Price's law on Black Hole Spacetimes

Daniel Tataru

University of California, Berkeley

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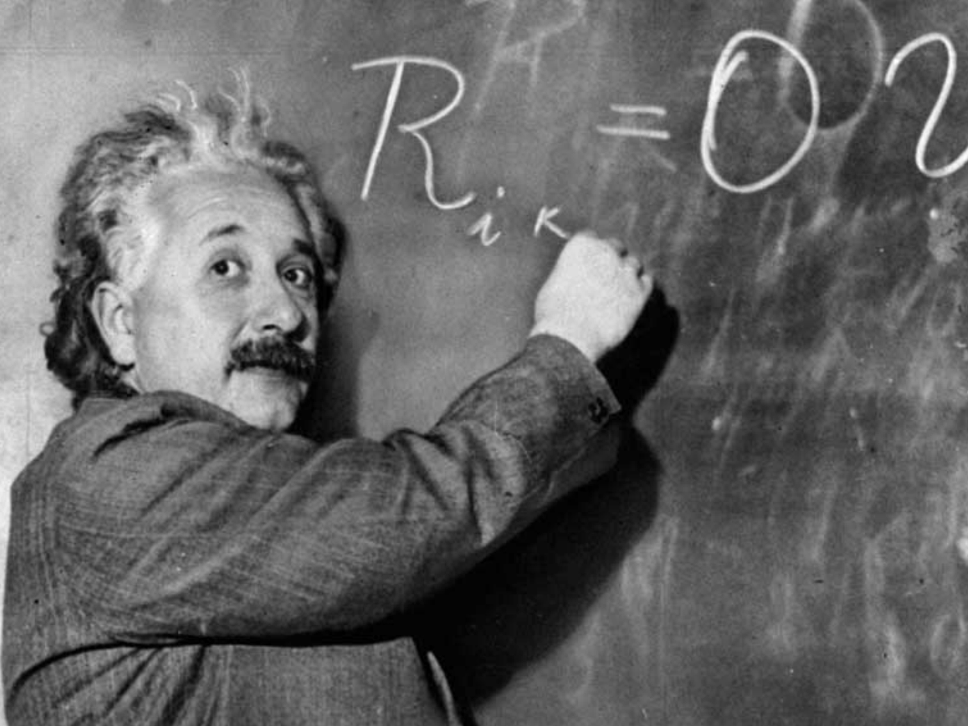
- Daniel Tataru. *Local decay of waves on asymptotically flat stationary space-times.* arXiv:0910.5290., to appear, American Journal of Mathematics

- Jason Metcalfe, Daniel Tataru and Mihai Tohaneanu *Price's Law on Nonstationary Spacetimes* arXiv:1104.5437
- Jason Metcalfe, Jacob Sterbenz, Daniel Tataru and Mihai Tohaneanu *Local decay of electromagnetic waves on asymptotically flat space-times.* in preparation


$R_{ik} = 0$

Vacuum Einstein Equations

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- cosmological constant $\Lambda = 0$
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- 4 relations between equations $\nabla^\alpha R_{\alpha\beta} = 0 \implies$ 6 independent equations
- 4 degrees of (gauge) freedom = choice of coordinates

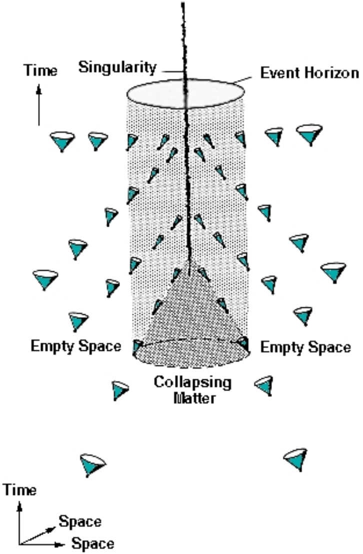
The Minkowski space-time (1907)

-mathematical setting of special relativity

$$M = \mathbb{R} \times \mathbb{R}^3$$

$$ds^2 = -dt^2 + dx_1^2 + dx_2^2 + dx_3^2$$

The Schwarzschild space time (1915)



- spherically symmetric stationary black hole
- parametrized by the mass M
- event horizon at $r = 2M$.
- image by Penrose (Scientific American)

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\omega^2$$

The Kerr space time (1963)

- rotating axisymmetric black hole
- parametrized by mass M and angular momentum aM
- Kerr(M,a=0) = Schwarzschild(M)

In polar Boyer-Lindquist coordinates

$$
\begin{aligned}
ds^2 = & -\frac{\Delta - a^2 \sin^2\theta}{\rho^2} dt^2 + -\frac{4aMr\sin^2\theta}{\rho^2} dt d\phi + \frac{\rho^2}{\Delta} dr^2 \\
& + \frac{(r^2+a^2)^2 - a^2 \Delta \sin^2\theta}{\rho^2} \sin^2\theta d\phi^2 + \rho^2 d\theta^2
\end{aligned}
$$

with

$$
\Delta = r^2 - 2Mr + a^2, \qquad \rho^2 = r^2 + a^2\cos^2\theta.
$$

The stability question

Theorem (Christodoulou-Klainerman '90)
The Minkowski space-time is nonlinearly stable as a solution to the vacuum Einstein equation.

Open Problem
Are the Schwarzschild/Kerr solutions stable ?

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Open Problem

Are the Schwarzschild/Kerr solutions linearly stable ?

A hierarchy of equations:

1. **Scalar waves:**

$$g^{\alpha\beta}\nabla_\alpha\nabla_\beta\phi = 0$$

2. **Maxwell's equations (spin 1)**

$$g^{\alpha\beta}\nabla_\alpha\nabla_\beta V_\gamma = 0, \qquad \nabla^\gamma V_\gamma = 0$$

- 4 equations for 4 unknowns (electromagnetic potential),
- one relation between equations $\implies$ 3 independent equations + 1 degree of freedom (gauge choice)

3. **Linearized gravity (spin 2)**

$$g^{\alpha\beta}\nabla_\alpha\nabla_\beta h_{\gamma\delta} = 2g^{\alpha\beta}g^{\mu\nu}R_{\gamma\mu\delta\alpha}h_{\nu\beta} \qquad \nabla^\gamma h_{\gamma\delta} - \frac{1}{2}g^{\alpha\beta}\nabla_\delta h_{\alpha\beta} = 0$$

- 10 equations for 10 unknowns
- 4 relations between equations $\implies$ 6 independent equations + 4 degrees of freedom (gauge choice)

A hierarchy of equations:

1. **Scalar waves:** ... this talk ...

$$g^{\alpha\beta}\nabla_\alpha\nabla_\beta\phi = 0$$

2. **Maxwell's equations (spin 1)** ... work in progress ...

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- 4 equations for 4 unknowns (electromagnetic potential),
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3. **Linearized gravity (spin 2)**... not too distant future ...

$$g^{\alpha\beta}\nabla_\alpha\nabla_\beta h_{\gamma\delta} = 2g^{\alpha\beta}g^{\mu\nu}R_{\gamma\mu\delta\alpha}h_{\nu\beta} \qquad \nabla^\gamma h_{\gamma\delta} - \frac{1}{2}g^{\alpha\beta}\nabla_\delta h_{\alpha\beta} = 0$$

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- Separation of variables for scalar waves in Kerr: Carter ’67
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S.Chandraseckar
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S.Chandraseckar
The Mathematical Theory of Black Holes ′85, p.497:

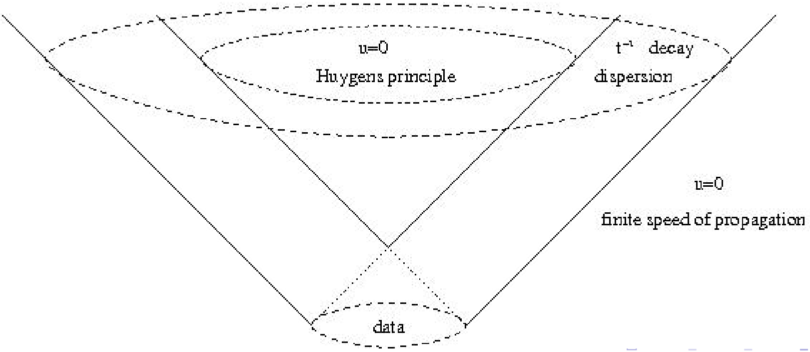
"The analysis is addressed to the problem of [linearized] gravitational perturbations of the Kerr space-time .... In the Newman-Penrose formalism ... we have to solve for ... fifty real quantities. For the solution we have .... seventy-six real equations. The solutions must be consistent with ten degrees of gauge freedom ..."

Linear wave decay in flat space

$$\Box u(t,x) = 0, \qquad u(0,x) = u_0(x),\ \partial_t u(0,x) = u_1(x), \qquad \Box = \partial_t^2 - \Delta_x$$

nice localized initial data (u_0, u_1)

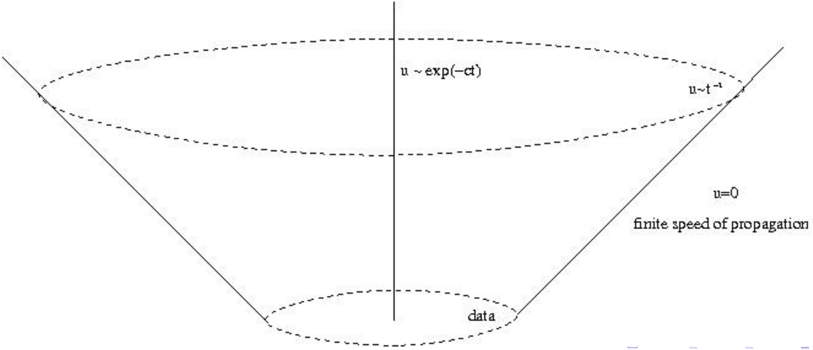
Conserved energy: $$E(u) = \int_{\mathbb{R}^3} |\nabla u(t,x)|^2 + |\partial_t u(t,x)|^2 dx$$



Flat space with localized potential

$$\Box u(t,x) = V(x)u(t,x), \qquad u(0,x) = u_0(x), \;\; \partial_t u(0,x) = u_1(x)$$

with nice, small, compactly supported V.
Local exponential decay (Morawetz, Lax 70's):

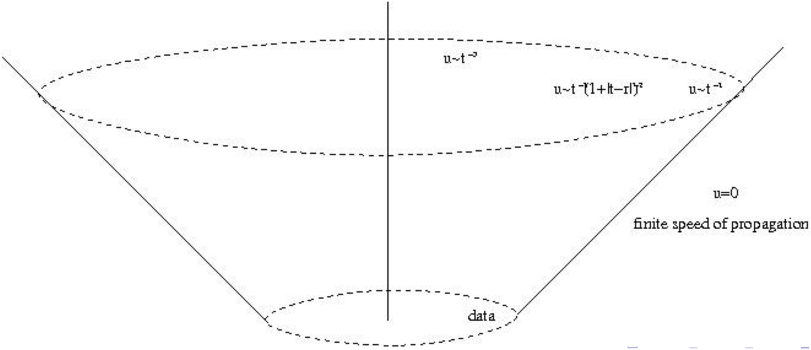


Flat space with polynomial potential

$$\Box u(t,x) = V(x)u(t,x), \qquad u(0,x) = u_0(x), \;\; \partial_t u(0,x) = u_1(x)$$

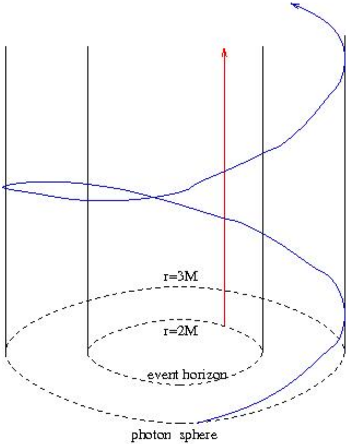
$$|V(x)| \lesssim \epsilon \langle r \rangle^{-3}$$

Direct iteration:



Linear wave decay in Kerr/Schwarzschild

Near infinity: $g = m + O_{rad}(1/r) + O(1/r^2)$, $V \sim r^{-3}$.



Trapped light rays in a compact set:

- Along the event horizon
- Along the photon sphere

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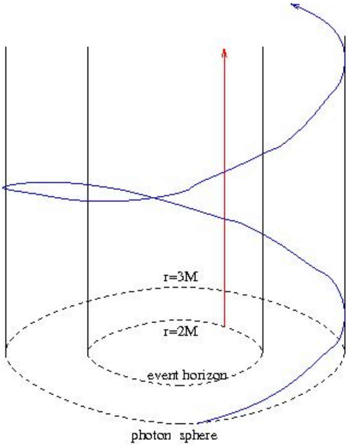

Trapped light rays in a compact set:

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- red shift $\Longrightarrow$ exponential decay
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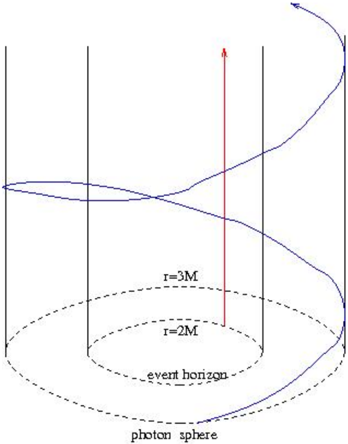


Trapped light rays in a compact set:

- Along the event horizon
- red shift $\Longrightarrow$ exponential decay
- Along the photon sphere
- Uncertainty principle + hyperbolic instability

R. Price '74 Heuristics for t^{-3} local decay for scalar waves in Schwarzschild. (Price's Law)

Further work:

R. Price '74 Heuristics for t^{-3} local decay for scalar waves in Schwarzschild. (Price's Law)

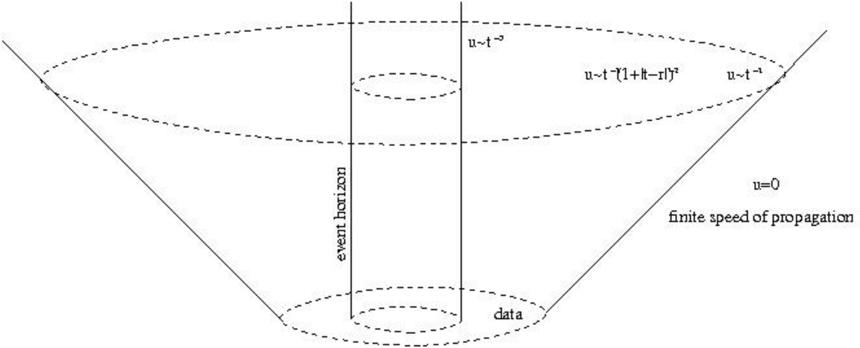
Further work:

- Schwarzschild/Kerr heuristics: Ching-Leung-Suen-Young
- Schwarzschild uniform boundedness: Wald, Kay-Wald
- Schwarzschild t^{-1} decay, Blue-Sterbenz and Dafermos-Rodnianski $t^{-3/2}$ decay Luk
- Schwarzschild: t^{-3-2l} for spherical modes, Kronthaler(radial $l = 0$), Donninger-Schlag-Soffer (all modes $l > 0$)
- Kerr: first decay results by Finster-Kamran-Smoller-Yau
- Kerr: t^{-1} decay, Dafermos-Rodnianski and Andersson-Blue

The main result: Take 1

Theorem (T. '09, Metcalfe-T.-Tohaneanu '11)
$t^{-1}(|t-r|+1)^{-2}$ *local decay for scalar waves in Schwarzschild, Kerr with small angular momentum, and small perturbations thereof:*

$$g - g_K = \epsilon O(\langle r \rangle^{-2}), \qquad g - g_K = \epsilon O(t^{-1-}) \text{ near photon sphere}$$



Set-up, the geometry at infinity

Equation:

$$(CP) \qquad (\Box_g + V)u = 0, \qquad u_{|\Sigma_0} = u_0, \quad \frac{\partial u}{\partial \nu}_{|\Sigma_0} = u_1$$

where g has smooth coefficients and geometry at infinity:

$$g = m + g_{sr} + g_{lr},$$

where m is the Minkowski metric, g_{lr} is a long range spherically symmetric component, with $S^Z_{rad}(r^{-1})$ coefficients,

$$g_{lr} = g_{lr,tt}(r)dt^2 + g_{lr,tr}(r)dtdr + g_{lr,rr}(r)dr^2 + g_{lr,\omega\omega}(r)r^2 d\omega^2$$

and g_{sr} is a short range component, with $S^Z(r^{-2})$ coefficients,

$$g_{sr} = g_{sr,tt}dt^2 + 2g_{sr,ti}dtdx_i + g_{sr,ij}dx_i dx_j$$

while V has the form

$$V = V_{lr} + V_{sr}, \qquad V_{lr} \in S^Z(r^{-3})$$

Vector fields $Z = (\partial_\alpha, \Omega = x_i\partial_j - x_j\partial_i, S = t\partial_t + x\partial_x)$.

Set-up: the local geometry

Case A: Minkowski-like.
Domain $D = \mathbb{R}^3 \times \mathbb{R}^+$, foliation $\Sigma_t = \mathbb{R}^3 \times \{t\}$,

(i) The surfaces Σ_t are space-like.

Case B: Black hole. (e.g. Schwarzschild with $R_0 < 2M$
Domain $D = \mathbb{R}^3/B(0, R_0) \times \mathbb{R}^+$, foliation $\Sigma_t = \mathbb{R}^3/B(0, R_0) \times \{t\}$,

(i) The surfaces Σ_t are space-like.

(ii) The lateral boundary $S(0, R_0) \times \mathbb{R}^+$ is outgoing space-like.

(e.g. Schwarzschild with $R_0 < 2M$, small perturbations thereof)

Case C: Exterior problem.
Domain $D = \mathbb{R}^3/B(0, R_0) \times \mathbb{R}^+$, foliation $\Sigma_t = \mathbb{R}^3/B(0, R_0) \times \{t\}$,

(i) The surfaces Σ_t are space-like.

(iii) The lateral boundary $S(0, R_0) \times \mathbb{R}^+$ is time-like, with Dirichlet or Neuman boundary condition.

The main result: Take 2

Theorem (T'09)

Assume Case A, Case B or Case C. Suppose that the evolution (CP) has the following properties:

- *uniform forward energy bounds.*
- *weak local energy decay estimates.*
- *stationary local energy decay estimates.*

Then in normalized coordinates the solution u satisfies the bounds

$$|u(t,x)| \lesssim \frac{1}{\langle t\rangle\langle t-|x|\rangle^2}\|\nabla u(0)\|_{H^m}$$

$$|\partial_t u(t,x)| \lesssim \frac{1}{\langle t\rangle\langle t-|x|\rangle^3}\|\nabla u(0)\|_{H^m}$$

$$|\partial_x u(t,x)| \lesssim \frac{1}{\langle r\rangle\langle t-|x|\rangle^3}\|\nabla u(0)\|_{H^m}$$

Normalized coordinates

Motivation: Fix the geometry of null cones.
General form of the metric g:

$$g = m + g_{lr} + g_{sr},$$

g_{lr} = long range spherically symmetric, with $S_{rad}(r^{-1})$ coefficients,

$$g_{lr} = g_{lr,tt}(r)dt^2 + g_{lr,tr}(r)dtdr + g_{lr,rr}(r)dr^2 + g_{lr,\omega\omega}(r)r^2d\omega^2$$

Normal form:

$$g_{lr} = g_{lr,\omega\omega}(r)r^2d\omega^2$$

Tools:

- Conformal transformations.
- Changes of coordinates $t \to t + O(\log r)$, $r \to r + O(\log r)$.

Example: the Regge-Wheeler coordinates in Schwarzschild/Kerr space-times

Energy bounds

Definition: The evolution (CP) is forward bounded if the following estimates hold:

$$\|\nabla u(t_1)\|_{H^k} \leq c_k \|\nabla u(t_0)\|_{H^k}, \qquad t_1 > t_0 \geq 0, \quad k \geq 0$$

Difficulty in Kerr: The positivity of the conserved ∂_t energy fails near the event horizon (ergosphere).
Redeeming feature = red shift: exponential energy decay near bicharacteristics along the event horizon.

Theorem (Dafermos-Rodnianski '07)
Uniform energy bounds for Schwarzchild and Kerr with small angular momentum (and a larger class of small perturbations of Schwarzschild).

Alternate view: Uniform energy bounds are a side effect of local energy decay.
Earlier work by Laba-Soffer, Blue-Soffer, Twainy, Finster-Smoller, Finster-Kamran-Smoller-Yau

The linear wave equation:local energy decay

$$\Box\phi = 0 \qquad \text{in } \mathbb{R}^{n+1}$$

Local energy decay (also known as *Morawetz estimates*):

$$\|\nabla_{x,t}\phi(x,t)\|_{L^2(\mathbb{R}\times B_R)} \lesssim R^{\frac{1}{2}}\|\nabla_{x,t}\phi(x,0)\|_{L^2}$$

Heuristics: A speed 1 wave spends at most $O(R)$ time inside B_R. Morawetz's proof uses the positive commutator method. If P and Q are selfadjoint, respectively skewadjoint operators then

$$2\Re\langle P\phi, Q\phi\rangle = \langle [Q,P]\phi, \phi\rangle$$

Apply this with

$$P = \Box, \qquad Q = \partial_r + \frac{n-1}{2r}.$$

to obtain

$$\|r^{-\frac{1}{2}}\not\nabla\phi(x,t)\|_{L^2} + \|\phi(0,t)\|_{L^2} \lesssim \|\nabla_{x,t}\phi(x,0)\|_{L^2}, \qquad n = 3$$

The local energy norms

At the L^2 level we set

$$\|u\|_{LE} = \sup_k \|\langle r\rangle^{-\frac{1}{2}} u\|_{L^2(\mathbb{R}\times A_k)}, \qquad A_k = \{|x| \approx 2^k\} \times \mathbb{R}$$

We also define its H^1 counterpart, as well as the dual norm

$$\|u\|_{LE^1} = \|\nabla u\|_{LE} + \|\langle r\rangle^{-1} u\|_{LE} \quad \|f\|_{LE^*} = \sum_k \|\langle r\rangle^{\frac{1}{2}} f\|_{L^2(\mathbb{R}\times A_k)}$$

Sharp formulation of local energy decay:

$$(LE) \qquad \|u\|_{LE^1} \lesssim \|\Box u\|_{LE^*} + \|\nabla u(0)\|_{L^2}$$

Theorem (Metcalfe-T 07)
(LE) holds for small $S(r^{-\epsilon})$ perturbations of the Minkowski metric.

Extensive work on this, e.g by Strauss, Keel-Smith-Sogge, Burq-Planchon-Stalker-Tahvildar-Zadeh, Metcalfe-Sogge, etc.

Local energy decay in geometries with trapping

Example: Schwarzschild space-time, with trapped set = all null geodesics tangent to the photon sphere $r = 3M$.

Redeeming feature: hyperbolic flow around trapped null geodesics.

Heuristics: frequency λ waves will stay localized up to time $\log \lambda$ (Ehrenfest time) near the trapped set, then disperse.

Consequence: $|\log \lambda|^{\frac{1}{2}}$ loss in (LE) at frequency λ on trapped set.

Theorem

Weak local energy decay holds for Schwarzschild, also for Kerr with small angular momentum.

Schwarzschild space-time: work by Laba-Soffer, Blue-Soffer, Blue-Sterbenz, Dafermos-Rodnianski and Marzuola-Metcalfe-T.-Tohaneanu.
Kerr space time: T.-Tohaneanu (also related work by Dafermos-Rodnianski and Andersson-Blue)

Stationary local energy decay

Definition:

$$(SLE) \qquad \|u\|_{LE^1[t_0,t_1]} \lesssim \|\Box u\|_{LE^*} + \|\nabla u(t_0)\|_{L^2} + \|\nabla u(t_1)\|_{L^2} + \|\partial_t u\|_{LE}$$

$$\text{(LE)} \implies \text{(SLE), but (WLE)} \not\Longrightarrow \text{(SLE)}$$

Example: Kerr waves with $\partial_t u = 0$. Then (SLE) is elliptic outside the ergosphere but not inside !

Theorem (Metcalfe-T.-Tohaneanu '11)
Weak local energy decay holds for small perturbations of Schwarzschild, $g = g_S + \epsilon O(r^{-1})$.

Proof of the main theorem

Klainerman's vector field method, vector fields:

$$Z = (\partial_\alpha, \Omega = x_i\partial_j - x_j\partial_i, S = t\partial_t + x\partial_x)$$

Weak local energy decay + commuting with vector fields:

$$\|Z^\alpha u\|_{LE^1} \lesssim \|\nabla u(0)\|_{H^m}, \qquad |\alpha| \ll m$$

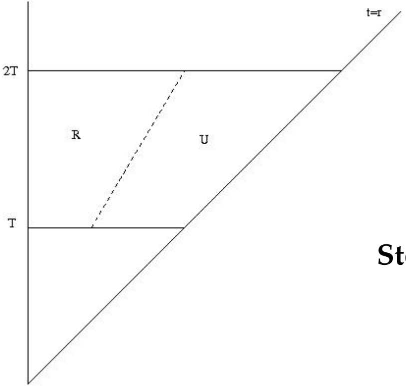
It remains to show that

$$|u(t,x)| \lesssim \frac{1}{\langle t\rangle\langle t - |x|\rangle^2}\|u_{\lesssim m}\|_{LE^1}$$

where

$$u_{\lesssim m} = \{u_\alpha\}_{|\alpha \leq m}$$

Two step iteration



Step 1: Improved bounds in region U.

$$\Box u_\alpha = Q_{sr} u_{\leq \alpha+2}$$

- Use fundamental solution for $\Box$
- Use Cauchy-Schwarz
- Elliptic estimate for ∇u_α in terms of u_α and $S u_\alpha$.

Step 2: Improved bounds in region R.

- Stationary local energy decay
- Sobolev embeddings
- Elliptic estimate for ∇u_α in terms of u_α and $S u_\alpha$.

Maxwell equations

A - electromagnetic potential, $F = dA$ -electromagnetic field

$$dF = 0, \qquad \nabla^i F_{ij} = 0$$

Kerr space-time and perturbations thereof:

Energy estimate:

$$\|F(t)\|_{L^2} \lesssim \|F(0)\|_{L^2}$$

Local energy decay:

$$\|F\|_{LE_K} \lesssim \|F(0)\|_{L^2}$$

Pointwise decay (conjectured by Penrose)[peeling estimates]:

$$|F(\bar{L}, e)| \lesssim \frac{1}{\langle t \rangle \langle t - |x| \rangle^2}$$

$$|F(\bar{L}, L)| + |F(e, e)| \lesssim \frac{1}{\langle t \rangle^2 \langle t - |x| \rangle}$$

$$|F(L, e)| \lesssim \frac{1}{\langle t \rangle^3}$$