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An example of surfaces ruled in conics

June, the 27^{th} 2002

Introduction.

This example is related to the forthcoming paper 'On rational surfaces ruled in conics' and we keep the notation there introduced.

Let $S \subset \mathbb{P}^n$ be a smooth geometric surface ruled in conics having exactly δ degenerate fibers $f_1, \dots, f_\delta$. Let $U \subset S$ be a directrix of minimum self–intersection and set $r := -U^2$.

Let us denote by $l_1, \dots, l_\delta$ the lines of the degenerate fibers that meet U and by $m_1, \dots, m_\delta$ the remaining lines; hence $f_i = l_i + m_i$, for $i = 1, \dots, \delta$.

In the cited paper we performed the following construction: if we consider the contraction σ of the lines $l_1, \dots, l_\delta$, we clearly obtain a geometrically ruled surface $\overline{S}$. Therefore $\overline{S} \cong R_{1,t+1}$ for a suitable $t \geq 0$.

In the paper we proved that, if $r \geq \delta$ or if $r < \delta$ and $t < 3r - \delta$ then U is unique and hence also the model $\overline{S}$ is unique. In particular, if $r \geq \delta$, then $t = r - \delta$.

Assume that
$$1 \leq r < \delta \quad \text{and} \quad t \geq 3r - \delta.$$
The forthcoming example shows that, in this case, the uniqueness does not hold anymore. More precisely, we are going to determine two directrices U_1 and U_2 of S such that $U_1^2 = U_2^2 = -r$. Furthermore , setting
$$l(U_1) := \{\text{the lines meeting } U_1\} = \{l_1, \dots, l_\delta\}$$
and
$$l(U_2) := \{\text{the lines meeting } U_2\}$$
and considering the two corresponding contractions $\sigma_i := Con(l(U_i))$, for $i = 1, 2$:
$$\begin{array}{ccc} & S & \\ {}^{\sigma_1}\swarrow & & \searrow^{\sigma_2} \\ R_{1,t_1+1} \cong \overline{S} & & \overline{\overline{S}} \cong R_{1,t_2+1} \end{array}$$
it turns out that $t_1 \neq t_2$.

We need first to introduce some notation: let $\overline{C}_0 \subset \overline{S}$ and $\overline{\overline{D}}_0 \subset \overline{\overline{S}}$ be the directrices of minimum self–intersection, i.e. $\overline{C}_0^2 = -t_1$ and $\overline{\overline{D}}_0^2 = -t_2$. Moreover set
$$\overline{U}_i := \sigma_1(U_i) \subset \overline{S}, \quad \overline{\overline{U}}_i := \sigma_2(U_i) \subset \overline{\overline{S}}.$$
It is clear that, denoting by
$$\alpha := \sharp\ (l(U_1) \cap l(U_2))$$

then $\overline{U}_2^2 = U_2^2 + \alpha = \alpha - r$; analogously $\overline{\overline{U}}_1^2 = \alpha - r$. Hence, if $\alpha \leq r$ then $\overline{U}_2^2 = \overline{\overline{U}}_1^2 \leq 0$, so $\overline{U}_2 = \overline{C}_0$ on $\overline{S}$ and $\overline{\overline{U}}_1 = \overline{\overline{D}}_0$ on $\overline{\overline{S}}$. Therefore $t_1 = \alpha - r = t_2$. Therefore it is clear that the example we are looking for do exist under the further assumption:

$$1 \leq r < \alpha < \delta.$$

The following picture describes the situation:

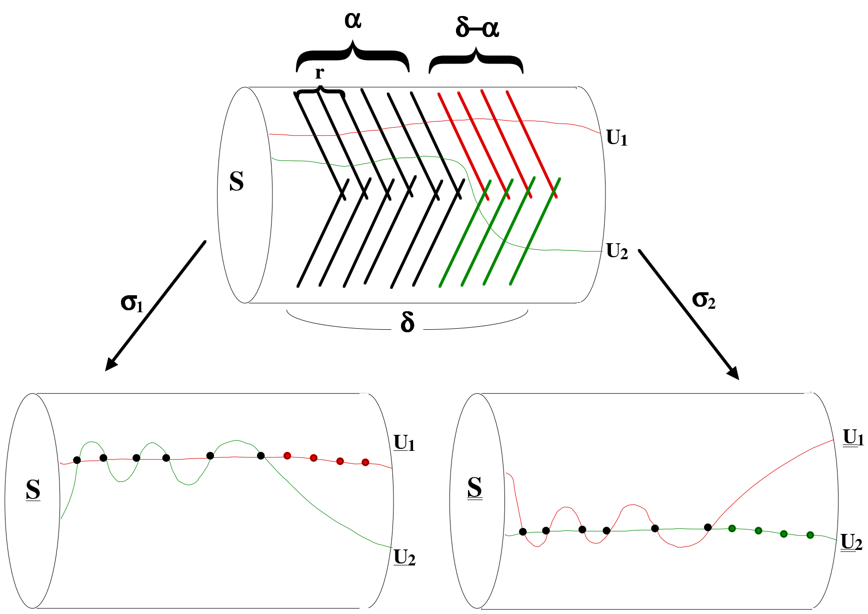

Figure 1

In order to do this, we are going to factor each σ_i via another map, say σ_r (defined in the cited paper), as follows: let σ_r be the contraction of the lines $l_1, \dots, l_r, m_{r+1}, \dots, m_\delta$. Since $\sigma_r(U_1)^2 = U_1^2 - r = 0$, it follows that

$$\sigma_r : S \longrightarrow R_{1,1}.$$

Let us set $\widetilde{U}_1 := \sigma_r(U_1)$ and $\widetilde{U}_2 := \sigma_r(U_2)$ and l, l' be the generators of $Pic(R_{1,1})$; then $\widetilde{U}_1 \sim l$.
If we put: $L_i := \sigma_r(l_i)$ for $i = 1, \dots, r$ and $M_j := \sigma_r(m_j)$, for $j = r+1, \dots, \delta$, we have that

$$L_1, \dots, L_r \in \widetilde{U}_1 \cap \widetilde{U}_2, \quad M_{r+1}, \dots, M_\alpha \notin \widetilde{U}_1 \cup \widetilde{U}_2, \quad M_{\alpha+1}, \dots, M_\delta \in \widetilde{U}_2. \quad (1)$$

Moreover, we obtain the following commutative diagram

$$\begin{array}{ccccc} & & S & & \\ & {}^{\sigma_1}\swarrow & \downarrow{\scriptstyle \sigma_r} & \searrow^{\sigma_2} & \\ R_{1,t_1+1} \cong \overline{S} & \stackrel{\varphi_1}{\longleftarrow} & R_{1,1} & \stackrel{\varphi_2}{\longrightarrow} & \overline{\overline{S}} \cong R_{1,t_2+1} \end{array}$$

where φ_i, for $i = 1, 2$ are the suitable map. Namely, since

$$\begin{array}{ll} \sigma_r & = Con(l_1, \ldots, l_r, m_{r+1}, \ldots, m_\delta) \\ \sigma_1 & = Con(l_1, \ldots, l_\delta) \\ \sigma_2 & = Con(l_1, \ldots, l_\alpha, m_{\alpha+1}, \ldots, m_\delta) \end{array}$$

then

$$\varphi_1 = Con(f_{M_{r+1}}, \ldots, f_{M_\delta}) \circ Bl(M_{r+1}, \ldots, M_\delta)$$

i.e. one has to blow–up first the points $M_{r+1}, \ldots, M_\delta \in R_{1,1}$ and then to contract the corresponding fibers f_{M_i}, for $i = r + 1, \ldots, \delta$ (substancially the lines $l_{r+1}, \ldots, l_\delta$). Analogously,

$$\varphi_2 = Con(f_{M_{r+1}}, \ldots, f_{M_\alpha}) \circ Bl(M_{r+1}, \ldots, M_\alpha).$$

The following picture shows the factorization

$$\begin{array}{ccc} & & S \\ & {}^{\sigma_1}\swarrow & \downarrow{\scriptstyle \sigma_r} \\ \overline{S} & \stackrel{\varphi_1}{\longleftarrow} & R_{1,1} \end{array}$$

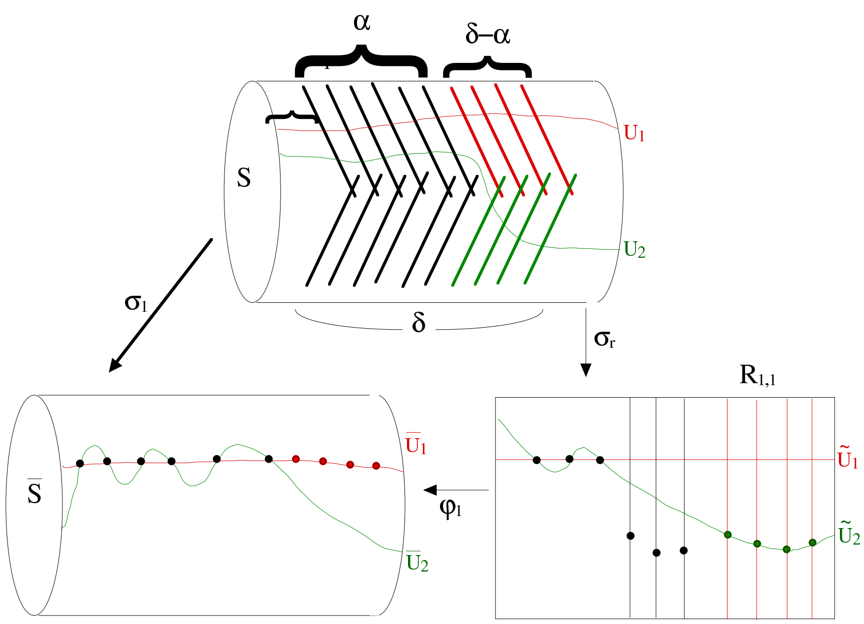

Figure 2

Finally, set $\widetilde{C}_0, \widetilde{D}_0 \subset R_{1,1}$ as follows: $\widetilde{C}_0 := \varphi_1^{-1}(\overline{C}_O)$ and $\widetilde{D}_0 := \varphi_2^{-1}(\overline{\overline{D}}_O)$.

The example Let us begin from the quadric surface $R_{1,1}$: we are going to choose 16 points on it in the following way.
Here $\widetilde{U}_1 \sim l$. Moreover, since $\widetilde{U}_2^2 = U_2^2 + r + \delta - \alpha = \delta - \alpha = 6$, then $\widetilde{U}_2 \sim l + 3l'$. Therefore $\widetilde{U}_1 \cdot \widetilde{U}_2 = L_1 + L_2 + L_3$ (in particular, this means that $U_1 \cdot U_2 = 0$ on S).
Let us choose $\delta - \alpha = 6$ further points on $\widetilde{U}_2$, say $M_{11}, \ldots, M_{16}$. Let us choose a directrix belonging to the linear system $|l + 4l'|$ passing through $M_{11}, \ldots, M_{16}$: it is possible since $\dim |l + 4l'| = 7$. Denote this directrix by $\widetilde{C}_0$.
Now let us take an irreducible curve $\widetilde{D}_0 \in |l + 2l'|$: clearly $\widetilde{C}_0 \cdot \widetilde{D}_0 = 6$; let us choose 5 points among these common points and denote them by $M_4, \ldots, M_8$. Finally choose two more points, say M_9, M_{10}, on $\widetilde{D}_0 \setminus \widetilde{C}_0$.

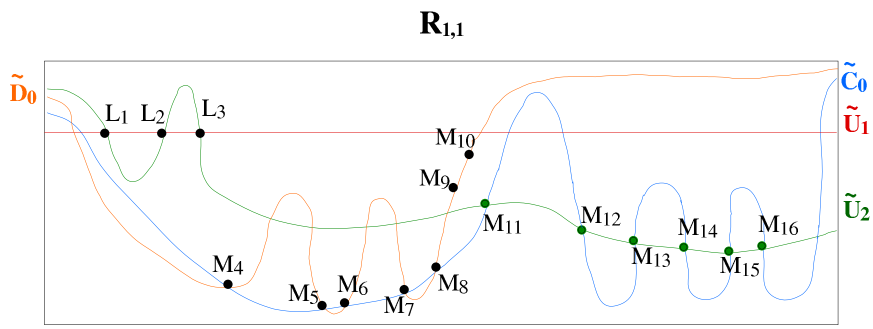

Figure 3

With this data, if we blow–up $R_{1,1}$ at all the above 16 points, i.e. the center of the blowing up is

$$\Delta := \{L_1, L_2, L_3, M_4, \ldots, M_{16}\}$$

we obtain that:

- the surface S obtained in this way has exactly 16 degenerate fibres;
- since $\widetilde{U}_1^2 \cap \Delta = L_1 + L_2 + L_3$, then $U_1^2 = \widetilde{U}_1^2 - 3 = -3$ on S;
- since $\widetilde{U}_2^2 \cap \Delta = L_1 + L_2 + L_3 + M_{11} + M_{12} + M_{13} + M_{14} + M_{15} + M_{16}$, then $U_2^2 = \widetilde{U}_2^2 - 9 = -3$ on S;
- on S the directrices U_1 and U_2 both meet exactly $\alpha = 10$ lines $(l_1, \ldots, l_{10})$;
- the geometrically ruled surface $\overline{S}$ has invariant $t_1 = -\overline{C}_0^2$ (where $\overline{C}_0 = \varphi_1(\widetilde{C}_0)$). Taking into account that

$$\varphi_1 = Con(f_{M_4}, \ldots, f_{M_{16}}) \circ Bl(M_4, \ldots, M_{16})$$

and that only $M_4, \dots, M_8$ and $M_{11}, \dots, M_{16}$ belong to $\widetilde{C}_0$, we obtain: $\overline{C}_0^2 = \widetilde{C}_0^2 - 11 + 2 = -1$; so $t_1 = 1$. In particular $\overline{S} \cong R_{1,2}$.

- the geometrically ruled surface $\overline{\overline{S}}$ has invariant $t_2 = -\overline{D}_0^2$ (where $\overline{D}_0 = \varphi_2(\widetilde{D}_0)$). Taking into account that

$$\varphi_2 = Con(f_{M_4}, \dots, f_{M_{10}}) \circ Bl(M_4, \dots, M_{10})$$

and that all these 7 points belong to $\widetilde{D}_0$, we obtain: $\overline{D}_0^2 = \widetilde{D}_0^2 - 7 = -3$; so $t_2 = 3$. In particular $\overline{\overline{S}} \cong R_{1,4}$.

This show how the same natural procedure from S to a geometrically ruled surface depends on the directrix we deal with. More precisely we can obtain two geometrically ruled surfaces, both birational to S, having different invariant.