ANALYSE MATHÉMATIQUE

Variational problems at resonance without monotonicity

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Abstract. — An answer is given to a problem raised in [6] and [8]. We prove the existence of solutions for some boundary value problems in variational form at resonance without monotonicity assumptions on the nonlinearity.

1. INTRODUCTION

The existence of solutions for semicoercive semilinear problems in variational form follows from a coercivity condition of its action on the kernel of the linear part when the nonlinearity is bounded or has a convex potential (cf. [1, 6, 8]).

In [6] and [8], the question of weakening for example the convexity assumption was raised in the context of a system of ordinary differential equations.

This paper provides a positive answer to this question. In particular, for the periodic problem for the system

$$u''(x) + D_u G(x, u(x)) = 0,$$

we are able to prove the existence of solutions if, in particular, G satisfies, besides the coercivity condition along the null space of the linear part, a generalized convexity assumption of the form

$$G(x,u) \geqslant G(x,v) + \langle D_u G(x,v), u - v \rangle - \gamma \qquad (0)$$

for a certain $\gamma \geqslant 0$ and all x, u and v.

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More generally, we can also deal with the Dirichlet or the Neumann problems associated to an elliptic operator at resonance with the first eigenvalue. The coercivity assumption on the kernel goes back to Ahmad, Lazer and Paul, while the assumption in (0) is only a particular case of a more general condition which will be introduced in Section 3.

We will show that our condition generalizing the convexity is in particular satisfied if the function is convex outside a sufficiently large ball, and satisfies a certain coercivity condition. Many examples can be given of non-convex functions satisfying assumption (0).

The paper is organized as follows. In Section 2 we prove an abstract theorem which is the backbone of the paper. In Section 3 the setting for an abstract differential problem is given, taking in mind the type of applications we are going to give. Some general existence results are given, which generalize also some results from [7]. Finally, in Section 4, we apply our theorems to three different kinds of problems: the Dirichlet and Neumann problems associated to an elliptic differential system at resonance with the first eigenvalue, and the periodic problem for a sistem of ODE's, at resonance, as well.

2. The abstract theorem

Let L and H be two Banach spaces such that $H \subset L$ algebraically and topologically, with norms $\| \cdot \|$ and $\| \cdot \|_L$, respectively. Let L^* be the topological dual of L, with the usual norm $\| \cdot \|_{L^*}$ and denote by $(.,.)$ the bilinear pairing between L^* and L.

Let H_0 and H_1 be vector subspaces of H, such that $H = H_0 \oplus H_1$. For $u \in H$, we will write $u = u_0 + u_1$, with $u_0 \in H_0$ and $u_1 \in H_1$.

Let $a : H \to \mathbb{R}$ be a map such that there exists an $\alpha : \mathbb{R}_+ \to \mathbb{R}$ bounded below, satisfying $\alpha(t) \to +\infty$ as $t \to +\infty$ and

$$a(u) \geqslant \alpha(\| u_1 \|) \| u_1 \| \tag{1}$$

for all $u \in H$.

Let $g : H \to \mathbb{R}$ be bounded on bounded subsets of H_1 and such that there exist

$\Phi_1 : H_0 \to \mathbb{R}$ bounded below with the property

$$\Phi_1(u_0) \to +\infty \quad \text{as } \| u_0 \| \to \infty \text{ in } H_0 \tag{2}$$

and

$\Phi_2 : H_0 \to L^*$ continuous and bounded on bounded sets such that

$$g(u) \geqslant \Phi_1(u_0) + (\Phi_2(u_0), u - u_0) \tag{3}$$

for every $u \in H$ and $u_0 \in H_0$.

THEOREM 1. — *Let $f = a + g$. Under the above assumptions, every minimizing sequence of f is bounded. If moreover f is weakly lower semi-continuous, then f has a minimum.*

PROOF. – Let (u^k) be a minimizing sequence for f in H. Since $f(u^k) \to \inf_H f$, we have that $f(u^k)$ is bounded from above.

Taking in (3) $u = 0$, we have by (2) that

$$(\Phi_2(u_0), u_0) \to +\infty \quad \text{as } \| u_0 \| \to \infty \quad \text{in } H_0.$$

Hence there is an $R > 0$ such that, whenever $\| u_0 \| = R$, we have $(\Phi_2(u_0), u_0) > 0$. If $u_0^k \neq 0$, we define the function $h_k : \mathbb{R} \to \mathbb{R}$ by

$$h_k(t) = (\Phi_2(tRu_0^k / \| u_0^k \|), Ru_0^k / \| u_0^k \|).$$

Then h_k is continuous by the continuity of Φ_2 and $h_k(-1) < 0 < h_k(1)$. Hence there exists a $t_k \in]-1,1[$ such that $h_k(t_k) = 0$. Set $U_0^k = t_k R u_0^k / \| u_0^k \|$. Then, for all $\gamma \in \mathbb{R}$,

$$(\Phi_2(U_0^k), \gamma u_0^k) = 0. \tag{4}$$

If $u_0^k = 0$, set $U_0^k = 0$. Then (4) holds in this case, too. By (1), (3) and (4) we have

$$\begin{aligned} f(u^k) &\geqslant \alpha(\| u_1^k \|) \| u_1^k \| + \Phi_1(U_0^k) + (\Phi_2(U_0^k), u^k - U_0^k) \\ &\geqslant \alpha(\| u_1^k \|) \| u_1^k \| + C + (\Phi_2(U_0^k), u_1^k + \bar{\gamma} u_0^k) \\ &\geqslant \alpha(\| u_1^k \|) \| u_1^k \| + C + C' r \| u_1^k \| \end{aligned}$$

where $C = \inf_{H_0} \Phi_1$, $C' \doteq \sup_{\| U_0 \| \leqslant R} \| \Phi_2(U_0) \|_{L^*}$ and r is such that

$$\| \cdot \|_L \leqslant r \| \cdot \|.$$

Since the above constants are independent of k, we kan conclude that there exists a $C_1 > 0$ such that, for all k,

$$\| u_1^k \| \leqslant C_1. \tag{5}$$

Consider now (3), and take $u, w \in H$ and $t \in [0, 1]$ such that

$u_0 := tu + (1 - t)w$ belongs to H_0. We then have

$$g(u) \geqslant \Phi_1(u_0) + (1-t)(\Phi_2(u_0), u - w)$$
$$g(w) \geqslant \Phi_1(u_0) + t(\Phi_2(u_0), w - u)$$

from which we obtain

$$tg(u) + (1-t)g(w) \geqslant \Phi_1(tu + (1-t)w). \tag{6}$$

Taking in (6) $t = 1/2$, $u = u^k$ and $w = -u_1^k$, we have by (1), (5) and the boundedness properties of a and g,

$$f(u^k) \geqslant a(\| u_1^k \|) \| u_1^k \| + g(u^k)$$
$$\geqslant C'' - g(-u_1^k) + 2\Phi_1((1/2)u_0^k)$$
$$\geqslant C'' - C''' + 2\Phi_1((1/2)u_0^k)$$

By (2), there exists a $C_2 > 0$ such that, for all k,

$$\| u_0^k \| \leqslant C_2.$$

Hence (u^k) is bounded, and the proof is complete.

3. A NONLINEAR SYSTEM

Let Ω be an open bounded subset of $\mathbb{R}^n$, and H a Hilbert space such that $(H_0^1(\Omega))^m \subset H \subset (H^1(\Omega))^m$ algebraically and topologically, $H_0^1(\Omega)$ and $H^1(\Omega)$ being the classical Sobolev spaces, and $m \geqslant 1$. We will assume the boundary $\partial\Omega$ to be sufficiently smooth in order to have the Sobolev imbeddings. Let $\| \cdot \|$ denote the norm of H.

We consider the following problem.

$$(\mathscr{L}u)(x) + D_u G(x,u(x)) = 0, \tag{7}$$

where $\mathscr{L}$ is an unbounded semipositive definite self-adjoint linear operator on $(L^2(\Omega))^m$and $G : \Omega \times \mathbb{R}^m \to \mathbb{R}$ is a Caratheodory function, Gateaux differentiable in its second variable.

Let us denote by H_0 the kernel of $\mathscr{L}$ and, for every $u \in H$, let us write $u = u_0 + u_1$, with $u_0 \in H_0$ and $u_1 \in H_0^\perp$.

We will consider the following assumption of $\mathscr{L}$.

ASSUMPTION 1. — *There is a $\delta > 0$ such that, for all $u \in H$,*

$$(\mathscr{L}u \mid u) \geqslant \delta \| u_1 \|^2 \tag{8}$$

where $(\cdot \mid \cdot)$ denotes the scalar product in $(L^2(\Omega))^m$.

REMARK 1. – The applications we are going to give in Section 4 deal with problems at resonance with λ_1, the first eigenvalue. In those cases, Assumption 1 is verified by taking $\delta = \lambda_2 - \lambda_1$.

We will also consider the following assumption on G.

ASSUMPTION 2. – *There exist two Caratheodory functions* $\varphi_1 : \Omega \times \mathbb{R}^m \to \mathbb{R}$ *and* $\varphi_2 : \Omega \times \mathbb{R}^m \to \mathbb{R}^m$ *such that, for a.e.* $x \in \Omega$ *and all* $u, v \in \mathbb{R}^m$,

$$G(x, u) \geqslant \varphi_1(x, v) + \langle \varphi_2(x, v), u - v \rangle, \tag{9}$$

where $\langle \cdot , \cdot \rangle$ *denotes the scalar product in* $\mathbb{R}^m$.

If $n \geqslant 2$, *we require the following growth restrictions.*

$$| G(x,u) | \leqslant b_0(x) + a_0 | u |^p \tag{10}$$

$$| \varphi_1(x,u) | \leqslant b_1(x) + a_1 | u |^p \tag{11}$$

$$| \varphi_2(x,u) | \leqslant b_2(x) + a_2 | u |^{p-1} \tag{12}$$

where, for $i = 0, 1$ and 2, *we have* $a_i \geqslant 0$, $p \geqslant 1$ *and, if* $n \geqslant 3$, $p < 2n / (n-2)$, *and* $b_i \in L^q(\Omega)$, *q being the conjugate exponent of p (i.e.* $(1/p) + (1/q) = 1$).

We also assume that, for $u_0 \in H_0$,

$$\lim_{\| u_0 \| \to \infty} \int_\Omega \varphi_1(x,u_0(x))\, dx = +\infty. \tag{13}$$

REMARK 2. — The hypothesis (9) is in particular satisfied if $G(x, \cdot)$ is convex for a.e. x, or, more generally, when G satisifies (0), by taking $\varphi_1 = G - \gamma$ and $\varphi_2 = D_u G$. In this case, (13) is precisely the Ahmad-Lazer-Paul coercivity condition on G [1]. The hypothesis (10), (11) and (12) are needed in order to be able to apply the classical Sobolev's imbedding theorems.

Let us now define the following functionals.

$$a : H \to \mathbb{R}, \quad a(u) = (1/2)(\mathscr{L}u \mid u)$$

$$g : H \to \mathbb{R}, \quad g(u) = \int_\Omega G(x,u(x))\, dx$$

The map a is weakly lower semicontinuous, because, being semi-positive definite and self-adjoint, we have that if $u^k \rightharpoonup u$,

$$0 \leqslant (\mathscr{L}(u^k - u) \mid u^k - u) = (\mathscr{L}u \mid u) - 2(\mathscr{L}u \mid u^k) + (\mathscr{L}u^k \mid u^k)$$

and then

$$\liminf_{k\to+\infty} (\mathscr{L}u^k \mid u^k) \geqslant (\mathscr{L}u \mid u).$$

The map g is well defined, weakly continuous and bounded on bounded subsets of H because, if $n = 1$, H is continuously and compactly imbedded in the space of continuous functions, while if $n \geqslant 2$, H is continuously and compactly imbedded in $(L^p(\Omega))^m$, with p chosen as in Assumption 2, and because of (10) (cf. [5]).

It is readily seen that, setting $f = a + g$, the solutions of (7) correspond to the critical points of f. It is also well known that if f is weakly lower semicontinuous and possesses a bounded minimizing sequence, then f attains its infimum at a point of H, which is obviously a critical point of f.

THEOREM 2. — *Suppose Assumptions 1 and 2 hold. Then* (7) *has a solution* $u \in$ H.

PROOF. — We define L to be the Banach space $(L^p(\Omega))^m$, where $p = 2$ if $n = 1$, and p is as in Assumption 2 if $n \geqslant 2$. We will identify L^* with $(L^q(\Omega))^m$, q being the conjugate exponent of p. Define the functions Φ_1 and Φ_2 as follows.

$$\Phi_1 : H_0 \to \mathbb{R}, \quad \Phi_1(u_0) = \int_\Omega \varphi_1(x,u_0(x))\, dx$$

$$\Phi_2 : H_0 \to L^*, \quad \Phi_2(u_0) = \varphi_2(\cdot\,, u_0(\cdot))$$

By (11) and (12), we have that Φ_1 and Φ_2 are well defined, continuous and send bounded sets into bounded sets (cf. [4, 5]). Condition (2) is indeed assumed in (13) and, by (9),

$$\begin{aligned} g(u) &= \int_\Omega G(x,u(x))\, dx \\ &\geqslant \int_\Omega \varphi_1(x,u_0(x))\, dx + \int_\Omega \langle \varphi_2(x,u_0(x)), u(x) - u_0(x) \rangle\, dx \\ &= \Phi_1(u_0) + (\Phi_2(u_0), u - u_0) \end{aligned}$$

so that (3) holds as well. Hence Theorem 1 can be applied, and since f is weakly lower semicontinuous, the result follows from the above remarks.

Let us now introduce the following

ASSUMPTION 3. — $H_0 = \text{span } \{\eta^1, \ldots, \eta^m\}$, and for every $i, j \in \{1, \ldots, m\}$, $\eta^i_i(x) > 0$ *for a.e.* $x \in \Omega$, *while* $\eta^i_j \equiv 0$ *when* $i \neq j$.

REMARK 3. — Assumption 3 will be satisfied in all our applications. For the Neumann and the periodic problems, it is sufficient to take $\eta^i_i \equiv 1$. For the Dirichlet problem, it is a consequence of the fact that the first eigenfunction has a definite sign in Ω.

The following theorem is an illustration of a case when the Assumption 2 is satisfied. As far as we know it is not included in any known result.

THEOREM 3. — *Let Assumptions 1 and 3 hold. Let* $G : \Omega \times \mathbb{R}^m \to \mathbb{R}$ *be a Caratheodory map of the form*

$$G(x,u) = c(x)\psi(x,u),$$

where $c : \Omega \to \mathbb{R}$ *is continuous, bounded, nonnegative and positive on a set* Ω_0 *of positive measure, and* $\psi : \Omega \times \mathbb{R}^m \to \mathbb{R}$ *is continuously differentiable in its second variable. Assume moreover :*

(k) $\exists\, R \geqslant 0 : \psi(x, \cdot)$ *is convex outside the ball* B_R, *for a.e.* $x \in \Omega_0$;

(kk) $\liminf_{|u| \to \infty}[\psi(x,u)/|u|] > \sup_{|u| < R} |D_u\psi(x,u)|$ *unif. a.e. in* x.

(kkk) *If* $n = 1$, *then* $\exists\, b_R \in L^1(\Omega)$:

$$|D_u\psi(x,u)| \leqslant b_R(x)$$

for a.e. $x \in \Omega$ *and all* $u \in B_R$.

If $n \geqslant 2$, *then*

$$|D_u\psi(x,u)| \leqslant b(x) + a|u|^{p-1}$$

for a.e. $x \in \Omega_0$ *and all* $u \in \mathbb{R}^m$, *where* $a \geqslant 0$, $b \in L^q(\Omega)$, $p \geqslant 2$ *and, if* $n \geqslant 3$, $p < 2n/(n-2)$, $(1/p) + (1/q) = 1$.

Then Assumption 2 holds, and the equation

$$(\mathscr{L}u)(x) + c(x)D_u\psi(x,u(x)) = 0$$

has a solution $u \in H$.

PROOF. — We will prove that there is a L^1 – map $\gamma : \Omega \to \mathbb{R}_+$ such that Assumption 2 is satisfied for φ_1 and φ_2 as follows :

$$\varphi_1(x,u) = c(x)\psi(x,u) - \gamma(x)$$

$$\varphi_2(x,u) = c(x)D_u\psi(x,u).$$

Notice first of all that if such a γ exists, then (10), (11) and (12) hold when $n \geqslant 2$, by (kkk). Moreover, by Assumption 3, if $|u_0| \to \infty$ in H_0, then $|u_0(x)| \to \infty$ for a.e. $x \in \Omega$, and since

$$\int_{\Omega} \varphi_1(x,u_0(x))\, dx \geqslant \int_{\Omega_0} c(x)\psi(x,u_0(x))\, dx - \|\gamma\|_{L^1}$$

so that (13) holds by (kk), as well. Further, (i) or (iii) in Proposition 1 holds.

What we still need to show in order to be able to apply Theorem 2 is the existence of a $\gamma \in L^1(\Omega)$ such that the map $\theta : \Omega_0 \times \mathbb{R}^{2m} \to \mathbb{R}$ *defined by*

$$\theta(x,u,v) = \psi(x,u) - \psi(x,v) - \langle D_u\psi(x,v), u - v \rangle$$

is bounded below by $-\gamma(x)$. We will consider four different cases.

CASE 1. — $u \in B_R$, $v \in B_R$. The result follows by (kkk).

CASE 2. — $u \notin B_R$, $v \notin B_R$. Then $\theta(x,u,v) \geqslant 0$ for a.e. $x \in \Omega$, because of (k).

CASE 3. — $u \in B_R$, $v \notin B_R$. In this case, define $\hat{u} \in \mathbb{R}^m$ such that $|\hat{u}| = R$ and

$$(\hat{u} - u)/|\hat{u} - u| = (v - \hat{u})/|v - \hat{u}|$$

Then

$$\begin{aligned}\theta(x,u,v) = {} & [\psi(x,\hat{u}) - \psi(x,v) - \langle D_u\psi(x,v), \hat{u} - u \rangle + \\ & + \psi(x,u) - \psi(x,\hat{u}) + \langle D_u\psi(u,v), \hat{u} - u \rangle\end{aligned}$$

By (k) and (kkk),

$$\begin{aligned}\theta(x,u,v) &\geqslant -c(x) + \langle D_u\psi(x,v), \hat{u} - u \rangle \\ &= -c(x) + \langle D_u\psi(x,v), v - \hat{u} \rangle\, |\hat{u} - u|/|v - \hat{u}| \\ &\geqslant -c(x) + \langle D_u\psi(x,\hat{u}), v - \hat{u} \rangle\, |\hat{u} - u|/|v - \hat{u}| \\ &= -c(x) + \langle D_u\psi(x,\hat{u}), \hat{u} - u \rangle \\ &\geqslant -\gamma(x),\end{aligned}$$

where $\gamma \in L^1(\Omega)$.

CASE 4. — $u \notin B_R$, $v \in B_R$. In this case we write

$$\theta(x,u,v) = [\psi(x,u)/|u| - \langle D_u\psi(x,v), u/|u| \rangle]\,|u| - \psi(u,v) - \langle D_u\psi(x,v), v \rangle$$

and the result follows from (kk) and (kkk).

4. APPLICATIONS

Theorem 3 and, more generally, Theorem 2, have been written in order to be applied to boundary value problems. In this section we illustrate three type of problems: the Dirichlet, the Neumann and the periodic solutions problem. They are considered to be at resonance with the first eigenvalue of the corresponding differential operator.

We denote by $\Delta^{(m)}$ the operator $\Delta \mathrm{Id}^{(m)}$, where $\mathrm{Id}^{(m)}$ is the identity $m \times m$ matrix. By λ_1 we denote the first eigenvalue of $(-\Delta)$ subject to the Dirichlet condition.

THEOREM 4. — *Let* $G : \Omega \times \mathbb{R}^m \to \mathbb{R}$ *be Caratheodory, Gateaux differentiable in its second variable, and verifying Assumption 2. Then the following problems have a solution.*

1) DIRICHLET PROBLEM.

$$-\Delta^{(m)}u(x) - \lambda_1 u(x) + D_u G(x,u(x)) = 0, \quad x \in \Omega$$

$$u(x) = 0, \quad x \in \partial\Omega$$

2) NEUMANN PROBLEM

$$-\Delta^{(m)}u(x) + D_u G(x,u(x)) = 0, \quad x \in \Omega$$

$$u(x) = 0, \quad x \in \partial\Omega$$

3) PERIODIC PROBLEM

$(n = 1,\ \Omega =]a, b[)$

$$-u''(x) + D_u G(x,u(x)) = 0, \quad x \in]a,b[$$

$$u(a) = u(b) \quad u'(a) = u'(b).$$

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