Invertible Harmonic Mappings in the Plane

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The Basic Question

Let $B \subset \mathbb{R}^2$ be the unit disk. Let $D \subset \mathbb{R}^2$ be a Jordan domain.
Given a homeomorphism $\Phi : \partial B \mapsto \partial D$, consider the solution $U = (u_1, u_2) : B \mapsto \mathbb{R}^2$ to the following Dirichlet problem

$$\begin{cases} \Delta U = 0, & \text{in} \quad B, \\ U = \Phi, & \text{on} \quad \partial B. \end{cases}$$

Under which conditions on Φ do we have that

U is a homeomorphism of $\overline{B} \mapsto \overline{D}$?

The Classical Results

$$\Phi : \partial B \mapsto \partial D,$$

$$\begin{cases} \Delta U = 0, & \text{in} \quad B, \\ U = \Phi, & \text{on} \quad \partial B. \end{cases}$$

Theorem (H. Kneser '26)

If D is convex, then U is a homeomorphism of $\overline{B}$ onto $\overline{D}$.

Posed as a problem by Radó ('26), rediscovered by Choquet ('45).

Theorem (H. Lewy '36)

If $U : B \mapsto \mathbb{R}^2$ is a harmonic homeomorphism, then it is a diffeomorphism.

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Natural questions.

- What happens in higher dimensions?
- Can we replace Δ with other elliptic operators?
- Can we replace the diagonal Δ system with other elliptic systems?
- Can we dispense with the convexity of the target D?

Motivations

- Minimal surfaces.
- Inverse problems.
- Homogenization.
- Variational grid generation.

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Higher Dimensions.

- Wood ('74): There exists a harmonic homeomorphism $U:\mathbb{R}^3 \mapsto \mathbb{R}^3$ such that $\det DU(0)=0$.
- Melas ('93): There exists a harmonic homeomorphism $U:\overline{B}\mapsto\overline{B}$, $B\subset\mathbb{R}^3$ unit ball, such that $\det DU(0)=0$.
- Laugesen ('96): $\forall\varepsilon>0\ \exists\Phi:\partial B\mapsto\partial B$ homeomorphism, such that $|\Phi(x)-x|<\varepsilon, \forall x\in\partial B$ and the solution U to

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Elliptic Operators.

- Bauman-Marini-Nesi ('01):

$$\mathrm{div}(\sigma \nabla u_i) = 0, i = 1, 2,$$

$$\sigma = \{\sigma_{ij}\}\ ,\ K^{-1} I \leq \sigma \leq K I\ ,\ \sigma \in C^{\alpha}\ .$$

the Kneser and the Lewy theorems continue to hold.

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the Kneser theorem holds true
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Theorem
If $U : B \mapsto \mathbb{R}^2$ is a $\sigma-$harmonic homeomorphism, then

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- Harmonic mappings between Riemann surfaces, Shoen and Yau ('78), Jost ('81).
- $$\begin{cases} \operatorname{div}(M\nabla u_1 + N\nabla u_2) = 0, \\ \operatorname{div}(P\nabla u_1 + Q\nabla u_2) = 0. \end{cases}$$

  M, N, P, Q are 2×2 real constant symmetric matrices. Legendre–Hadamard condition

  $$\eta_1^2 M\xi\cdot\xi + \eta_1\eta_2(N+P)\xi\cdot\xi + \eta_2^2 Q\xi\cdot\xi > 0, \quad \forall \xi, \eta \in \mathbb{R}^2\setminus\{0\}.$$

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Equivalence.

We say

$$\begin{cases} \operatorname{div}(M\nabla u_1 + N\nabla u_2) = 0, \\ \operatorname{div}(P\nabla u_1 + Q\nabla u_2) = 0, \end{cases} \sim \begin{cases} \operatorname{div}(M'\nabla u_1 + N'\nabla u_2) = 0, \\ \operatorname{div}(P'\nabla u_1 + Q'\nabla u_2) = 0, \end{cases}$$

if there exists a non-singular 2×2 matrix $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ such that

$$\begin{pmatrix} M & N \\ P & Q \end{pmatrix} = \begin{pmatrix} \alpha \mathrm{Id} & \beta \mathrm{Id} \\ \gamma \mathrm{Id} & \delta \mathrm{Id} \end{pmatrix} \begin{pmatrix} M' & N' \\ P' & Q' \end{pmatrix}.$$

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The Kneser theorem fails.

A.-Nesi ('09). Either

$$\begin{cases} \operatorname{div}(M\nabla u_1 + N\nabla u_2) = 0, \\ \operatorname{div}(P\nabla u_1 + Q\nabla u_2) = 0, \end{cases} \sim \begin{cases} \operatorname{div}(M\nabla u_1) = 0, \\ \operatorname{div}(M\nabla u_2) = 0, \end{cases}$$ *(pure diag.)*

or

there exists a polynomial solution U to

$$\begin{cases} \operatorname{div}(M\nabla u_1 + N\nabla u_2) = 0, \\ \operatorname{div}(P\nabla u_1 + Q\nabla u_2) = 0, \end{cases}$$

and a convex set D such that $\Phi = U|_{\partial B}$ is a homeomorphism onto ∂D but $U : B \mapsto \mathbb{R}^2$ is not one-to-one.

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Examples.

- For any $\varepsilon > 0$ the system

$$\begin{cases} u_{1,xx} + u_{1,yy} = 0, \\ (1+\varepsilon)u_{2,xx} + u_{2,yy} = 0, \end{cases}$$

is not equivalent to a pure diagonal system.

- The Lamé system

$$\mu \operatorname{div}((DU)^T + DU) + \lambda \nabla(\operatorname{div} U) = 0.$$

$\mu, \lambda \in \mathbb{R}$ with $\mu > 0$ and $\mu + \lambda > 0$ is not equivalent to a pure diagonal system.

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The Kneser theorem fails for Lamé, $\mu = \lambda = 1$

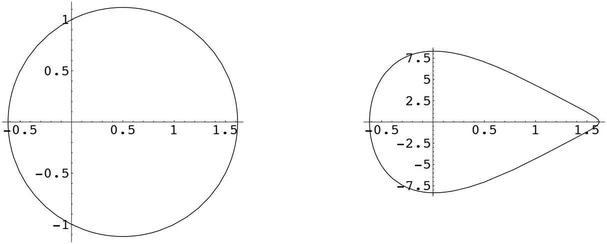

Figure: ∂B and its image $\Phi(\partial B)$.

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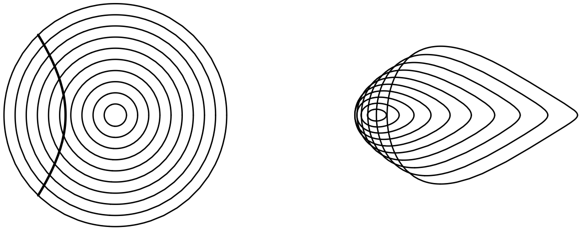

Figure: Left: circles C_r of varying radii and the nodal line of the Jacobian (an hyperbola) drawn within B. Right: the images $U(C_r)$.

Non–convex Target.

$$\Phi : \partial B \mapsto \partial D,$$

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- A:-Nesi ('09): another example.

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Given D, possibly non–convex,

- what are the additional conditions on the homeomorphism

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such that the solution U to

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is a homeomorphism of $\overline{B} \mapsto \overline{D}$?

- assume in addition $U \in C^1(\overline{B}; \mathbb{R}^2)$, under which conditions on Φ do we have that U is a diffeomorphism of $\overline{B} \mapsto \overline{D}$?

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The necessary condition.

If U is an orientation preserving diffeomorphism then, in particular,

$$\det DU > 0 \quad \text{everywhere on} \quad \partial B. \tag{1}$$

Set $\Phi = (\varphi, \psi)$, and denote

$$Hg(\theta) = \frac{1}{2\pi} \, \mathrm{P.V.}\int_0^{2\pi} \frac{g(\tau)}{\tan\left(\frac{\theta-\tau}{2}\right)} d\tau, \qquad \theta \in [0, 2\pi],$$

(1) is equivalent to

$$\frac{\partial\phi}{\partial\theta} H\left(\frac{\partial\psi}{\partial\theta}\right) - \frac{\partial\psi}{\partial\theta} H\left(\frac{\partial\phi}{\partial\theta}\right) > 0 \quad \text{everywhere on } \partial B. \tag{2}$$

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The main theorem.

Theorem (A.- Nesi '09)

Let $\Phi : \partial B \mapsto \partial D$ be an orientation preserving diffeomorphism of class C^1. Let U be the solution to

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and assume, in addition, that $U \in C^1(\overline{B}; \mathbb{R}^2)$.
The mapping U is a diffeomorphism of $\overline{B}$ onto $\overline{D}$ if and only if

$$\det DU > 0 \quad everywhere\ on \quad \partial B.$$

Non–convex Target.

The main theorem, remark.

Let $co(D)$ be the convex hull of D.
We define the convex part of ∂D as the closed set

$$\gamma_c = \partial D \cap \partial(co(D)).$$

We define the non–convex part of ∂D as the open set

$$\gamma_{nc} = \partial D \setminus \partial(co(D)).$$

Lemma

Let $\Phi : \partial B \mapsto \partial D$ be an orientation preserving diffeomorphism of class C^1, and assume that $U \in C^1(\overline{B}; \mathbb{R}^2)$. We always have

$$\det DU > 0 \quad \textit{everywhere on } \Phi^{-1}(\gamma_c).$$

Proof: Hopf lemma

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$$\det DU > 0 \quad \text{everywhere on } \Phi^{-1}(\gamma_c).$$

Proof: Hopf lemma

Non–convex Target.

The main theorem, improved.

Theorem (A.- Nesi '09)

Let $\Phi : \partial B \mapsto \partial D$ be an orientation preserving diffeomorphism of class C^1. Let U be the solution to

$$\begin{cases} \Delta U = 0, & in \quad B, \\ U = \Phi, & on \quad \partial B. \end{cases}$$

and assume, in addition, that $U \in C^1(\overline{B}; \mathbb{R}^2)$.
The mapping U is a diffeomorphism of $\overline{B}$ onto $\overline{D}$ if and only if

$$\det DU > 0 \quad everywhere\ on \quad \Phi^{-1}(\gamma_{nc}).$$

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Non–convex Target.

The main theorem, proof (i).

We assume

$$\det DU > 0 \quad \text{everywhere on} \quad \partial B.$$

The crucial point is to prove that

$$\det DU > 0 \text{ everywhere in } \quad B.$$

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The Jacobian may change sign.

a polynomial example (i)

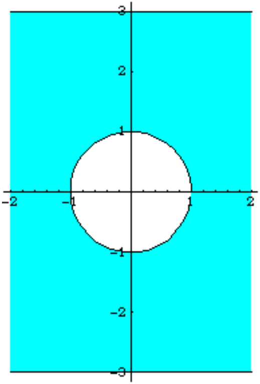

Figure: $u_1 = \Re\{\frac{(z+1)^2-1}{2}\}, u_2 = \Im\{\frac{1-(z-1)^2}{2}\}$

The Jacobian may change sign.

a polynomial example (ii)

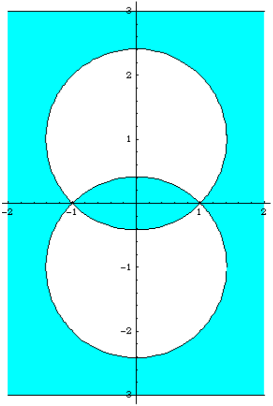

Figure: $u_1 = \Re\{(z+1)^3\}, u_2 = \Im\{(z-1)^3\}$

Non–convex Target.

The main theorem, proof (ii).

The condition

$$\det DU > 0 \text{ everywhere in } \quad B,$$

is equivalent to

$$\nabla(au_1 + bu_2) \neq 0 \text{ everywhere in } \quad B.$$

for every $(a, b) \neq (0, 0)$.

Non–convex Target.

The main theorem, proof (iii).

- Fix (a, b) and denote $u = au_1 + bu_2$, $\tilde{u}$ its harmonic conjugate and

$$f = u + i\tilde{u}$$

- Denote

$$\mathrm{WN}(f(\partial B)) = \frac{1}{2\pi}\int_{\partial B} \mathrm{d}\arg\left(\frac{\partial f}{\partial \theta}\right).$$

- The argument principle says

$$\mathrm{WN}(f(\partial B)) = \sharp \text{ critical points of } u + 1.$$

- We prove

$$\mathrm{WN}(f(\partial B)) = \mathrm{WN}(\Phi(\partial B)) = 1.$$

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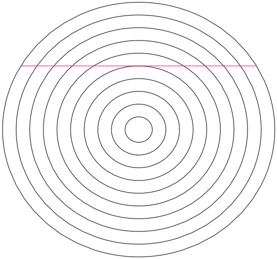
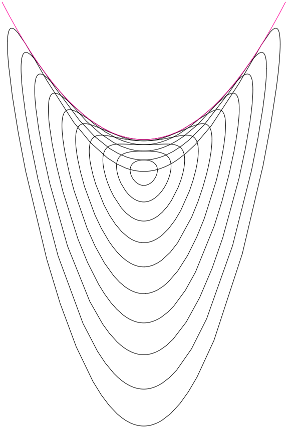
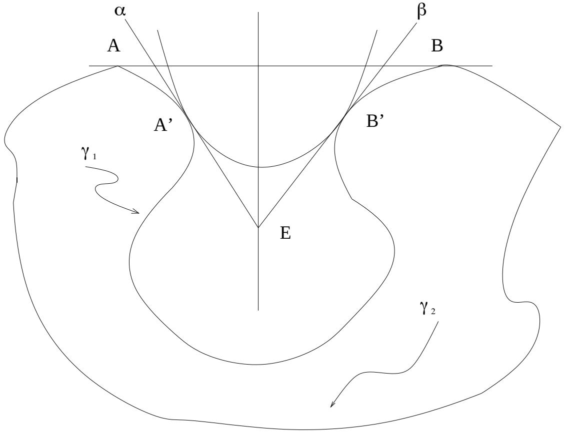
The Counterexample. $U(x, y) = (x, x^2 - y^2)$.





Non–convex Target.

The Counterexample, continued.



Open issues.

- What if $\Phi : \partial B \mapsto \partial D$ is only a homeomorphism?
- Can we replace Δ with $\mathrm{div}(\sigma\nabla\cdot)$?
- Higher dimensions?

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Thanks!

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