), \\ \alpha(t) = (a(t), c(t), p(t), r(t)), \\ \beta(t) = (b(t), d(t), q(t), s(t)) \end{gathered}$$

has property P;

(iv) *there exists some constant* $u^0 \in \mathbb{R}^4$ *such that*

$$\alpha(t) \leqslant u^0 \leqslant \beta(t).$$

Then the problem (46) *has at least one solution.*

The proof is identical to the proof of Theorem 2 but uses Proposition 8 instead of Proposition 5. Conditions ensuring (iii) are given in Propositions 6 and 7.

6.3. Consider the third order problem

$$\dddot{x} + a\ddot{x} + b\dot{x} + g(t, x) = e(t, x, \dot{x}, \ddot{x}),$$
$$x(0) = x(2\pi), \qquad \dot{x}(0) = \dot{x}(2\pi), \qquad \ddot{x}(0) = \ddot{x}(2\pi). \tag{47}$$

THEOREM 4. *Assume:*

(i) *the functions* $g(t, x)$ *and* $e(t, x, y, z)$ *are continuous functions defined for* $t \in [0, 2\pi]$, $x \in \mathbb{R}$, $y \in \mathbb{R}$, $z \in \mathbb{R}$;

(ii) *Assumption* A3 *holds*;

(iii) *the triplet*

$$G(x, y, z, u) = (y, z, -u_2 x_+, +u_3 x_- - by - az),$$
$$\alpha(t) = (p(t), r(t)),$$
$$\beta(t) = (q(t), s(t))$$

has property P;

(iv) *there exists some constant* $u^0 \in U(a, b)$, *where* $U(a, b)$ *is defined as in* 5.3, *such that*

$$\alpha(t) \leqslant u^0 \leqslant \beta(t).$$

Then there exists $\varepsilon_0 > 0$ *such that if* $\Delta_0 \leqslant \varepsilon_0$, *the problem* (47) *has a solution.*

The proof of this theorem goes as the proof of Theorem 2. One has only to notice that from Corollary 5, there exists $\varepsilon_0 > 0$ such that the triplet

$$\hat{G}(x, y, z, u) = (y, z, u(|x| + |y| + |z|) - u_2 x_+, +u_3 x_- - by - az),$$
$$\hat{\alpha}(t) = (-\varepsilon_0, p(t), r(t)),$$
$$\hat{\beta}(t) = (\varepsilon_0, q(t), s(t))$$

has property P.

Assumption (iii) can be obtained from Propositions 10 and 11. Theorem 4 generalizes then a result of O. C. Ezeilo and M. N. Nkashama [5].

6.4. Let us remark that in Theorems 2 and 3, assumption (iv) can be replaced by:

(iv′) *there exists some functions*

$$u^0(t) = (u_1^0(t), u_1^0(t), u_3^0(t), u_3^0(t))$$

such that

$$\alpha(t) \leqslant u^0(t) \leqslant \beta(t).$$

In this case, the proof uses Corollary 1 instead of Theorem 1.
A similar statement holds for Theorem 4.

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