CURVES AND SYMMETRIC SPACES, I

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Dedicated to Professor Heisuke Hironaka on His Sixtieth Birthday

We have announced some results on the canonical model of a compact Riemann surface in [M3]. In this article we prove them for a complete algebraic curve of genus 7. Curves of genus 8 and 9 will be treated in the forthcoming Part II.

Let $\wedge^\nu U$ be the even part of the exterior algebra over a $\nu$-dimensional vector space $U$. The exponential map embeds the affine space $\wedge^2 U$ into the projective space $\mathbb{P}_* (\wedge U)$. The closure $X$ of this image is a homogeneous space of the special orthogonal group $SO(2\nu)$ and parametrizes all Lagrangean subspaces $U'$ of the $2\nu$-dimensional quadratic space

$$\left(U \oplus U^{\vee}, \begin{pmatrix} 0 & 1 \nu \\ 1 \nu & 0 \end{pmatrix}\right)$$

with dim $U' \cap U \equiv \nu \mod 2$. This projective variety $X \subset \mathbb{P}_* (\wedge U)$ is called the (even) orthogonal Grassmannian. In the case $\nu = 5$, $X \subset \mathbb{P}^{15}$ is the 10-dimensional projective variety defined by the following 10 quadratic forms:

$$N_1 = \xi_0 \xi_{2345} - \xi_{23} \xi_{45} + \xi_{24} \xi_{35} - \xi_{25} \xi_{34},$$
$$N_{-1} = \xi_{12} \xi_{1345} - \xi_{13} \xi_{1245} + \xi_{14} \xi_{1235} - \xi_{15} \xi_{1234},$$
$$N_2 = \xi_0 \xi_{1345} - \xi_{13} \xi_{45} + \xi_{14} \xi_{35} - \xi_{15} \xi_{34},$$
$$N_{-2} = \xi_{12} \xi_{2345} - \xi_{23} \xi_{245} + \xi_{24} \xi_{1235} - \xi_{25} \xi_{1234},$$
$$N_3 = \xi_0 \xi_{1245} - \xi_{12} \xi_{45} + \xi_{14} \xi_{25} - \xi_{15} \xi_{24},$$
$$N_{-3} = \xi_{13} \xi_{2345} - \xi_{23} \xi_{1345} + \xi_{34} \xi_{1235} - \xi_{35} \xi_{1234},$$
$$N_4 = \xi_0 \xi_{1235} - \xi_{12} \xi_{35} + \xi_{13} \xi_{25} - \xi_{15} \xi_{23},$$

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\begin{align*}
N_{-4} &= \xi_{14}\xi_{2345} - \xi_{24}\xi_{1345} + \xi_{34}\xi_{1245} - \xi_{45}\xi_{1234}, \\
N_5 &= \xi_0\xi_{1234} - \xi_{12}\xi_{34} + \xi_{13}\xi_{24} - \xi_{14}\xi_{23}, \\
N_{-5} &= \xi_{15}\xi_{2345} - \xi_{25}\xi_{1345} + \xi_{35}\xi_{1245} - \xi_{45}\xi_{1235}.
\end{align*}

In [M1], we have observed that a transversal linear subspace of dimension 6 cuts out a (canonical) curve of genus 7 from $X \subset \mathbf{P}^{15}$ and proved that the generic curve of genus 7 is obtained in this way (over the complex number field). Here we make this earlier result into a final form:

**Main Theorem.** A curve $C$ of genus 7 is a transversal linear section of the 10-dimensional orthogonal Grassmannian $X \subset \mathbf{P}^{15}$ if and only if $C$ has no $g^1_4$. Moreover, the transversal linear subspaces which cut out $C$ are unique up to the action of $SO(10)$.

By this theorem and (0.1), the system of defining equations of the canonical curve $C_{12} \subset \mathbf{P}^6$ of genus 7 is now quite explicit in the non-tetragonal case. This result will be applied to the classification of Gorenstein Fano 3-folds (cf. [M2]). When $C$ is tetragonal, i.e., has a linear pencil $g^1_4$ of degree 4, its canonical model $C_{12} \subset \mathbf{P}^6$ is easier to describe by virtue of the presence of a ‘significant’ subvariety which contains $C_{12}$. See §6 and Table at the end of this article.

The proof of the ‘if’ part is better understood when compared with a result on a quintic normal elliptic curve $E_5 \subset \mathbf{P}^4$. Let $V$ be the space $H^0(\mathbf{P}^4, I_E(2))$ of quadratic forms on $\mathbf{P}^4$ which vanish identically on $E_5$. Then dim $V = 5$ and $E_5$ is the common zero locus of forms in $V$. Hence, for each point $p \in E_5$, the space $V_p$ of all forms $f \in V$ doubly vanishing at $p$ is of dimension 2. Therefore, we obtain a morphism $\rho_E$ to the 6-dimensional Grassmannian $G(2, V)$, which is a projective variety of $\mathbf{P}^9$ by the Plücker coordinate.

**Proposition 0.2.** The morphism $\rho_E : E_5 \longrightarrow G(2, V) \subset \mathbf{P}^9, p \mapsto [V_p]$, is an isomorphism onto a transversal linear section.

The proof is easy since a similar statement is almost obvious for a set of five points of $\mathbf{P}^4$ in general position. By the proposition, every quintic del Pezzo manifold $M_5 \subset \mathbf{P}^{m+3}$, $m = \dim M$, is a transversal linear section of the 6-dimensional Grassmannian. (Consult [Fuj] for more details on such manifolds.)

For a canonical curve $C_{12} \subset \mathbf{P}^6$ of genus 7, we argue similarly. Set $W = H^0(\mathbf{P}^6, I_C(2))$ and $W_p = \{f \in W \mid f(x_1, \cdots, x_7) = 0 \text{ is singular at } p \}$ for $p \in C_{12}$. When $C$ is not trigonal, $W$ is of dimension 10 and $W_p$ of dimension 5. Hence we obtain a morphism $\rho_C : C \longrightarrow G(5, W) \simeq G(5, W^\vee)$ to the 25-dimensional Grassmannian. Using the quadratic identity

\begin{equation}
N_1N_{-1} + N_2N_{-2} + N_3N_{-3} + N_4N_{-4} + N_5N_{-5} = 0
\end{equation}

among the 10 quadratic forms in (0.1), we show that the multiplication map
$S^2W \rightarrow H^0(\mathbb{P}^6, I_C^2(4))$ is not injective. Moreover, its kernel is generated by a nondegenerate symmetric tensor $\sigma$ if $C$ has no $g^1_4$ (Theorem 4.2). By our choice of $\sigma$, $W^\perp_p$ is a Lagrangean of the quadratic space $(W^\vee, \sigma)$ for every point $p \in C_{12}$, and the image of $\rho_C$ is contained in the 10-dimensional orthogonal Grassmannian $X$. We actually prove the following:

**Theorem 0.4.** If a curve $C$ of genus 7 has no $g^1_4$, then the Grassmannian morphism $\rho_C : C \rightarrow X \subset G(5, W^\vee)$, $p \mapsto [W^\perp_p]$, is an isomorphism onto a transversal linear section of $X \subset \mathbb{P}^{15}$.

By the construction and the uniqueness of $\sigma$, we have also

**Corollary 0.5.** If a curve $C$ of genus 7 is cut out from $X \subset \mathbb{P}^{15}$ by a transversal linear subspace $P$, then every automorphism of $C$ is the restriction of an automorphism of $X \subset \mathbb{P}^{15}$ which preserves $P$.

A key of our proof is the self-duality of $X \subset \mathbb{P}^{15}$: Its projective dual, or discriminant, is again a 10-dimensional orthogonal Grassmannian. More precisely, the projective dual is naturally identified with the odd orthogonal Grassmannian, which parametrizes all Lagrangean subspaces with $\dim U \cap U'' \neq \nu \text{ mod } 2$ (Proposition 2.7). We prove Theorem 0.4 in §5 using this duality. The ‘only if’ part of our Main Theorem is proved in §2 using the prehomogeneity of the 16-dimensional spin representation (Proposition 1.13).

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**Notation and conventions.** Varieties are considered over an algebraically closed field $k$ of arbitrary characteristic. A smooth complete variety of dimension one is simply called a curve. A $g^1_d$ on a curve $C$ is a line bundle $\xi$ of degree $d$ with $\dim H^0(C, \xi) \geq r + 1$. For a vector space $V$, $G(s, V)$ is the Grassmannian of $s$-dimensional subspaces of $V$ and $G(V, r)$ that of $r$-dimensional quotient spaces. Two projective spaces $G(1, V)$ and $G(V, 1)$ associated to $V$ are denoted by $\mathbb{P}_s(V)$ and $\mathbb{P}^s(V)$, respectively. The dual vector space of $V$ is $V^\vee$. More generally, for a vector bundle $E$, $E^\vee$ is the dual vector bundle.

1. **10-dimensional orthogonal Grassmannian.** Let $V$ be a $2\nu$-dimensional vector space with a quadratic form $q : V \rightarrow k$. We assume that $q$ is nondegenerate, that is, the associated symmetric bilinear form $B(x, y) = q(x + y) - q(x) - q(y)$ is so. A $\nu$-dimensional subspace over which $q$ is identically zero is called a Lagrangean of $(V, q)$. We denote the set of Lagrangeans by $\mathcal{L}(V, q)$, which is a subset of the Grassmannian $G(\nu, V)$. We fix a Lagrangean $U_\infty$ and set $Z = \{[U] \mid U \cap U_\infty = 0\}$ in $G(\nu, V)$. When $[U_0] \in Z$ is fixed, $Z$ is isomor-
phic to the \( \nu^2 \)-dimensional affine space associated with \( \text{Hom}(U_0, U_\infty) \) by the map

\[
\text{Hom}(U_0, U_\infty) \ni f \mapsto [\Gamma_f] \in G(\nu, V),
\]

where \( \Gamma_f \subset U_0 \times U_\infty = V \) is the graph of \( f \). Since \( V \simeq U_0 \oplus U_\infty, \wedge^\nu \! V \) is isomorphic to \( \bigoplus_i \text{Hom}(\wedge^i U_0, \wedge^i U_\infty) \). The composite of (1.1) and the Plücker embedding \( G(\nu, V) \subset \mathbf{P}^* (\wedge^\nu \! V) \) is equal to

\[
\text{Hom}(U_0, U_\infty) \ni f \mapsto 1 + f + (f \wedge f) + (f \wedge f \wedge f) + \cdots \in \mathbf{P}^* \left( \bigoplus_{i=1}^\nu \text{Hom} \left( \wedge^i U_0, \wedge^i U_\infty \right) \right).
\]

Assume that \( U_0 \) is also a Lagrangean. Then \( U_0 \) and \( U_\infty \) are dual to each other by the bilinear form \( B \).

**Proposition 1.3.** For \( f \in \text{Hom}(U_0, U_\infty) \simeq U_\infty \otimes U_\infty, \Gamma_f \) is a Lagrangean if and only if \( f \) lies in the kernel of the natural map \( U_\infty \otimes U_\infty \rightarrow S^2 U_\infty \).

By the proposition, \( \mathcal{L}(V, q) \) is a smooth subscheme of \( G(\nu, V) \). Moreover, by the map \( \wedge^2 U_\infty \rightarrow U_\infty \otimes U_\infty, a \wedge b \mapsto a \otimes b - b \otimes a \), the intersection \( \mathcal{L}(V, q) \cap Z \) is isomorphic to the \( \nu(\nu - 1)/2 \)-dimensional affine space \( \wedge^2 U_\infty \).

We take the exterior algebra \( \wedge^* \! U_\infty \simeq \wedge^\text{ev} \! U_\infty \oplus \wedge^{\text{odd}} \! U_\infty \) as the space \( S \) of spinors of the quadratic space \( (V, q) \). \( S = S^+ \oplus S^- \) is a \( \mathbb{Z}/2 \)-graded vector space with a Clifford map \( V \rightarrow \text{End} S, v \mapsto \varphi_v \). The endomorphism \( \varphi_v \) is a wedge product, or a creation operator, for \( v \in U_\infty \subset V \) and a derivation, or an annihilation operator, for \( v \in U_0 \simeq U_\infty^\vee \). The Clifford map is linear and satisfies

\[
\varphi_v(S^\pm) \subset S^\mp \quad \text{and} \quad \varphi_v^2 = q(v) \cdot 1_S
\]

for every \( v \in V \). For a Lagrangean \( U \), there exists a nonzero half spinor \( s_U \), i.e., an element of \( S^+ \cup S^- \), which satisfies \( \varphi_u(s_U) = 0 \) for every \( u \in U \). Such \( s_U \)'s are unique up to constant multiplication and called the pure spinor associated with \( U \). For example, \( s_U \) is equal to \( 1 \in \wedge^0 U_\infty \) if \( U = U_0 \) and a volume element in \( \wedge^\nu \! U_\infty \) if \( U = U_\infty \). The uniquely determined point \( [s_U] \) of the projective space \( \mathbf{P}^* (S^\pm) \) is called the spinor coordinate of \( U \).

**Proposition 1.5.** If \( \alpha \in \wedge^2 U_\infty \) is a bivector corresponding to a Lagrangean \( U \) in \( \mathbb{Z} \) via (1.1), then the exponential \( \exp (-\alpha) \in \wedge^\text{ev} \! U_\infty = S^+ \) is the pure spinor associated with \( U \).

**Proof.** Let \( \{e_1, \ldots, e_\nu\} \) be a basis of \( U_\infty \) and \( \{e_{-1}, \ldots, e_{-\nu}\} \subset U_0 \) its dual. We put \( -\alpha = \sum_{i<j} a_{ij} e_i \wedge e_j \) and let \( A \) be the skew-symmetric matrix \( (a_{ij})_{1 \leq i \leq \nu} \) with \( a_{ij} + a_{ji} = a_{ii} = 0 \). The Lagrangean \( U \) is generated by \( \nu \) vectors \( u_i = e_{-i} - \sum_{j=1}^\nu a_{ij} e_j, 1 \leq i \leq \nu \), of \( V \). For a subset \( I = \{i < j < \cdots < \ell\} \) of
$\{1, 2, \ldots, \nu\}$, let $e_I$ be $e_i \wedge e_j \wedge \cdots \wedge e_\ell$ and $A_I$ the principal minor $(a_{ij})_{i,j \in I}$ of $A$. Then $\exp(-\alpha)$ is equal to $\sum_{|I|:\text{even}} (\text{Pfaff} A_I) e_I$ (by definition in positive characteristic). By the expansion theorem of Pfaffian, e.g.,

$$\text{Pfaff} A = \sum_{i=2}^\nu (-1)^i a_{1i} \text{Pfaff} A_{\{1,i\}},$$

we have $\varphi_{\nu_i}(\exp(-\alpha)) = 0$ for every $i$. \qed

More generally, a half spinor $s \in S = \wedge^* U_\infty$ is pure if and only if it is representable in the form $(\exp \alpha) \wedge x_1 \wedge \cdots \wedge x_h$ for some bivector $\alpha \in \wedge^2 U_\infty$ and vectors $x_1, \ldots, x_h \in U_\infty$ (see [Ch] §3.1).

We put $\mathcal{L}^+(V, q) = \{[U] \in \mathcal{L}(V, q) \mid \dim U \cap U' = \nu \bmod 2\}$, which is a connected component of $\mathcal{L}(V, q)$. Both $\mathcal{L}^+(V, q)$ and its complement $\mathcal{L}^-(V, q)$ are homogeneous spaces of the special orthogonal group $SO(V, q)$, which we call the orthogonal Grassmannians associated with $(V, q)$. The following is easy to verify:

**Proposition 1.6.** Two Lagrangeans $U$ and $U'$ belong to the same component $\mathcal{L}^\pm(V, q)$ if and only if $\dim U \cap U' \equiv \nu \bmod 2$.

The pure spinor $s_U$ belongs to $S^+$ if and only if $U$ belongs to $\mathcal{L}^+(V, q)$. Since the annihilator $\{v \in V \mid \varphi_v(s_U) = 0\}$ of $s_U$ is $U$, the map

$$\mathcal{L}^+(V, q) \ni [U] \mapsto [s_U] \in \mathbf{P}_\ast(S^+)$$

is injective. Both $S_+$ and $S_-$ are irreducible representations of $Spin(V, q)$, a central extension of $SO(V, q)$ by $\{\pm 1\}$. The map is equivariant under the action of $SO(V, q)$, and an embedding by Proposition 1.5, which we call the spinor embedding. Since the determinant of a skew-symmetric matrix is the square of its Pfaffian, we have the following by (1.2) and Proposition 1.5:

**Proposition 1.7.** The hyperplane section of the composite $\mathcal{L}^+(V, q) \subset G(\nu, V) \subset \mathbf{P}_\ast(\wedge^\nu V)$ is linearly equivalent to twice the hyperplane section of the spinor embedding $\mathcal{L}^+(V, q) \subset \mathbf{P}_\ast(S^+)$.  

Let $\kappa$ be the projection of $\wedge U_\infty$ to the top part $\wedge U_\infty \cong k$ and define the bilinear form $\beta$ on $\wedge U_\infty$ by

$$(1.8) \quad \beta(\xi, \xi') = (-1)^{p(p+1)/2} \kappa(\xi \wedge \xi'), \quad p = \deg \xi$$

for every homogeneous element $\xi \in \wedge U_\infty$ and $\xi' \in \wedge U_\infty$ ([Ca] §101). This pairing $\beta$ is invariant under the action of $Spin(V, q)$, and called the fundamental polar form.

Now we put $\nu = 5$ and $X = \mathcal{L}^+(V, q)$. Let $\{e_{\pm 1}, \ldots, e_{\pm 5}\}$ and $A$ be as in the proof of Proposition 1.5. $\mathcal{L}^+(V, q)$ is a 10-dimensional projective variety and its
open subset is the image of
\[ \wedge^2 U_\infty \quad \longrightarrow \quad \mathbb{P}^{15}, \]
\[ \alpha = - \sum_{i<j} a_{ij} e_i \wedge e_j \quad \exp (-\alpha) = (1 : a_{ij} : \text{Pfaff}_{ijkl} A). \]

For every \( v \in V \), we define the quadratic form \( N_v \) on \( S^+ \) by \( N_v(s) = \beta(s, \varphi_v(s)) \) where \( \beta \) is the fundamental polar form in (1.8). We abbreviate \( N_{e_{\pm i}} \) by \( N_{\pm i} \) for \( i = 1, \ldots, 5 \). Then \( N_i(s) = \beta(s, e_i \wedge s) \) and \( N_{-i}(s) = \beta(s, \frac{\partial s}{\partial e_i}) \), and we have (0.1). By Proposition 1.5, \( X \) is the common zero locus of the five quadratic forms \( N_1, \ldots, N_5 \) on the affine open subset \( \wedge^2 U_\infty \). Since the embedding is \( SO(10) \)-equivariant, we have

**Proposition 1.9.** The 10-dimensional orthogonal Grassmannian \( X \subset \mathbb{P}^{15} \) is the common zero locus of the 10 quadratic forms in (0.1).

For an even spinor \( s \in S^+ \), we define a vector \( v(s) \in V \) by
\[ v(s) = N_{-5}(s)e_5 + \cdots + N_{-1}(s)e_1 + N_1(s)e_{-1} + \cdots + N_5(s)e_{-5}. \]

Then \( s \) is pure if and only if \( v(s) = 0 \). Consider the odd spinor
\[ \varphi_{v(s)}(s) = N_{-5}(s)e_5 \wedge s + \cdots + N_{-1}(s)e_1 \wedge s + N_1(s)\frac{\partial s}{\partial e_1} + \cdots + N_5(s)\frac{\partial s}{\partial e_5}. \]

A direct computation shows

**Proposition 1.10.** \( \varphi_{v(s)}(s) = 0. \)

By (1.3) or by a direct computation, we have

**Corollary 1.11.** \( q(v(s)) = 0 \), that is, \( N_1(s)N_{-1}(s) + \cdots + N_5(s)N_{-5}(s) = 0 \).

Assume that an even spinor \( s \in S^+ \) is not pure. The endomorphism \( \varphi_{v(s)} : S \to S \) is not zero but square zero. Its kernel coincides with its image and is identified with the space of spinors of the 8-dimensional quadratic space \( (v(s)^{1/2}/k \cdot v(s), \bar{q}) \). Let \( F \) be a 7-dimensional linear subspace of \( P_*(S^+) \) corresponding to \( \text{Ker} \ \varphi_{v(s)} \cap S^+ \). Then the intersection \( F \cap X \) is the orthogonal Grassmannian of \( (v(s)^{1/2}/k \cdot v(s), \bar{q}) \), which is a hyperquadric in \( F \cong \mathbb{P}^7. \) Hence we have

**Proposition 1.12.** For every point \( [s] \in \mathbb{P}^{15} \setminus X \), there exists a 7-dimensional linear subspace \( F_{[s]} \) such that \( F_{[s]} \owns [s] \) and \( F_{[s]} \cap X \) is a 6-dimensional quadric.

The special orthogonal group \( SO(V, q) \) acts on \( Q^8 = \{ [v] \mid q(v) = 0 \} \subset P_*(V) \) transitively and the stabilizer group \( SO(v^{1/2}/kv, \bar{q}) \) acts on \( F_{[s]} \setminus X \) transitively. Hence we have

**Proposition 1.13.** (II) The special orthogonal group \( SO(10) \) acts on the complement \( \mathbb{P}^{15} \setminus X \) of the 10-dimensional orthogonal Grassmannian \( X \) transitively.
We consider the quadric hull \( \bigcap_{Q \supset X \cup \{p\}} Q \) of the union of \( X \) and a point \( p \notin X \). Every secant line of \( X \) passing through \( p \) is contained in the hull. Hence \( F_p \) in Proposition 1.12 is contained in the hull.

**Proposition 1.14.** For every point \( p \notin X \), the quadric hull of \( X \cup \{p\} \) is the union of \( X \) and the 7-dimensional linear subspace \( F_p \) in Proposition 1.12.

**Proof.** By Proposition 1.13, we may assume that \( p \) corresponds to the even spinor \( s = 1 + e_1 \wedge e_2 \wedge e_3 \wedge e_4 \). Since \( v(s) = e_{-5} \), \( F_p \) is defined by

\[
(*) \quad \xi_{15} = \xi_{25} = \xi_{35} = \xi_{45} = \xi_{2345} = \xi_{1345} = \xi_{1245} = \xi_{1235} = 0.
\]

Assume that \( q \) corresponding to \( t = \sum \xi_i e_i \) belongs to the quadric hull of \( X \cup \{p\} \). Then \( t \) satisfies

\[
N_{-5}(t) = \cdots = N_{-1}(t) = N_1(t) = \cdots = N_4(t) = 0.
\]

If \( N_5(t) = 0 \), then \( t \) is pure and \( q \) belongs to \( X \). Hence it suffices to show that \( t \) satisfies \((*)\) assuming \( N_5(t) \neq 0 \). We denote the five \( 4 \times 4 \) principal minors of \( A = (\xi_{ij}) \), \( \xi_{ij} + \xi_{ji} = \xi_{ii} = 0 \), by \( A^{(1)}, \ldots, A^{(5)} \).

**Case 1.** Assume that \( \text{rk} A = 2 \), i.e., Pfaff \( A^{(i)} = 0 \) for \( 1 \leq i \leq 5 \). Since \( N_1(s) = \cdots = N_4(s) = 0 \neq N_5(s) \), we have \( \xi_0 \xi_{2345} = \cdots = \xi_0 \xi_{1235} = 0 \neq \xi_0 \xi_{1234} \) and hence \( \xi_{2345} = \xi_{1345} = \xi_{1245} = \xi_{1235} = 0 \). Since \( N_{-1}(s) = \cdots = N_{-4}(s) = 0 \), we have \( \xi_{15} = \xi_{25} = \xi_{35} = \xi_{45} = 0 \).

**Case 2.** Assume that \( \text{rk} A = 4 \). Then the radical of \( A \) is spanned by \((\text{Pfaff} A^{(1)}, \ldots, \text{Pfaff} A^{(5)})\). Since \( (\xi_{2345}, \ldots, \xi_{1234}) \) is also contained in the radical by the condition \( N_{-1}(t) = \cdots = N_{-5}(t) = 0 \), there exists a constant \( c \) such that

\[
(\xi_{2345}, \ldots, \xi_{1234}) = c(\text{Pfaff} A^{(1)}, \ldots, \text{Pfaff} A^{(5)}).
\]

Since \( \xi_0 \xi_{1234} - \text{Pfaff} A^{(5)} = N_5(t) \neq 0 \), we have \( (c\xi_0 - 1) \text{Pfaff} A^{(5)} \neq 0 \). Similarly, by the condition \( N_1(t) = N_2(t) = N_3(t) = N_4(t) = 0 \), we have \( (c\xi_0 - 1) \text{Pfaff} A^{(i)} = 0 \) and \( \text{Pfaff} A^{(i)} = 0 \) for \( 1 \leq i \leq 4 \). This implies \( \xi_{2345} = \xi_{1345} = \xi_{1245} = \xi_{1235} = 0 \). So the radical of \( A \) contains \((0, 0, 0, 0, 1)\) and we have \( \xi_{15} = \xi_{25} = \xi_{35} = \xi_{45} = 0 \). \( \square \)

**Corollary 1.15.** The 7-dimensional linear subspace \( F_p \) is the union of all secant lines of \( X \) passing through \( p \).

Let \( R \) be a 2-plane in \( \mathbf{P}^{15} \) such that the intersection \( X \cap R \) contains at least four points. Since \( X \) is an intersection of quadrics, we have either: 1) \( X \cap R \) consists of the four points; 2) \( X \cap R \) is a conic in \( R \); or 3) \( R \subset X \). If 1) holds, then there
are two secant lines of $X$ which intersect at a point $p \not\in X$. This is impossible by Proposition 1.12 and the corollary. Hence we have

**Proposition 1.16.** The 10-dimensional orthogonal Grassmannian $X \subset \mathbb{P}^{15}$ has no 4-secant 2-plane; that is, if a 2-plane $R$ has at least four common points with $X$, then the intersection $X \cap R$ is of positive dimension.

2. Linear section of the orthogonal Grassmannian. Let $(V, q)$ be a $2\nu$-dimensional nondegenerate quadratic space and $0 \to \mathcal{E} \to V \otimes \mathcal{O}_G \to \mathcal{F} \to 0$ be the universal exact sequence on the Grassmannian $G(\nu, V)$. Since $T_G \simeq \mathcal{E}^\vee \otimes \mathcal{F}$, the anti-canonical class $-K_G$ of the $G(\nu, V)$ is $2\nu$ times the Plücker (hyperplane section) class. By Proposition 1.3, $\mathcal{L}(V, q)$ is a complete intersection in $G(\nu, V)$ with respect to the vector bundle $S^2\mathcal{E}^\vee$. Since $c_1(S^2\mathcal{E}^\vee)$ is $(\nu + 1)$ times the Plücker class, The anti-canonical class $-K_\mathcal{L}$ of $\mathcal{L}(V, q)$ is the restriction of $(\nu - 1)$ times the Plücker class by the adjunction formula. By Proposition 1.7, we have

**Proposition 2.1.** The anti-canonical class of the $\nu(\nu - 1)/2$-dimensional orthogonal Grassmannian $\mathcal{L}^+(V, q)$ is equal to $2(\nu - 1)$ times a hyperplane class of the spinor embedding $\mathcal{L}^+(V, q) \subset \mathbb{P}(S^+)$.}

Now we put $\nu = 5$ and $X = \mathcal{L}^+(V, q)$. Let $P$ be a 6-dimensional linear subspace of $\mathbb{P}(S^+) = \mathbb{P}^{15}$ which intersects $X$ transversally, and put $C = P \cap X$. Since $P$ is an intersection of 9 hyperplanes, we have $K_C = (9H + K_X)|_C = H|_C$ by the proposition. By the Schubert calculus, $X \subset \mathbb{P}^{15}$ is of degree 12. Hence the genus of $C$ is equal to $\frac{1}{2}\deg X + 1 = 7$. Moreover, since $H^0(P, \mathcal{O}_P(1)) \to H^0(C, \mathcal{O}_C(1))$ is injective, the natural inclusion $C \subset P = \mathbb{P}^6$ is the canonical embedding. By Proposition 1.16, $C \subset \mathbb{P}^6$ has no 4-secant planes. Therefore, by the geometric Riemann-Roch theorem (see [GH] §2.3), we have the ‘only if’ part of our Main Theorem, that is,

**Proposition 2.2.** The 10-dimensional orthogonal Grassmannian $X \subset \mathbb{P}^{15}$ has a canonical curve of genus 7 as its linear section and every smooth curve $C = P \cap X$ of genus 7 obtained in this way has no $g_4$.

We need two more properties of the orthogonal Grassmannian for the proof of Theorem 0.4. Let $R_X$ be the space of quadratic forms on $\mathbb{P}^{15}$ which vanish identically on $X$. The number of linearly independent quadrics which pass through canonical curve of genus 7 is equal to 10 (see (3.1)). Hence $R_X$ is isomorphic to $V$ by the linear map

$$\alpha: V \to R_X, \quad v \mapsto N_v.$$  

(2.3)

If $v$ is nonzero nulvector, i.e., $q(v) = 0$, then the quadratic form $N_v$ is equal to the pull-back of the fundamental polar form by the linear map

$$S^+ = S^+(V, q) \to S^+(v^\perp / k \cdot v, q) = \text{Im} \varphi_v.$$
Hence $N_v$ is of rank 8 and the singular locus $F$ of $N_v = 0$ is the projectivization of $\text{Ker} \, \varphi_v$. Therefore, the intersection $F \cap X$ consists of all $[U] \in X$ with $v \in U$. Hence we have proved

**Proposition 2.4.** Let $[U]$ be a point of $X$ corresponding to a Lagrangean of $(V, q)$ and $R_{X, [U]}$ the subspace of $R_X$ consisting of quadratic forms $f$ such that $f(s) = 0$ is singular at the point $[U]$. Then $R_{X, [U]}$ coincides with the image of $U$ by the linear map $\alpha$ in (2.3). In particular, the following diagram is commutative:

$$
\begin{array}{ccc}
X & \subset & G(5, V) \\
\rho_X & \downarrow & \alpha_* \\
& & G(5, R_X)
\end{array}
$$

Here $\rho_X$ is the map associating $R_{X, [U]} \subset R_X$ for each $[U] \in X$.

Let $C \subset \mathbb{P}^6$ be a transversal linear section of $X \subset \mathbb{P}^{15}$ and define $R_C$ and $R_{C, p}$ in the same way as above. Since the restriction map $R_X \to R_C$ is an isomorphism, we have

**Corollary 2.5.** Let $\rho_C$ be the Grassmannian morphism associating the subspace $R_{C, p} \subset R_C$ for each $p \in C$. Then the following diagram is commutative:

$$
\begin{array}{ccc}
C & \subset & X \subset G(5, V) \\
\rho_C & \downarrow & \alpha_* \\
& & G(5, R_C)
\end{array}
$$

Here $\alpha'$ is the composite of $\alpha$ and the restriction map $R_X \to R_C$.

The two spaces of half spinors, $S^+ = \wedge^{e\nu} U_\infty$ and $S^\nu, \, \nu = (--)^\nu$, are dual to each other with respect to the fundamental polar form $\beta$ in (1.8). Hence a Lagrangean $U$ in $\mathcal{L}^\nu(V, q) \subset \mathbf{P}_* (S^\nu)$ defines a hyperplane of $\mathbf{P}_* (S^+)$, which we denote by $H_U$.

**Proposition 2.6.** The multiplicity of the hyperplane section $H_U \cap \mathcal{L}^+ (V, q)$ at the point $[U_0]$ is equal to $\frac{1}{2} \dim U_0 \cap U$.

**Proof.** For a subset $I$ of $\{1, 2, \ldots, \nu\}$, let $U_I$ be the Lagrangean generated by $e_i$ with $i \in I$ and $e_{-j}$ with $j \notin I$. Then its pure spinor is $e_I \in S$ defined in the proof of Proposition 1.5. Hence $H_{U_I}$ is defined by $\xi_\nu = 0$ under the coordinate system $\sum_j \xi_j e_j$ of $S$. Hence, the proposition holds for $U_I$ with $|I| \equiv \nu \mod 2$ by Proposition 1.5, since dim $U_0 \cap U_I = |I|$. It holds for every $U$, since $\mathcal{L}^{\pm} (V, q)$ is a homogeneous space of $SO(V, q)$. $\Box$
In the case $\nu = 5$, combining with Proposition 1.13, we have

**Proposition 2.7.** The two 10-dimensional orthogonal Grassmannians $\mathcal{L}^+(V, q) \subset \mathbf{P}_*(S^+)$ and $\mathcal{L}^-(V, q) \subset \mathbf{P}_*(S^-)$ are each other’s projective dual variety with respect to the fundamental polar form.

The spinor embedding is similarly defined for an odd dimensional quadratic space. Let $(V, q)$ be a $2\nu$-dimensional quadratic space and $V \longrightarrow \text{End} S$ its Clifford map as in §1. Let $\varphi_1$ be the involution of $S$ which is $\pm 1$ on $S^\pm$. Then the Clifford map extends to $V' \longrightarrow \text{End} S$, $(v, a) \mapsto \varphi_{(v, a)} := \varphi_v + a\varphi_1$, so that $\varphi_{(v, a)}^2 = q'(v, a) \cdot 1_S$ for every $(v, a) \in V' := V \oplus k$, where we put $q'(v, a) = q(v) + a^2$. For a Lagrangean $U$ of $(V', q')$, i.e., a $\nu$-dimensional subspace with $q'|_U \equiv 0$, its pure spinor $s_U \in S$ is defined in the same way as $(V, q)$. The set $\mathcal{L}(V', q')$ of Lagrangeans is a $\nu(\nu + 1)/2$-dimensional homogeneous space of $\text{SO}(V', q')$ and equivariantly embedded into $\mathbf{P}_*(S)$ by the correspondence $[U] \mapsto [s_U]$. By the fundamental polar form $\beta$, a nonzero spinor defines a hyperplane of $\mathbf{P}_*(S)$. We denote by $H_U$ the hyperplane defined by the pure spinor $s_U$. Since $\beta(s_U, s_{U'}) = 0$ if and only if $U \cap U' \neq 0 ([\text{Ca}] \S 111)$, we have

**Proposition 2.8.** The hyperplane section $H_U \cap \mathcal{L}(V', q') \subset \mathbf{P}_*(S)$ corresponding to a Lagrangean $U$ of $(V', q')$ consists of all Lagrangeans $U'$ with $U' \cap U \neq 0$.

In the case $\nu = 4$, $\mathcal{L}(V', q') \subset \mathbf{P}_*(S)$ is the 10-dimensional orthogonal Grassmannian $X \subset \mathbf{P}^{15}$ regarded as a homogeneous space of $\text{SO}(9)$. The singular locus of $H_U \cap \mathcal{L}(V', q')$ consists of $[U']$ with $\dim U' \cap U \geq 3$ and $\mathcal{L}(V', q')$ is self-dual with respect to the fundamental polar form by Proposition 1.13.

3. Quadrics passing through a canonical curve. Let $C \subset \mathbf{P}^{g-1}$ be a canonical curve of genus $g$ and set $W = H^0(\mathbf{P}^{g-1}, I_C(2))$. By Noether’s theorem, we have

$$
(3.1) \quad \dim W = g(g + 1)/2 - \dim H^0(\mathcal{O}_C(2K)) = (g - 2)(g - 3)/2.
$$

Let $E$ be the bicanonical twist $N^\vee_{C/P} \otimes \mathcal{O}(2)$ of the conormal bundle of $C \subset \mathbf{P}^{g-1}$. $E$ is a vector bundle of rank $g - 2$. By the exact sequence

$$
0 \longrightarrow N^\vee_{C/P} \longrightarrow \Omega_P|_C \longrightarrow \mathcal{O}_C(K) \longrightarrow 0,
$$

det $E$ is isomorphic to $\mathcal{O}_C((g - 5)K)$. Since $N^\vee_{C/P} = I_C/I^2_C$, we obtain a linear map $W \longrightarrow H^0(C, E)$ and the homomorphism $W \otimes_k \mathcal{O}_C \longrightarrow E$. For a point $p \in C$, we denote the kernel of $W \longrightarrow E_p$ by $W_p$, where $E_p$ is the fibre of $E$ at $p$. $W_p$ is the kernel of the natural linear map $S^2H^0(\mathcal{O}_C(K - p)) \longrightarrow H^0(\mathcal{O}_C(2K - 2p))$. If $p \neq q$, then the intersection $W_p \cap W_q$ is the kernel of $S^2H^0(\mathcal{O}_C(K - p - q)) \longrightarrow H^0(\mathcal{O}_C(2K - 2p - 2q))$. 
**Theorem 3.2.** ([GL]) Let $D$ be a divisor with $\dim H^0(\mathcal{O}_C(D)) = 1$. If $\deg D$ is smaller than the Clifford index of $C$, then the linear map

$$S^2H^0(\mathcal{O}_C(K - D)) \longrightarrow H^0(\mathcal{O}_C(2K - 2D))$$

induced by multiplication is surjective.

Let $C \subset \mathbb{P}^6$ be a canonical curve of genus 7. Then $E$ is a rank 5 vector bundle with $\det E \simeq \mathcal{O}_C(2K)$ and $\dim W = 10$. If $C$ is not trigonal, $C$ is a (scheme-theoretic) intersection of $W$. Hence, $\dim W_p = 5$ for every point $p \in C$. The Clifford index of $C$ is at most 3, and equal to 3 if and only if $C$ has no $g^1_4$. By the above theorem, we have

**Proposition 3.3.** If $C$ has no $g^1_4$, then

$$S^2H^0(\mathcal{O}_C(K - p - q)) \longrightarrow H^0(\mathcal{O}_C(2K - 2p - 2q))$$

is surjective and $\dim W_p \cap W_q = 1$ for every pair of distinct points $p$ and $q$ of $C$.

The following lemma, together with the above proposition, plays an important role in the next two sections.

**Lemma 3.4.** Let $C$ be as above and fix a point $p \in C$. Then the union of $W_q \cap W_p$, $q \neq p \in C$, generates $W_p$.

**Proof.** Assume the contrary, that is, there exists a 4-dimensional subspace $V$ of $W_p$ which contains $W_q \cap W_p$ for every $q \neq p$. Let $C_p \subset \mathbb{P}^5$ be the image of the projection of the canonical curve from $p$. We regard $W_p$ as a space of quadratic forms on $\mathbb{P}^5$. By Proposition 3.3, $C_p$ is a connected component of the common zero locus of $W_p$ (cf. the remark below). Let $S$ be a component of the common zero locus of $V \subset W_p$ containing $C_p$, and $L$ the linear web of quadrics in $\mathbb{P}^5$ corresponding to $V$. Since $C_p$ is the intersection of $S$ and a quadric $Q$, the dimension of $S$ is equal to 1 or 2. For every $q \neq p$, $L$ contains a member which is singular at $q$. Hence in the case $\dim S = 1$, $S$ is singular along $C_p$ and we have

$$2^4 \geq \deg S \geq 2 \deg C_p = 22,$$

by the refined Bezout’s theorem [Ful], which is a contradiction. In the case $\dim S = 2$, the intersection $S \cap Q = C_p$ is complete and we have $2 \deg S = \deg C_p = 11$, which is absurd. \hfill \Box

**Remark 3.5.** The curve $C_p \subset \mathbb{P}^5$ in the proof coincides with the common zero locus of $W_p$ (see [L] §2.4).
4. Quadratic relation among quadratic forms. Let $C \subset \mathbb{P}^6$ be a canonical curve of genus 7. Let $W$, $E$ and $W_p$ be as in the preceding section. If $C$ is a transversal linear section of the 10-dimensional orthogonal Grassmannian, then by Corollary 1.11 there is a quadratic relation among quadratic forms, that is, the multiplication map $\mu: S^2W \to H^0(\mathbb{P}^6, I^2_C(4))$ is not injective. By the commutative diagram

\[
\begin{array}{ccc}
S^2W & \longrightarrow & S^2H^0(C, E) \\
\mu \downarrow & & \downarrow \\
H^0(\mathbb{P}^6, I^2_C(4)) & \longrightarrow & H^0(C, S^2E)
\end{array}
\]

the natural map $f: S^2W \to H^0(S^2E)$ is not injective, either.

In this section we prove the following:

**Theorem 4.2.** If $C$ has no $g_4^1$, then every nonzero tensor in $\text{Ker} f$ is nondegenerate.

**Corollary 4.3.** $\dim \text{Ker} f \leq 1$.

Let $\sigma$ be a nonzero degenerate symmetric tensor in the kernel of $f$ and denote its rank by $r \leq 9$. Let $R$ be the unique $r$-dimensional subspace of $W$ such that $S^2R$ contains $\sigma$. Since $\sigma$ is nondegenerate as a quadratic form on $R^\vee$, we have

\[
\dim R \cap W_p \geq \frac{r}{2}
\]

for every $p \in C$. By Proposition 3.3, we have

\[
\dim R \cap W_p + \dim R \cap W_q \leq r + 1
\]

for every pair of distinct points of $p$ and $q$ of $C$. Hence we have either

(4.6a) $r$ is odd (resp. even) and $\dim R \cap W_p = \frac{r+1}{2}$ (resp. $= \frac{r}{2}$) for every point $p$ of $C$, or

(4.6b) $r$ is even, there exists a point $p \in C$ with $\dim R \cap W_p = \frac{r}{2} + 1$

and $\dim R \cap W_q = \frac{r}{2}$ for every point $q \neq p$.

In the case (4.6a), $(R \cap W_p)^\perp$ is a Lagrangean of the quadratic space $(R^\vee, \sigma)$. If $r$ is odd, $R \cap W_p \cap W_q$ is nonzero for every $p$ and $q$. Hence, by Proposition 3.3 and Lemma 3.4, we have $R \cap W_p = W_p$. It follows from (4.5) that $r = 9$. If $r$ is even, then

\[
\dim R \cap W_p \cap W_q \equiv \frac{r}{2} \mod 2
\]
by Proposition 1.6. In particular, if \( r = 2 \) or 6, then \( R \cap W_p \cap W_q \neq 0 \) for every \( p \neq q \), which contradicts Lemma 3.4. Hence we have \( r = 4 \) or 8.

**Claim.** (4.6a) does not occur.

Let \( F \) be the subsheaf of \( E \) generated by \( R \subset W \subset H^0(E) \) and \( \Phi \) the morphism of \( C \) to Grassmannian induced by \( R \otimes \mathcal{O}_C \to F \). The image of \( \Phi \) is contained in the orthogonal Grassmannian \( Y \) associated with \((R^\vee, \sigma)\). In the case \( r = 9 \), \( W_p \) is contained in \( R \). Let \( H_p \) be the hyperplane associated with \( \Phi(p) \) by the fundamental polar form. Then, applying Proposition 2.8 to \((R^\vee, \sigma)\), we have \( \Phi^{-1}(H_p \cap Y) = \{p\} \) by Proposition 3.3. Hence we have \( np \sim \Phi^* \mathcal{O}(1) \) for every point \( p \) of \( C \), where \( n = \deg \Phi > 0 \), which contradicts the finiteness of \( n \)-torsion points of \( \text{Pic} \, C \).

In the case \( r = 8 \), \( Y \) is a smooth hyperquadric in \( \mathbb{P}^7 \). By Proposition 3.3 and (4.7), \( R \cap W_p \cap W_q = 0 \) for every \( p \neq q \). We have the contradiction \( np \sim \Phi^* \mathcal{O}(1) \) in a similar way to \( r = 9 \). In the case \( r = 4 \), \( Y \) is \( \mathbb{P}^1 \) and \( \Phi \) is injective by Proposition 3.3 and (4.7). This is absurd and completes the proof of the claim.

In the case (4.6b), \( R \cap W_p \cap W_q \) is nonzero for every \( q \) different from \( p \). Hence \( R \cap W_p = W_p \), that is, \( W_p \subset R \) by Lemma 3.4 and we have \( r = 8 \). \( W_p^\perp \) is a 3-dimensional subspace of \( R^\vee \) over which \( \sigma \) is identically zero. There are two Lagrangeans of \((R^\vee, \sigma)\) which contains \( W_p^\perp \) as a subspace. Let them be \( U_+^\perp \) and \( U_-^\perp \) for 4-dimensional subspaces \( U_\pm \) of \( R \). By Proposition 1.6, one of \( U_\pm \), say \( U_- \), satisfies that \( \dim U_- \cap W_q \) is odd for every point \( q \neq p \) of \( C \), which contradicts Lemma 3.4. So (4.6b) is impossible, either, and the proof of Theorem 4.2 is completed.

**5. Spinor embedding of curves of genus 7.** Let \( X \subset \mathbb{P}^{15} \) be the 10-dimensional orthogonal Grassmannian and \( R(N) \equiv \sum_{i=1}^5 N_iN_{-i} = 0 \) the quadratic relation in Corollary 1.11. Let \( \Xi \) be the open subset of Grassmannian consisting of 6-dimensional linear subspaces \( P \) which intersect \( X \) transversally. By Proposition 2.2, we obtain a morphism

\[
(5.1) \quad \alpha: \Xi/\text{SO}(10) \longrightarrow \mathcal{M}_7
\]

to the moduli space \( \mathcal{M}_7 \) of curves of genus 7. If \( C \) belongs to the image of \( \alpha \), then \( C \) has no \( g_2^+ \). By Corollary 4.3, the kernel of \( f: S^2W \to H^0(S^2E) \) is generated by the restriction of \( R(N) \). Hence we can recover the original embedding \( C \hookrightarrow X \) from \( W \subset H^0(E) \) and \( f \) by Corollary 2.5. It follows that \( \alpha \) has its inverse on the image of \( \alpha \). So we have

**Proposition 5.2.** The morphism \( \alpha \) is injective.

Since \( \text{Aut} \, C \) is finite, we have \( \dim \Xi/\text{SO}(10) = \dim \Xi - \dim \text{SO}(10) = 7(16 - 7) - 45 = 18 = \dim \mathcal{M}_7 \). Hence the image of \( \alpha \) contains a non-empty Zariski open subset. By the irreducibility of \( \mathcal{M}_7 \) ([DM]) and Corollary 4.3, we have
Corollary 5.3. The map $f$: $S^2W \rightarrow H^0(S^2E)$ is not injective for any curve $C$ of genus 7. Moreover, $\dim\ker f = 1$ if $C$ has no $g^1_4$.

Proof of Theorem 0.4. Let $C \subset \mathbf{P}^6$ be a non-tetragonal canonical curve of genus 7 and $\sigma$ a generator of the kernel of $f$: $S^2W \rightarrow H^0(S^2E)$. The Grassmannian morphism $\rho_C: C \rightarrow G(W, 5)$ is injective by Proposition 3.3. Since $\sigma$ vanishes at each fibre of $E$, its image is contained in one, say $X^+$, of the two orthogonal Grassmannians $X^\pm$ associated with $(W, \sigma)$. Let $\xi$ be the pull-back of the tautological line bundle $\mathcal{O}_\mathbf{P}(1)$ by

$$\Phi: C \rightarrow X^+ \subset \mathbf{P}(S^+) = \mathbf{P}^{15}.$$ 

By Proposition 1.7, $\xi^2$ is isomorphic to $\det E \cong \omega_C^2$. Hence $\deg \xi = \deg \omega_C$ and $\dim H^0(\xi) \leq 7$. Therefore, the linear span $P$ of $\Phi(C)$ is of dimension $\leq 6$.

Claim. The intersection $P \cap X$ is transversal at every point of $\Phi(C)$.

It suffices to show that every hyperplane $H$ containing $P$ intersects $X$ transversally along $\Phi(C)$. If the intersection $H \cap X$ is transversal (everywhere), then there is nothing to prove. So we may assume that $H = H_U$ for some $[U] \in X^-$ by virtue of Proposition 2.7. Since $\Phi(C) \subset H$, $W_p \cap U$ is nonzero for every $p$ by Proposition 2.6. Since $\dim W_p \cap U$ is even, $\dim W_p \cap U = 2$ or 4. If $\dim W_p \cap U = 4$, then $W_p \cap W_q \cap U$ is nonzero for every $q \in C$, which contradicts Lemma 3.4. Hence $H \cap X$ is transversal along $\Phi(C)$ again by Proposition 2.6.

By the claim, we have $\dim P = 6$ and hence $\xi \cong \omega_C$. Moreover, since the morphism $C \rightarrow P$ is canonical, $\Phi$ is an embedding. By the claim, the intersection $X \cap P$ contains $C \cong \Phi(C)$ as a connected component. Since $X \cap P$ is connected by Enriques-Severi-Zariski’s lemma ([H], p.244), $X \cap P$ coincides with $\Phi(C)$, which completes the proof of Theorem 0.4 and hence the ‘if’ part of our Main Theorem.

6. Tetragonal curves of genus 7. Let $C$ be a curve of genus 7 which has a $g^1_4$. We assume that $C$ is neither hyperelliptic nor trigonal. Let $\xi$ be a $g^1_4$ and $\eta = \omega_C \xi^{-1}$ its Serre adjoint. By the Riemann-Roch theorem $\eta$ is a $g^3_5$. The complete linear system $|\eta|$ has no fixed points since $C$ has no $g^2_3$. Let $\pi: C \rightarrow \mathbf{P}^1$ and $\tau: C \rightarrow \mathbf{P}^3$ be the morphisms associated to $|\xi|$ and $|\eta|$, respectively.

We first consider the case in which $C$ has no $g^2_3$. If two points $p$ and $q$ lay in the same fibre of $\tau$, then $\eta(-p-q)$ would be a $g^2_6$. Hence $\tau$ is an embedding. By the genus formula, the image of $\tau$ is not contained in a quadric. In particular, $\eta$ is not a product of two $g^1_4$'s and we have $\dim H^0(\xi^{-1}\eta) \leq 1$. Therefore, by the exact sequence

$$[0 \rightarrow \xi^{-1} \rightarrow \mathcal{O}^{\oplus 2} \rightarrow \xi \rightarrow 0] \otimes \eta,$$
We have

**Lemma 6.1.** The map \( \mu: H^0(\xi) \otimes H^0(\eta) \to H^0(\omega_C) \) induced by multiplication is surjective.

The map \( \mu \) induces the linear embedding

\[ \mu^*: \mathbb{P}^6 = \mathbb{P}^*(H^0(\omega_C)) \to \mathbb{P}^*(H^0(\xi) \otimes H^0(\eta)) = \mathbb{P}^7 \]

and we have the following commutative diagram:

\[
\begin{array}{ccc}
C & \xrightarrow{\mu^*} & \mathbb{P}^1 \times \mathbb{P}^3 \\
\text{canonical} \cap & \subset & \text{Segre} \\
\mathbb{P}^6 & \xrightarrow{\pi, \tau} & \mathbb{P}^7 \\
\end{array}
\]

(6.2)

The morphism \((\pi, \tau)\) is an embedding and its image \(\tilde{C}\) is contained in the intersection \(W\) of \(\mu^*(\mathbb{P}^6)\) and \(\mathbb{P}^1 \times \mathbb{P}^3\). \(W\) is an irreducible divisor of bidegree \((1,1)\) in \(\mathbb{P}^1 \times \mathbb{P}^3\). Consider the restriction map

\[ H^0(W, O_W(1, 2)) \to H^0(C, \xi \eta^2). \]

The source is of dimension 16 and the target of dimension 14. Hence there exists a pencil of divisors \(D_t \subset W, t \in \mathbb{P}^1\), of bidegree \((1,2)\) which contain \(\tilde{C}\). By the surjectivity of \(\mu\), \(\tilde{C}\) is not contained in a divisor of bidegree \((1,1)\). Since \(\tau(C)\) is not contained in a quadric, \(\tilde{C}\) is not contained in a divisor of bidegree \((0,2)\), either. Therefore, every divisor \(D_t, t \in \mathbb{P}^1\), is irreducible and the intersection \(D_0 \cap D_\infty\) is of dimension 1. Since the degree of \(D_0 \cap D_\infty \subset W \subset \mathbb{P}^6\) is equal to

\[ (a + 2b)^2(a + b)^2 = (a^2 + 3ab + 2b^2)^2 = 12 = \deg C, \]

we have \(C = D_0 \cap D_\infty\). So we have proved

**Proposition 6.3.** If \(C\) is tetragonal and has no \(g^2_6\), then \(C\) is isomorphic to a complete intersection of a divisor \(W\) of bidegree \((1,1)\) and two divisors of bidegree \((1,2)\) in \(\mathbb{P}^1 \times \mathbb{P}^3\).

Every member of \(|\xi|\) spans a 4-secant plane in the canonical model \(C_{12} \subset \mathbb{P}^6\). The divisor \(W\), which is a \(\mathbb{P}^2\)-bundle over \(\mathbb{P}^1\), is the union of these 4-secant planes.

Now we consider the case in which \(C\) has a \(g^2_6\). Let \(\alpha\) be a \(g^2_6\) of \(C\) and \(f: C \to \mathbb{P}^2\) the morphism associated to \(|\alpha|\). Since \(C\) is neither hyperelliptic nor trigonal, we have either

(a) \(f\) is a degree two morphism onto a smooth cubic \(E: f_3(x_0, x_1, x_2) = 0\), or
(b) \(f\) is birational onto its image.
Table 1. Canonical models of curves of genus 7.

<table>
<thead>
<tr>
<th></th>
<th># of $g_1^1$'s</th>
<th># of $g_2^1$'s</th>
<th># of $g_4^1$'s</th>
<th># of moduli</th>
<th>Complete intersection</th>
<th>Canonical model $C_{12} \subset \mathbb{P}^6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1)</td>
<td>non-tetragonal</td>
<td>0</td>
<td>0</td>
<td>18</td>
<td>Linear section of the 10-dim. spinor variety $X_{12}^{10} \subset \mathbb{P}^{15}$</td>
<td></td>
</tr>
<tr>
<td>2)</td>
<td>tetragonal</td>
<td>2</td>
<td>1</td>
<td>17</td>
<td>$(1, 1) \cap (1, 2) \cap (1, 2)$ in $\mathbb{P}^1 \times \mathbb{P}^3$</td>
<td>The union of 4-secant planes is a 3-fold of degree 4 in $\mathbb{P}^6$</td>
</tr>
<tr>
<td>3)</td>
<td>1, 2 or 3</td>
<td>1</td>
<td>16</td>
<td>Hyperquadric section of a sextic del Pezzo surface $S_6 \subset \mathbb{P}^6$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4)</td>
<td>1</td>
<td>$\infty$</td>
<td>15</td>
<td>Hyperquadric section of the cone of a sextic elliptic curve $E_6 \subset \mathbb{P}^5$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5)</td>
<td>$\infty$</td>
<td>$\infty$</td>
<td>12</td>
<td>The union of trisecant lines is a surface of degree 5 in $\mathbb{P}^6$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6)</td>
<td>trigonal</td>
<td>2</td>
<td>$\infty$</td>
<td>15</td>
<td>$(1, 1) \cap (3, 3)$ in $\mathbb{P}^1 \times \mathbb{P}^2$</td>
<td></td>
</tr>
<tr>
<td>7)</td>
<td>1</td>
<td>$\infty$</td>
<td>13</td>
<td>$(9) \subset \mathbb{P}(1 : 1 : 3) \subset \mathbb{P}^6$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>8)</td>
<td>hyperelliptic</td>
<td>$\infty$</td>
<td>$\infty$</td>
<td>13</td>
<td>$(16) \subset \mathbb{P}(1 : 1 : 8)$</td>
<td></td>
</tr>
</tbody>
</table>
In the former case, $C$ is bielliptic. The canonical line bundle $\omega_C$ is the pull-back of a line bundle $\zeta$ of degree 6 on $E$. Choose a $g_6^2$ $\xi$ of $C$ so that $\xi^2 \cong \omega_C$. Then the branch locus of $C \to E$ is cut out by a quartic $g_4(x_0,x_1,x_2) = 0$. Hence $C$ is defined by the two equations $f_3(x) = 0$ and $y^2 = g_4(x)$. Therefore, we have

**Proposition 6.4.** A bielliptic curve of genus 7 is a complete intersection of two surfaces of degree 3 and 4 in the weighted projective space $\mathbf{P}(1:1:1:2)$.

In the latter case, the image of $f$ is a sextic curve without triple points. Since the arithmetic genus of $f(C)$ is equal to 10, $C$ is contained in the blow-up $S$ of $\mathbf{P}^2$ at three points $p, q$ and $r$. Moreover, $C$ belongs to the anti-bicanonical linear system $|-2K_S|$ of $S$ and the canonical linear system $|K_C|$ is the trace of $|-K_S|$. The blowing-up $\tilde{f}: S \to \mathbf{P}^2$ is an extension of $f: C \to \mathbf{P}^2$. Let $\beta$ be the Serre adjoint of $\alpha$. $\beta$ is also a $g_6^2$. Let $g: C \to \mathbf{P}^2$ be the morphism associated to $|\beta|$. By the adjunction formula, $|\beta|$ is induced from the net of conics passing through the center $\{p, q, r\}$ of the blowing-up $\tilde{f}$. Hence $g$ extends to a morphism $\tilde{g}: S \to \mathbf{P}^2$, which is also a blowing-up at three points. $\beta$ is isomorphic to $\alpha$ if and only if the three points $p, q$ and $r$ are collinear. If $\alpha \not\cong \beta$, then the image of $(\tilde{f}, \tilde{g}): S \to \mathbf{P}^2 \times \mathbf{P}^2$ is a complete intersection of two divisors of bidegree $(1, 1)$. In the case $\alpha \cong \beta$, take a system of homogeneous coordinate $(x_0 : x_1 : x_2)$ of $\mathbf{P}^2$ so that $\{p, q, r\}$ is defined by $x_0 = f_3(x_1, x_2) = 0$. Then the anti-canonical morphism $S \to \mathbf{P}^6$ factors through the map

$$\mathbf{P}^2 \longrightarrow S \longrightarrow \mathbf{P}(1 : 1 : 1 : 2)$$

$$(x_0 : x_1 : x_2) \longmapsto (x_0 : x_1 : x_2 : y)$$

with $y = f_3(x_1, x_2)/x_0$. The image of $S$ in $\mathbf{P}(1 : 1 : 1 : 2)$ is a cubic surface

$$x_0 y + f_3(x_1, x_2) = 0.$$

Hence we have proved

**Proposition 6.5.** Assume that $C$ is neither hyperelliptic, trigonal, nor bielliptic, and that $C$ has a $g_6^2$, which we denote by $\alpha$. Then $C$ is isomorphic to a complete intersection of three divisors of bidegree $(1,1), (1,1) \text{ and } (2,2)$ in $\mathbf{P}^2 \times \mathbf{P}^2$ if $\alpha^2 \not\cong \omega_C$, and of two surfaces of degree 3 and 4 in the weighted projective space $\mathbf{P}(1 : 1 : 1 : 2)$ if $\alpha^2 \cong \omega_C$.

The canonical model $C_{12} \subset \mathbf{P}^6$ is a hyperquadric section of a sextic surface $S_6 \subset \mathbf{P}^6$, which is the cone of an elliptic curve if $C$ is bielliptic and the anti-canonical model of a rational surface otherwise.

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REFERENCES


