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A continuous version of Liapunov's convexity theorem

by

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ABSTRACT. — Given a continuous map \( s \mapsto \mu_s \) from a compact metric space into the space of nonatomic measures on \( T \), we show the existence of a family \( (A_\alpha^s)_{\alpha \in [0,1]} \) increasing in \( \alpha \) and continuous in \( s \), such that
\[
\mu_s(A_\alpha^s) = \alpha \mu_s(T) \quad (\alpha \in [0,1]).
\]

Key words: Liapunov's convexity theorem - Measure theory - Selections.

RÉSUMÉ. — Étant donnée une application continue \( s \mapsto \mu_s \), d'un espace métrique compact dans l'espace des mesures nonatomices sur \( T \), nous montrons l'existence d'une famille \( (A_\alpha^s)_{\alpha \in [0,1]} \) croissante avec \( \alpha \) et continue en \( s \), telle que
\[
\mu_s(A_\alpha^s) = \alpha \mu_s(T) \quad (\alpha \in [0,1]).
\]

I. INTRODUCTION

Let \( \mu \) be a non-atomic finite measure on a measurable space \( T \). A result of measure theory states the existence of a family \( (A_\alpha)_{\alpha \in [0,1]} \) of subsets of \( T \), increasing with \( \alpha \) in \( [0,1] \) and such that
\[
\mu(A_\alpha) = \alpha \mu(T).
\]
According to Liapunov's Convexity Theorem on the range of vector measures (see Halmos [2], [3] and Liapunov [4]) the above result holds for a finite family of nonatomic measures \( \mu_i, i=1, \ldots, n \): there exists an increasing family \( (A_\alpha)_{\alpha} \) such that

\[
\mu_i(A_\alpha) = \alpha \mu_i(T), \quad i=1, \ldots, n.
\]

In general, the above is not true for an infinite family \( (\mu_s)_s \) of measures (see Liapunov [5]). In this paper we consider a map \( s \to \mu_s \), continuous for \( s \) in a compact metric space \( S \). Denoting by \( \mathcal{A}(\mu) \) the set of increasing families \( (A_\alpha^s)_{\alpha} \) satisfying

\[
\mu_s(A_\alpha^s) = \alpha \mu_s(T),
\]

we show the existence of a selection \( (\hat{A}_\alpha^s)_{\alpha} \) of the multivalued map \( \mathcal{A}(\mu) \) continuously depending on \( s \) in the sense of Definition 2 of the following section.

2. NOTATIONS AND PRELIMINARY RESULTS

We consider a measure space \( (T, \mathcal{F}, \mu_0) \) where \( \mu_0 \) is a non-atomic positive measure on a \( \sigma \)-algebra \( \mathcal{F} \) and \( \mu_0(T)=1 \). Denote by \( \mathcal{M} \) the set of positive finite measures \( \mu \) on \( T \) which are absolutely continuous with respect to \( \mu_0 \), hence non-atomic. The metric in \( \mathcal{M} \) is induced by the norm \( \| \mu \| \) given by the variation of \( \mu \).

**Definition 1.** A family \( (A_\alpha)_{\alpha \in [0, 1]} \), \( A_\alpha \in \mathcal{F} \), is called increasing if

\[
A_\alpha \subseteq A_\beta \quad \text{when} \ \alpha \leq \beta.
\]

An increasing family is called refining \( A \in \mathcal{F} \) with respect to the measure \( \mu=(\mu_1, \ldots, \mu_n) \in \mathcal{M}^n \) if \( A_0=\emptyset, A_1=A \) and

\[
\mu(A_\alpha) = \alpha \mu(A) \quad (\alpha \in [0, 1]).
\]

The set of the families refining \( T \) with respect to \( \mu \) is denoted by \( \mathcal{A}(\mu) \).

The proofs of Lemmas 1 and 2 are based on Liapunov's theorem (see Fryszkowski [1]).

**Lemma 1.** Consider a vector measure \( \mu \in \mathcal{M}^n \). For each \( A \in \mathcal{F} \) there exists a family \( (A_\alpha)_{\alpha \in [0, 1]} \) refining \( A \) with respect to \( \mu \). In particular, the set \( \mathcal{A}(\mu) \) is nonempty.

In what follows, \( S \) is a compact metric space with distance \( d \).
LEMMA 2. — Let $s \to \mu_s$ be a continuous map from $S$ into $\mathcal{M}^n$. Then for every $\varepsilon > 0$ there exists an increasing family $(A_{\alpha})_\alpha$ satisfying

(i) $\mu_0(A_{\alpha}) = \alpha (\alpha \in [0, 1])$;

(ii) $|\mu_s(A_{\alpha}) - \alpha \mu_s(T)| < \varepsilon (\alpha \in [0, 1], s \in S)$.

DEFINITION 2. — A map $s \to (A_{\alpha})_\alpha$ is called continuous on $S$ if for every $s^0 \in S$ and $\varepsilon > 0$ there exists a $\delta > 0$ such that: $s$, $s'$ and $s''$ in $B(s^0, \delta)$ implies

$$\sup_{\alpha \in [0, 1]} \mu_s(A_{\alpha} \triangle A_{\alpha}'') < \varepsilon.$$ 

Analogously we set

DEFINITION 3. — The set valued map $s \to \mathcal{A}(\mu_s)$ is called continuous if for every $s^0 \in S$ and $\varepsilon > 0$ there exists a $\delta > 0$ such that: $s$, $s'$ and $s''$ in $B(s^0, \delta)$ implies $\forall (A_{\alpha}') \in \mathcal{A}(\mu_s)$, $\exists (A_{\alpha}'') \in \mathcal{A}(\mu_s')$ such that

$$\sup_{\alpha \in [0, 1]} \mu_s(A_{\alpha}' \triangle A_{\alpha}'') < \varepsilon.$$ 

We will use the symbol $\bigcup$ to denote the union of disjoint sets. Finally, we recall that $\rho(\cdot, \cdot)$ defined as $\rho(A, B) = \mu(A \Delta B) (\mu \in \mathcal{M})$ is a pseudometric on $\mathcal{F}$.

Remarks. — (a) In [5], Liapunov considers a sequence $\mu_n$ of measures on $[0, 2 \pi]$ defined by a family of densities $f_n$ converging strongly in $L^1$ to zero. He shows that there cannot exist any Borel subset $A$ of $[0, 2 \pi]$ such that for every $n$, $\mu_n(A) = \frac{1}{2} \mu_n([0, 2 \pi])$. By associating $\mu_n$ to the point $1/n$ and $\mu_\infty = 0$ to the point 0, we have a map $s \to \mu_s$ from the compact metric space $S = \{1/n : n \in \mathbb{N}\} \cup \{0\}$ into the space of nonatomic measures. The continuity at 0 follows from the strong convergence of $(f_n)$. This example shows that the assumptions of Theorem 1 below do not guarantee the existence of a constant selection.

(b) A further example is taken from Valadier [7]. Let $S$ and $T$ be the real interval $[0, 1]$, and set $\mu_s(A) = \int_A e^{-st} \, dt$. Assume there exists a set $\bar{A} \subseteq T$ such that

$$\forall s, \quad \mu_s(A) = \frac{1}{2} \mu_s(T).$$

Then

$$\int_{-\infty}^{+\infty} \chi_A(t) e^{-st} \, dt = \int_{-\infty}^{+\infty} \frac{1}{2} \chi_T(t) e^{-st} \, dt.$$
Since the Laplace transformations of $\chi_A$ and $\frac{1}{2}\chi_T$, both of compact support, are analytic and coincide on $[0,1]$, they are identical. By the injectivity of the Laplace Transformation, we have

$$\chi_A = \frac{1}{2}\chi_T,$$

a contradiction. Hence again we have an example where there exist no constant selections.

(c) It seems more natural to express the continuity in terms of the pseudometric $p(A, B) = \mu_0(A, B)$. However, Definition 2 is not necessarily equivalent to the continuity with respect to this pseudometric when $\mu_0$ is not absolutely continuous with respect to $\mu_{\delta_0}$.

3. MAIN RESULTS

In order to prove our main theorem we need three additional Lemmas.

**Lemma 3.** Consider a 1-dimensional measure $\mu \in \mathcal{M}$ and an increasing family $(A^4_\alpha)_\alpha$ such that for some $\varepsilon > 0$,

$$|\mu(A^4_\alpha) - \alpha \mu(T)| < \varepsilon \quad (\alpha \in [0, 1]).$$

There exists an increasing family $(A^2_\alpha)_\alpha$ such that

(i) $\mu(A^2_\alpha) = \alpha \mu(T) \quad (\alpha \in [0, 1])$

(ii) $\mu(A^1_\alpha \triangle A^2_\alpha) < 6 \varepsilon \quad (\alpha \in [0, 1])$.

**Proof.** Fix $M$ so that $\frac{1}{M} > \frac{\varepsilon}{\mu(T)} \geq \frac{1}{n+1}$. We begin by defining recursively an increasing family $(A^2_\alpha)_\alpha$ for $\alpha = i/M$, $i = 0, \ldots, M$, such that (i) holds and $A^2_{i/M} \subseteq A^1_{(i+1)/M}$. Set $A^2_0 = \emptyset$ and assume $A^2_{i/M}$ has been defined for $i = 0, \ldots, n < M$.

**Case 1.** When $\mu(A^1_{(n+1)/M}) \geq \frac{n+1}{M} \mu(T)$, define $A^2_{(n+1)/M}$ by Lemma 1, as a set such that $A^2_{n/M} \subseteq A^2_{(n+1)/M} \subseteq A^1_{(n+1)/M}$ and

$$\mu(A^2_{(n+1)/M}) = \frac{n+1}{M} \mu(T).$$

**Case 2.** When $\mu(A^1_{(n+1)/M}) < \frac{n+1}{M} \mu(T)$, we first notice that by the
choice of $M$ we have that $\mu(A_{(n+2)/M}^{1}) \geq \frac{n+1}{M} \mu(T)$; hence we can define $A_{(n+1)/M}^{2}$ as a set such that $A_{(n+1)/M}^{1} \subseteq A_{(n+1)/M}^{2} \subseteq A_{(n+2)/M}^{1}$ and

$$\mu(A_{(n+1)/M}^{2}) = \frac{n+1}{M} \mu(T).$$

Notice that $A_{(n+1)/M}^{2} \supseteq A_{n/M}^{2}$, since $A_{(n+1)/M}^{1} \supseteq A_{n/M}^{2}$ by the inductive hypothesis.

In either case, we have

$$\mu(A_{(n+1)/M}^{2} \triangle A_{(n+1)/M}^{1}) = \left| \mu(A_{(n+1)/M}^{2}) - \mu(A_{(n+1)/M}^{1}) \right|$$

$$\leq \left| \mu(A_{(n+1)/M}^{2}) - \frac{n+1}{M} \mu(T) \right|$$

$$+ \left| \mu(A_{(n+1)/M}^{1}) - \frac{n+1}{M} \mu(T) \right|$$

$$< \varepsilon.$$  

By Lemma 1 it is now easy to define a family $(A_{x}^{2})_{x \in [0, 1]}$ such that

(a) $A_{i/M}^{2} \subseteq A_{x}^{2} \subseteq A_{(i+1)/M}^{2}$ for $\frac{i}{M} \leq x \leq \frac{i+1}{M}$;

(b) $\mu(A_{x}^{2}) = \alpha \mu(T)$.

Now we check that (ii) holds for $\frac{i}{M} \leq x \leq \frac{i+1}{M}$. We can as well assume that $\mu(T) \geq \alpha \varepsilon$ otherwise (ii) trivially holds.

$$\mu(A_{x}^{1} \triangle A_{x}^{2}) = \mu(A_{x}^{1} \setminus A_{x}^{2}) + \mu(A_{x}^{2} \setminus A_{x}^{1})$$

$$\leq \mu(A_{(i+1)/M}^{1} \setminus A_{i/M}^{1}) + \mu(A_{i/M}^{1} \setminus A_{i/M}^{2})$$

$$+ \mu(A_{(i+1)/M}^{2} \setminus A_{i/M}^{2}) + \mu(A_{i/M}^{2} \setminus A_{i/M}^{1})$$

$$\leq \frac{1}{M} \mu(T) + 2 \varepsilon + \frac{1}{M} \mu(T) + \varepsilon$$

$$\leq 2 \frac{\varepsilon \mu(T)}{\mu(T) - \varepsilon} + 3 \varepsilon = \frac{2 \varepsilon}{1 - (\varepsilon/\mu(T))} + 3 \varepsilon$$

$$< \left( \frac{12}{5} + 3 \right) \varepsilon < 6 \varepsilon.$$  

**Corollary.** — The set-valued map $s \rightarrow \mathcal{A}(\mu)$ is continuous.
Proof. — Choose $s^0$ and $\varepsilon > 0$. Let $\delta > 0$ be such that $d(s, s^0) < \delta$ implies $\|\mu_s - \mu_s^0\| < \varepsilon/26$. Fix $s$, $s'$ and $s''$ in $B(s^0, \delta)$ and $A_s \in \mathcal{A}(\mu_s)$. Since
\[
\left| \mu_s(A_s) - \alpha \mu_s^0(T) \right| = \left| \mu_s(A_s) - \mu_s'(A_s) + \mu_s'(A_s') \right|
- \alpha \mu_s'(T) + \alpha \mu_s'(T) - \alpha \mu_s^0(T) \right| \\
\leq 2 \|\mu_s - \mu_s^0\| < \varepsilon/13,
\]
by Lemma 3 there exists $A_s^0 \in \mathcal{A}(\mu_s^0)$ such that $\mu_s^0(A_s^0 \triangle A_s^0) \leq 6 \varepsilon/13$. Analogously, given $A_s^0$, there exists $A_s' \in \mathcal{A}(\mu_s')$ such that $\mu_s'(A_s^0 \triangle A_s') \leq 6 \varepsilon/13$.

Hence
\[
\mu_s(A_s^0 \triangle A_s') \leq \|\mu_s - \mu_s^0\| + \mu_s^0(A_s^0 \triangle A_s^0) + \mu_s'(A_s^0 \triangle A_s') \leq \varepsilon/26 + 6 \varepsilon/13 + \|\mu_s - \mu_s'\| + \mu_s'(A_s^0 \triangle A_s') \leq \varepsilon.
\]

In the following Lemmas, the symbol $\sup_{\lambda_j(s) > 0}$ is a shorthand notation for $\sup_{\{j \in \mathbb{N} : \lambda_j(s) > 0\}}$.

Lemma 4. — Let $s \rightarrow \mu_s$ be a continuous map from a metric space $S$ into the space $\mathcal{M}$ and let $(B(s_j, \eta_j))_{j=1,\ldots,N}$ be a finite open covering of $S$. Let $(\lambda_j(\cdot))_{j=1,\ldots,N}$ be a continuous partition of unity subordinate to it such that $\lambda_j(s_j) = 1$.

For any center $s_j$, $j = 1, \ldots, N$, let be defined a finite increasing family $\left(\tilde{A}_{i/M}^j\right)_{i=0,\ldots,M}$ such that
\[
\mu_s(\tilde{A}_{i/M}^j) = \frac{i}{M} \mu_s^j(T) \quad (i \in \{0, \ldots, M\}).
\]

Then for each $s \in S$ there exists an increasing family $(A_s^0)$ that extends the family $\left(\tilde{A}_{i/M}^j\right)_{i=0,\ldots,M}$ in the sense that $A_{i/M}^j = \tilde{A}_{i/M}^j$ for every $i$ and $j$, and such that the following properties hold:

(i) $\left| \mu_s(A_s^0) - \alpha \mu_s^0(T) \right| \leq 6 \sup_{\lambda_j(s) > 0} \|\mu_s - \mu_s^j\|(\alpha \in [0, 1])$;

(ii) for $\alpha \in \left[\frac{i}{M}, \frac{i+1}{M}\right]$ and any center $s_j$,
\[
\mu_s(A_s^0 \triangle A_s^0) \leq \sup_{\lambda_k(s) > 0} \mu_s(\tilde{A}_{i+1/M}^j \triangle \tilde{A}_{(i+1)/M}^j)
\]

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\[ + \sup_{\lambda_k(s) > 0} \| \mu_j - \mu_k \| + \frac{1}{M} \left( \sup_{\lambda_k(s) > 0} \mu_k(T) + \mu_j(T) \right) \]

(iii) \[ \lim_{s \to s^*} \sup_{\alpha \in [0,1]} \mu_0(A^*_s \triangle A^*_s) = 0. \]

Proof. — For each \( s \in S \), first we will define the sets \((A^i_{i/M})_i\) by interpolating among the given families \((\bar{A}^{s_i}_{i/M})_i\), taking from each set a subset having measure proportional to the corresponding \(\lambda_i(s)\). Then we extend the construction for \(\alpha \in [i/M, (i+1)/M[\). Finally we check that (i)-(iii) hold.

I. For any set \( A \subseteq T \), we define \( A^1 = A \) and \( A^0 = T \setminus A \). We denote by \( \mathcal{X} \) the set of all \( N \times (M - 1) \) matrices \( \Gamma = (\gamma_{ij}) \) whose elements are in \( \{0, 1\} \).

Now we define

\[ A(\Gamma) = (\bar{A}^{s_1}_{1/M})^{\gamma_{11}} \cap \ldots \cap (\bar{A}^{s_N}_{1/M})^{\gamma_{1N}} \]

\[ \cap (\bar{A}^{s_1}_{2/M})^{\gamma_{21}} \cap \ldots \cap (\bar{A}^{s_N}_{2/M})^{\gamma_{2N}} \]

\[ \ldots \ldots \ldots \ldots \]

\[ \cap (\bar{A}^{s_1}_{(M-1)/M})^{\gamma_{M-1,1}} \cap \ldots \cap (\bar{A}^{s_N}_{(M-1)/M})^{\gamma_{M-1,N}}. \]

Note that:

(a) since the family \((\bar{A}^{s_i}_{i/M})_i\) is increasing in \( i \), \( A(\Gamma) = \emptyset \) if \( \exists i, j: \gamma_{ij} = 1 \), \( \gamma_{i+1,j} = 0 \); moreover, if \( \Gamma_1 \neq \Gamma_2 \), then \( A(\Gamma_1) \cap A(\Gamma_2) = \emptyset \);

(b) for any \( i, j \)

\[ \bar{A}^{s_j}_{i/M} = \bigcup_{\Gamma \in \mathcal{X} \ : \ \gamma_{ij} = 1} A(\Gamma), \]

i.e. the family at the r. h. s. is a partition of \( \bar{A}^{s_j}_{i/M} \);

(c) \[ \bigcup_{\Gamma \in \mathcal{X} \ : \ \gamma_{ij} = 0, \ \gamma_{ik} = 1} A(\Gamma) = A^{s_j}_{i/M} \setminus A^{s_j}_{i/M} \]

\[ \bigcup_{\Gamma \in \mathcal{X} \ : \ \gamma_{ij} = 1, \ \gamma_{ik} = 1} A(\Gamma) = A^{s_j}_{i/M} \cap A^{s_j}_{i/M}. \]

By lemma 1, for each \( \Gamma \in \mathcal{X} \) there exists a family \((A(\Gamma))_{\alpha \in [0,1]}\) refining \( A(\Gamma) \) with respect to the measure \((\mu_0, \mu_{s_1}, \ldots, \mu_{s_N})\). Define

\[ \beta_{i}(s) = \sum_{k=1}^{N} \gamma_{ik} \lambda_k(s) \]

and

\[ A^{s_j}_{i/M} = \bigcup_{\Gamma \in \mathcal{X}} A(\Gamma) \beta_{i}^{s_j}(s) \]  \hspace{1cm} (1)

(see Fig., where the case \( N = M = 3 \) is described).

The family \((A_i^{s_j})\) coincides with \((\bar{A}_i^{s_j})\) for \(s=s_j\); in fact we have \(\beta_{s}^r(s) = \gamma_{ij}\) so that, by (b),

\[
A_i^{s_j} = \bigcup_{\Gamma \in \mathcal{X}} A(\Gamma)_{\gamma_{ij}} = \bigcup_{\Gamma \in \mathcal{X}} A(\Gamma) = \bar{A}_i^{s_j}.
\]

Next we have:

\[
\mu_{s_j}(A_i^{s_j}) = \sum_{\Gamma \in \mathcal{X}} \mu_{s_j}(A(\Gamma)_{\beta_{s}^r(s)}) = \sum_{\Gamma \in \mathcal{X}} \beta_{s}^r(s) \mu_{s_j}(A(\Gamma))
= \sum_{\Gamma \in \mathcal{X}} \left( \sum_{k=1}^{N} \gamma_{ik} \lambda_k(s) \right) \mu_{s_j}(A(\Gamma))
= \sum_{k=1}^{N} \lambda_k(s) \sum_{\Gamma \in \mathcal{X}} \gamma_{ik} \mu_{s_j}(A(\Gamma))
\]

(2)
II. Set, for \( \alpha = (1 - t)i/M + t(i + 1)/M \) \((t \in [0, 1])\) and \( s \in S\),

\[
A^s_\alpha = \bigcup_{\Gamma \in \mathcal{X}} A(\Gamma)(1 - t)\beta^i_t(s) + t\beta^{i+1}_t(s).
\]

Remark that by the above definition and (1), it follows that

\[
\mu_{s_j}(A^s_\alpha) = (1 - t)\mu_{s_j}(A^s_{i/M}) + t\mu_{s_j}(A^s_{i+1/M}),
\]

We claim that

\[
\mu_{s_j}(A^s_\alpha) = \sum_{k=1}^{N} \lambda_k(s) \mu_{s_j}(A^s_{\alpha_k}) \quad (j = 1, \ldots, N; \alpha \in [0, 1]; s \in S).
\]

In fact, for \( \alpha \) as above, we have:

\[
\mu_{s_j}(A^s_\alpha) = \sum_{\Gamma \in \mathcal{X}} [((1 - t)\beta^i_t(s) + t\beta^{i+1}_t(s))\mu_{s_j}(A(\Gamma))
\]

\[= (1 - t)\sum_{k=1}^{N} \lambda_k(s) \sum_{\Gamma \in \mathcal{X}} \gamma_{ik} \mu_{s_j}(A(\Gamma))
\]

\[+ t\sum_{k=1}^{N} \lambda_k(s) \sum_{\Gamma \in \mathcal{X}} \gamma_{i+1, k} \mu_{s_j}(A(\Gamma))
\]

\[= (1 - t)\sum_{k=1}^{N} \lambda_k(s) \mu_{s_j}(A^s_{i/M}) + t\sum_{k=1}^{N} \lambda_k(s) \mu_{s_j}(A^s_{i+1/M})
\]

\[= \sum_{k=1}^{N} \lambda_k(s) [(1 - t)\mu_{s_j}(A^s_{i/M}) + t\mu_{s_j}(A^s_{i+1/M})]
\]

\[= \sum_{k=1}^{N} \lambda_k(s) \mu_{s_j}(A^s_{\alpha_k}).
\]

III. We are now in the position of proving (i). Fix \( s \in S \) and \( \alpha \in [0, 1] \) and set \( \omega_s = \sup \left\{ \|\mu_s - \mu_{s_j}\| : \lambda_j(s) > 0 \right\} \). We have:

\[
|\mu_s(A^s_\alpha) - \alpha \mu_s(T)| \leq |\mu_s(A^s_\alpha) - \mu_{s_j}(A^s_\alpha)|
\]

\[+ |\mu_{s_j}(A^s_\alpha) - \alpha \mu_{s_j}(T)| + \alpha |\mu_{s_j}(T) + \mu_s(T)|
\]
In order to prove (ii), note first that

\begin{equation}
\mu_{s_j}(\bigcup_{\Gamma \in \mathcal{X}} A(\Gamma)p_{1}^{i}(s_j)) = \sum_{k=1}^{N} \lambda_{k}(s) \mu_{s_j}(A_{i/M}^{s_k} \Delta A_{i/M}^{s_j}) \tag{4}
\end{equation}

\begin{equation}
\mu_{s_j}(\bigcup_{\Gamma \in \mathcal{X}} A(\Gamma)p_{1}^{i}(s_j)) = \sum_{k=1}^{N} \lambda_{k}(s) \mu_{s_j}(A_{i/M}^{s_j} \Delta A_{i/M}^{s_k}) \tag{5}
\end{equation}

and from (4), (5) this last expression is

\begin{equation}
\sum_{k=1}^{N} \lambda_{k}(s) \mu_{s_j}(A_{i/M}^{s_k} \Delta A_{i/M}^{s_j}) \geq \sup \{ \mu_{s_j}(A_{i/M}^{s_k} \Delta A_{i/M}^{s_j}) : \lambda_{k}(s) > 0 \}.
\end{equation}

Hence (ii) holds for \( \alpha = i/M \).

In order to prove (ii) for \( \alpha \in [i/M, (i+1)/M[ \), let us note that

\begin{equation}
A_{s}^{\alpha} \Delta A_{s}^{\alpha} \subseteq [(A_{(i+1)/M}^{s_j} \Delta A_{(i+1)/M}^{s_j}) \setminus A_{s}^{\alpha}] \cup [A_{(i+1)/M}^{s_j} \setminus A_{s}^{\alpha}] \subseteq (A_{(i+1)/M}^{s_j} \setminus A_{(i+1)/M}^{\alpha}) \cup (A_{(i+1)/M}^{s_j} \setminus A_{(i+1)/M}^{s_j}),
\end{equation}

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so that
\[ \mu_{s_j}(A^s_a \setminus A^s_j) \leq \mu_{s_j}(A^s_{(i+1)/M} \setminus A^s_j) + \mu_{s_j}(A^s_j \setminus A^s_{(i+1)/M}) \]
and
\[ \mu_{s_j}(A^s_j \setminus A^s_2) \leq \mu_{s_j}(A^s_{(i+1)/M} \setminus A^s_j) + \mu_{s_j}(A^s_j \setminus A^s_{(i+1)/M}). \]
Hence
\[ \mu_{s_j}(A^s_a \triangle A^s_2) \leq \mu_{s_j}(A^s_{(i+1)/M} \triangle A^s_j) \]
\[ + \mu_{s_j}(A^s_j \setminus A^s_{(i+1)/M}) + \mu_{s_j}(A^s_j \setminus A^s_{(i+1)/M}) \]
\[ \leq \sup_{\lambda_k(s) > 0} \{ \mu_{s_j}(A^s_{(i+1)/M} \triangle A^s_j) : \lambda_k(s) > 0 \} \]
\[ + (1/M) \mu_{s_j}(T) + \sum_{k=1}^{N} \lambda_k(s) \mu_{s_j}(A^s_k \setminus A^s_{(i+1)/M}) \]
\[ \leq \sup_{\lambda_k(s) > 0} \mu_{s_j}(A^s_{(i+1)/M} \triangle A^s_j) + (1/M) \mu_{s_j}(T) \]
\[ + \sum_{k=1}^{N} \lambda_k(s) \left| \mu_{s_j}(A^s_k \setminus A^s_{(i+1)/M}) - \mu_{s_k}(A^s_k \setminus A^s_{(i+1)/M}) \right| \]
\[ + \sum_{k=1}^{N} \lambda_k(s) \mu_{s_k}(A^s_k \setminus A^s_{(i+1)/M}) \]
\[ \leq \sup_{\lambda_k(s) > 0} \mu_{s_j}(A^s_{(i+1)/M} \triangle A^s_j) \]
\[ + \sup_{\lambda_k(s) > 0} \left\| \mu_{s_j} - \mu_{s_k} \right\| \]
\[ + (1/M) \sup_{k=1, \ldots, N} \mu_{s_k}(T). \]
This proves (ii).
Finally we prove (iii); for \( \alpha = (1-t) i/M + t (i+1)/M \) we have
\[ \mu_0(A^s_{(1-t) i/M + t (i+1)/M}) = \sum_{\Gamma \in \mathcal{A}} \mu_0(A(\Gamma)(1-t) \beta^{i,1}_\Gamma(s) + t \beta^{i+1}_\Gamma(s)) \]
\[ \triangle A(\Gamma)(1-t) \beta^{i,1}_\Gamma(s) + t \beta^{i+1}_\Gamma(s) \]
\[ = \sum_{\Gamma \in \mathcal{A}} \{ \left| (1-t) \beta^{i,1}_\Gamma(s) + t \beta^{i+1}_\Gamma(s) \right| \}
\[ \leq (1-t) \sum_{\Gamma \in \mathcal{A}} \left| \beta^{i,1}_\Gamma(s) - \beta^{i+1}_\Gamma(s) \right| \mu_0(A(\Gamma)) \]
\[ + t \sum_{\Gamma \in \mathcal{A}} \left| \beta^{i+1}_\Gamma(s) - \beta^{i+1}_\Gamma(s) \right| \mu_0(A(\Gamma)). \]
By taking the limit as s tends to $s^*$ we conclude the proof.

**Lemma 5.** Let $s \rightarrow \mu_s$ be a continuous map from a compact metric space $S$ into the space $\mathcal{M}$ and, for each $s \in S$, let $(\bar{A}_s^* )_s$ be an increasing family, continuous with respect to $s$ and such that, for some $\varepsilon > 0$,

$$|\mu_s(\bar{A}_s^*) - \alpha \mu_s(T)| < \varepsilon \quad (\alpha \in [0, 1], s \in S).$$

For every $s \in S$ there exists an increasing family $(A_s^*)_s$ continuous with respect to $s$ and such that

(i) $|\mu_s(A_s^*) - \alpha \mu_s(T)| < \varepsilon/10 \quad (\alpha \in [0, 1]);$

(ii) $\sup_{\alpha \in [0, 1]} \mu_s(A_s^* \triangle A_s^*) < 10 \varepsilon.$

**Proof.** By continuity, for each $s \in S$ there is a $\eta_s > 0$ such that $d(s, s') < 2 \eta_s$ implies $\|\mu_s - \mu_{s'}\| < \varepsilon/60$ and $\mu_s(\bar{A}_s^* \triangle \bar{A}_s^*) < \varepsilon$. The open balls $B(s, \eta_s)$ cover $S$. Let $\{ B(s_j, \eta_j) : j = 1, \ldots, N \}$ be a finite sub-covering and $\{ \lambda_j : j = 1, \ldots, N \}$ be a continuous partition of unity subordinate to it and such that $\lambda_j(s_j) = 1, j = 1, \ldots, N.$

Let $(A_s^*)_s$ be the families defined by Lemma 3 by taking $\mu = \mu_{s_j}$.

Fix $j$ such that $\mu_{s_j}(T) = \max \{ \mu_{s_k}(T) : k = 1, \ldots, N \}$ and choose $M \geq 2 \mu_{s_j}(T)/\varepsilon$. By Lemma 4, extend the collection $(\bar{A}_{s_k}^*)_{k=0,\ldots,M}$ to the family $(A_s^*)_s \in [0, 1] (s \in S)$.

The continuity of $s \rightarrow (A_s^*)_s \in [0, 1]$ follows from (iii) of Lemma 4, recalling that $\mu_s \leq \mu_0$ for each $s \in S$.

The choice of $\eta_s$ and (i) of Lemma 4 imply that (i) holds. Moreover

$$\mu_{s_j}(\bar{A}_{s_j}^* \triangle A_{s_j}^*) \leq \mu_{s_j}(\bar{A}_{s_j}^* \triangle \bar{A}_{s_j}^*) + \mu_{s_j}(\bar{A}_{s_j}^* \triangle A_{s_j}^*) + \mu_{s_j}(A_{s_j}^* \triangle A_{s_j}^*).$$

By the choice of $\eta_s$ and (ii) of Lemma 3, the r. h. s. is bounded by $\varepsilon + 6 \varepsilon + \mu_{s_j}(A_{s_j}^* \triangle A_{s_j}^*)$,

which, by (ii) of Lemma 4 and the choice of $M$, yields

$$\mu_{s_j}(\bar{A}_{s_j}^* \triangle A_{s_j}^*) \leq \left( 9 + \frac{1}{60} \right) \varepsilon.$$

Since $\|\mu_{s_j} - \mu_s\| < \varepsilon/60$, (ii) follows.

The following theorem shows the existence of a selection $(\bar{A}_s^*)_s \in \mathcal{M}$, continuously depending on $s$.

**Theorem 1.** Let $s \rightarrow \mu_s$ be a continuous map from a compact metric space $S$ into the space $\mathcal{M}$. For every $s \in S$ there an increasing family $(\bar{A}_s^*)_s$ of measurable subsets of $T$ satisfying

$$\mu_s(\bar{A}_s^*) = \alpha \mu_s(T) \quad (\alpha \in [0, 1])$$

(6)
and such that the map \( s \to (\tilde{A}_a^s) \) is continuous.

**Proof.** — We assume that we have defined for \( s \) in \( S \) an increasing family \((A_a^{s,n})_a\) which is continuous with respect to \( s \) and satisfies

\[
|\mu_s(A_a^{s,n}) - \alpha \mu_s(T)| < 10^{-n}.
\]

By Lemma 2, the above is true for \( n=1 \) taking a family \((A_a^{s,1})_a\) constant with respect to \( s \).

We obtain the existence of an increasing family \((A_a^{s,n+1})_a\) continuous with respect to \( s \) and such that

\[
|\mu_s(A_a^{s,n+1}) - \alpha \mu_s(T)| < 10^{-(n+1)}
\]

and

\[
\mu_s(A_a^{s,n+1} \triangle A_a^{s,n}) < 10^{-(n-1)}. \tag{8}
\]

In fact, set in Lemma 5 \( \tilde{A}_a^s \) to be \( A_a^{s,n} \) and \( \varepsilon \) to be \( 10^{-n} \) to infer the existence of a family, denoted by \((A_a^{s,n+1})_a\), satisfying (7) and (8).

Consider now the sequence \((A_a^{s,n})_n\) defined by the above recursive procedure: we wish to show that it converges to a family \((\tilde{A}_a^s)_a\) which is continuous with respect to \( s \) and satisfies (6).

Property (8) implies that the sequence \((A_a^{s,n})_n\) \((s \text{ and } a \text{ fixed})\) is a Cauchy sequence in \( \mathcal{F} \) supplied with the pseudometric \( \rho_s(A, B) = \mu_s(A \triangle B) \). The procedure in Oxtoby [6], Chap. 10, defines a limit family \((\tilde{A}_a^s)_a\), which is increasing: \( \tilde{A}_a^s = \bigcup_{n \in \mathbb{N}} \bigcap_{m \geq n} A_a^{s,m} \).

By the inequality

\[
|\mu_s(A) - \mu_s(B)| \leq \mu_s(A \triangle B)
\]

and (7) we have

\[
\mu_s(\tilde{A}_a^s) = \lim_{n \to \infty} \mu_s(A_a^{s,n}) = \alpha \mu_s(T).
\]

In order to check the continuity of the map \( s \to (\tilde{A}_a^s)_a \), fix \( \varepsilon > 0 \) and \( s^0 \in S \). Since the inequality (8) is uniform with respect to \( s \) and \( a \), there exists an \( \tilde{n} \) such that \( \mu_s(A_a^{s,n} \triangle A_a^{\tilde{s},\tilde{n}}) < \varepsilon/5 \) for every \( s \) in \( S \) and \( \alpha \) in [0, 1]. Let \( \delta > 0 \) be such that

\[
||\mu_s - \mu_{s^0}|| < \varepsilon/10 \quad [s \in B(s^0, \delta)]
\]

and

\[
\sup_{a \in [0, 1]} \mu_s(A_a^{s,n} \triangle A_a^{s',\tilde{n}}) < \varepsilon/5 \quad [s, s' \text{ and } s'' \text{ in } B(s^0, \delta)].
\]

Then for every \( \alpha \in [0, 1] \), \( s \), \( s' \) and \( s'' \) in \( B(s^0, \delta) \), we have:

\[
\mu_s(\tilde{A}_a^{s'} \triangle \tilde{A}_a^{s''}) \leq \mu_s(\tilde{A}_a^{s'} \triangle A_a^{s,n}) + \mu_s(A_a^{s,n} \triangle \tilde{A}_a^{s''}) \\
\quad \leq \mu_s(\tilde{A}_a^{s'} \triangle A_a^{s,n}) + \mu_s(A_a^{s,n} \triangle A_a^{s',\tilde{n}}) + \mu_s(A_a^{s',\tilde{n}} \triangle \tilde{A}_a^{s''})
\]

COROLLARY. - Under the same assumptions, for every $\eta > 0$ and for every increasing family $(A_a)_a$ satisfying

$$\mu_s(A) - \mu_s(T) < \eta \quad (\alpha \in [0, 1], s \in S),$$

the family $(\tilde{A}^s_a)_a$ of Theorem 1 can be chosen as to satisfy, in addition,

$$\mu_s(\tilde{A}^s_a \triangle A_a) < \eta \quad (\alpha \in [0, 1], s \in S).$$

Proof. — Set $A^{-1}_a$ to be $A_a$ in the proof of Theorem 1. ■

REFERENCES


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