Radially symmetric solutions of an anisotropic mean curvature equation modeling the corneal shape *

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Abstract

We prove existence and uniqueness of classical solutions of the anisotropic prescribed mean curvature problem

$$-\mathrm{div}\left(\nabla u/\sqrt{1+|\nabla u|^2}\right) = -au + b/\sqrt{1+|\nabla u|^2}, \quad \text{in } B, \quad u = 0, \quad \text{on } \partial B,$$

where $a, b > 0$ are given parameters and B is a ball in $\mathbb{R}^N$. The solution we find is positive, radially symmetric, radially decreasing and concave. This equation has been proposed as a model of the corneal shape in the recent papers [11, 12, 13, 14], where however a linearized version of the equation has been investigated.

2010 Mathematics Subject Classification: 35J25, 35J62, 35J93, 35A05, 35A09, 35B07, 35B09, 35A24, 34A12.

*This paper was written under the auspices of INdAM-GNAMPA. The first and the third named authors have been supported by the University of Trieste, in the frame of the 2013 FRA project "Nonlinear Ordinary Differential Equations: Qualitative Theory, Numerics and Applications".

Keywords and Phrases: anisotropic prescribed mean curvature equation; Dirichlet boundary condition; radially symmetric solution; positive solution; existence; uniqueness; shooting method.

1 Introduction

This short note is devoted to the study of the existence, the uniqueness and the qualitative properties of classical solutions of the anisotropic prescribed mean curvature problem

$$\begin{cases} -\mathrm{div}\left(\dfrac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = -au + \dfrac{b}{\sqrt{1+|\nabla u|^2}}, & \text{in } B, \\ u = 0, & \text{on } \partial B, \end{cases} \tag{1}$$

where $a > 0$ and $b > 0$ are given constants and $B = B(x_0, R)$ is the open ball in $\mathbb{R}^N$ of center x_0 and radius R. This problem has been recently proposed in [11, 12, 13, 14] as a mathematical model for the geometry of the human cornea: we refer to these articles for further references on the subject. However, in all these papers a simplified version of (1) has been investigated, where the curvature operator

$$\mathrm{div}\big(\nabla u/\sqrt{1+|\nabla u|^2}\big)$$

has been replaced by its linearization Δu around 0. In particular, it has been proved in [12] that, if

$$b \in \left]0, \frac{3\sqrt{3}}{2}\frac{\sqrt{a}I_0(\sqrt{a})}{I_1(\sqrt{a})(2I_0(\sqrt{a})-1)}\right[,$$

with I_n $(n = 0, 1)$ the n-order modified Bessel functions of the first kind, and B is a unit disk in $\mathbb{R}^2$, then the (physically relevant) problem

$$\begin{cases} -\Delta u = -au + \dfrac{b}{\sqrt{1+|\nabla u|^2}}, & \text{in } B, \\ u = 0, & \text{on } \partial B, \end{cases}$$

has a unique radially symmetric solution which is the uniform limit of a sequence of successive approximations. We stress that, in the one-dimensional case, these limitations on the parameters have later been removed in [14], where it has also been pointed out the interest of studying the complete model (1). Some numerical expo

As in [5], dealing with the one-dimensional case, we takle here the fully nonlinear problem (1) and we prove the existence of a unique solution for the whole range of positive parameters a, b and for any radius R. Precisely, we first prove a uniqueness result for a more general problem

$$\begin{cases} -\mathrm{div}\left(\dfrac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = -au + \dfrac{b}{\sqrt{1+|\nabla u|^2}}, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{2}$$

where the ball B is replaced by any bounded domain Ω in $\mathbb{R}^N$ with a Lipschitz boundary $\partial\Omega$. Then we establish the existence of a classical radially symmetric solution of (1), by solving the problem

$$\begin{cases} -\left(\dfrac{r^{N-1}\,v'}{\sqrt{1+v'^2}}\right)' = r^{N-1}\left(-av+\dfrac{b}{\sqrt{1+v'^2}}\right), & \text{in }]0,R[, \\ v'(0)=0, \ v(R)=0. \end{cases} \tag{3}$$

This radial solution is therefore the unique solution of (1). We also prove that it is positive, radially decreasing and concave. Our result is formally stated in the following theorem.

Theorem 1.1. *Let $a>0$ and $b>0$ be given and let $B=B(x_0,R)$ be the open ball in $\mathbb{R}^N$ of center x_0 and radius R. Then there exists a unique solution $u\in C^2(\overline{B})$ of* (1)*, which in addition satisfies:*

- *there exists a function $v\in C^2([0,R])$ such that $u(x)=v(|x-x_0|)$ for all $x\in\overline{B}$;*
- *$0<v(r)<b/a$ for all $r\in[0,R[$;*
- *$v'(r)<0$ for all $r\in\,]0,R]$;*
- *$v''(r)<0$ for all $r\in[0,R]$.*

It is well-known that in general the study of mean curvature problems requires much care because of the possible occurrence of derivative blow-up phenomena. However, in this case, we can show that an a priori bound in C^1 for a class of solutions of

$$-\left(\frac{r^{N-1}v'}{\sqrt{1+v'^2}}\right)' = r^{N-1}\left(-av+\frac{b}{\sqrt{1+v'^2}}\right) \tag{4}$$

can be obtained by an elementary argument which exploits the structure of the equation and the qualitative properties – positivity, monotonicity and concavity – of the solutions themselves. These estimates enable us to use a shooting method on a modification of equation (4) in order to prove the existence of a solution of (3) and hence of a radially symmetric solution of (1).

The proof of the uniqueness of solutions of (2) is instead based on converting, by a suitable change of variable, the original problem into a variational inequality, for which the uniqueness of solutions can be easily established by using a monotonicity argument.

We wish to mention that part of our results extends to the N-dimensional problem in a general domain: this topic, which requires a quite different approach even in the case of an annulus, will be discussed elsewhere (see [6]).

We finally recall that anisotropic prescribed mean curvature equations have been recently considered, driven by different motivations, in [7, 8, 1, 9, 3, 2, 4, 10].

2 Existence, uniqueness and qualitative properties

The proof of Theorem 1.1 is based on Proposition 2.1 and Proposition 2.2 below. We start with the uniqueness result.

Proposition 2.1. *Let $a > 0$ and $b > 0$ be given and let Ω be a bounded domain in $\mathbb{R}^N$ with a Lipschitz boundary $\partial\Omega$. Then problem* (2) *has at most one solution $u \in C^2(\overline{\Omega})$.*

Proof. The proof consists of two steps.

Step 1. An associated variational inequality. We show that if $u \in C^2(\overline{\Omega})$ is a solution of (2), then $v = \exp(-bu)$ satisfies

$$\int_\Omega \sqrt{w^2 + b^{-2}|\nabla w|^2}\,dx - \int_\Omega \sqrt{v^2 + b^{-2}|\nabla v|^2}\,dx \geq -\int_\Omega ab^{-2}\ln v\,(w-v)\,dx \tag{5}$$

for all $w \in C^1(\overline{\Omega})$ with $\min_{\overline{\Omega}} w > 0$ and $w = 1$ on $\partial\Omega$. Indeed, it is easy to verify that, if $u \in C^2(\overline{\Omega})$ is a solution of (2), then $v = \exp(-bu)$ satisfies

$$\begin{cases} -\mathrm{div}\left(\dfrac{\nabla v}{\sqrt{v^2 + b^{-2}|\nabla v|^2}}\right) + \dfrac{b^2 v}{\sqrt{v^2 + b^{-2}|\nabla v|^2}} = -a\ln v, & \text{in } \Omega, \\ v = 1, & \text{on } \partial\Omega. \end{cases} \tag{6}$$

Pick any $w \in C^1(\overline{\Omega})$, with $\min_{\overline{\Omega}} w > 0$ and $w = 1$ on $\partial\Omega$, multiply the equation in (6) by $w - v$ and integrate by parts. The convexity and the differentiability in $\mathbb{R}_0^+ \times \mathbb{R}^N$ of the map $(s, \xi) \mapsto b^2\sqrt{s^2 + b^{-2}|\xi|^2}$ then yields

$$\begin{aligned} -\int_\Omega a\ln v\,(w-v)\,dx &= \int_\Omega \frac{\nabla v\,\nabla(w-v)}{\sqrt{v^2 + b^{-2}|\nabla v|^2}}\,dx + \int_\Omega \frac{b^2\,v\,(w-v)}{\sqrt{v^2 + b^{-2}|\nabla v|^2}}\,dx \\ &\leq \int_\Omega b^2\sqrt{w^2 + b^{-2}|\nabla w|^2}\,dx - \int_\Omega b^2\sqrt{v^2 + b^{-2}|\nabla v|^2}\,dx. \end{aligned}$$

Step 2. Uniqueness. Let us show that problem (2) has at most one solution $u \in C^2(\overline{\Omega})$. Suppose that $u_1, u_2 \in C^2(\overline{\Omega})$ are solutions of (2). Then, as $v_1 = \exp(-bu_1), v_2 = \exp(-bu_2)$ satisfy (5), we have in particular

$$\int_\Omega \sqrt{v_2^2 + b^{-2}|\nabla v_2|^2}\,dx - \int_\Omega \sqrt{v_1^2 + b^{-2}|\nabla v_1|^2}\,dx \geq -\int_\Omega ab^{-2}\ln v_1\,(v_2 - v_1)\,dx$$

and

$$\int_\Omega \sqrt{v_1^2 + b^{-2}|\nabla v_1|^2}\,dx - \int_\Omega \sqrt{v_2^2 + b^{-2}|\nabla v_2|^2}\,dx \geq -\int_\Omega ab^{-2}\ln v_2\,(v_1 - v_2)\,dx.$$

Summing up and rearranging we get

$$0 \geq \int_\Omega ab^{-2}(\ln v_2 - \ln v_1)\,(v_2 - v_1)\,dx.$$

The strict monotonicity of the logarithm function yields $v_1 = v_2$ and hence $u_1 = u_2$. $\square$

Remark 2.1 We point out that, essentially by the same proof, we can obtain the following more general conclusion: if $u_1, u_2 \in W^{1,1}(\Omega)$ satisfy $u_1|_{\partial\Omega} = u_2|_{\partial\Omega}$ and, for $i = 1, 2$,

$$\int_\Omega \frac{\nabla u_i \, \nabla w}{\sqrt{1+|\nabla u_i|^2}} \, dx = -\int_\Omega a \, u_i \, w \, dx + \int_\Omega \frac{b \, w}{\sqrt{1+|\nabla u_i|^2}} \, dx$$

for all $w \in W_0^{1,1}(\Omega)$, then $u_1 = u_2$.

Proposition 2.1 above guarantees that problem (1) has at most one solution $u \in C^2(\overline{B})$. Proposition 2.2 below ensures that problem (3) has a solution $v \in C^2([0,R])$. Setting $u(x) = v(|x - x_0|)$ for all $x \in \overline{B}$, a simple calculation shows that $u \in C^2(\overline{B})$ is the unique solution of (1); thus Theorem 1.1 follows.

Proposition 2.2. *For any given a, $b > 0$ and any $R > 0$, problem* (3) *has a solution $v \in C^2([0,R])$. In addition, the following conditions hold:*

(i) $0 < v(r) < b/a$ *for all* $r \in [0, R[$*;*

(ii) $v'(r) < 0$ *for all* $r \in \,]0, R]$*;*

(iii) $v''(r) < 0$ *for all* $r \in [0, R]$.

Proof. The proof is divided into some steps.

Step 1: A modified problem. Set $c = \sqrt{\exp(2b^2/a) - 1}$ and define a function $\varphi : \mathbb{R} \to \mathbb{R}$ by

$$\varphi(s) = \begin{cases} \dfrac{s}{\sqrt{1+s^2}}, & \text{if } |s| \le c, \\ \dfrac{s + \operatorname{sgn}(s)\, c^3}{(1+c^2)^{3/2}}, & \text{if } |s| > c. \end{cases}$$

Note that

$$\varphi'(s) = \begin{cases} \dfrac{1}{(1+s^2)^{3/2}}, & \text{if } |s| \le c, \\ \dfrac{1}{(1+c^2)^{3/2}}, & \text{if } |s| > c \end{cases}$$

is bounded, bounded away from 0 and satisfies, for all $s \in \mathbb{R}$,

$$\varphi'(s) \ge \frac{1}{(1+s^2)^{3/2}}. \tag{7}$$

Let us introduce the initial value problem

$$\begin{cases} -(r^{N-1}\varphi(v'))' = r^{N-1}\Big(-av + \dfrac{b}{\sqrt{1+v'^2}}\Big), \\ v(0) = d, \; v'(0) = 0, \end{cases} \tag{8}$$

with $d \in \mathbb{R}$. Clearly $v \in C^2([0,\omega[)$, for some $\omega \in \mathbb{R}_0^+ \cup \{+\infty\}$, is a solution of (8) if and only if it is a solution of

$$\begin{cases} v'' = \left(av - \dfrac{b}{\sqrt{1+v'^2}} \right) \dfrac{1}{\varphi'(v')} - \dfrac{N-1}{r} \dfrac{\varphi(v')}{\varphi'(v')}, \\ \qquad v(0) = d, \ v'(0) = 0. \end{cases} \tag{9}$$

In addition, as

$$\lim_{r \to 0} \frac{\varphi(v'(r))}{r\, \varphi'(v'(r))} = v''(0),$$

one has

$$v''(0) = \frac{ad - b}{N}. \tag{10}$$

Step 2: Global existence, uniqueness and continuous dependence. We are going to show that, for any given $d \in \mathbb{R}$, the initial value problem (8), or equivalently (9), has a unique solution $v \in C^2([0,R])$. Moreover, v depends continuously on the initial datum d and satisfies (10).

Indeed, local existence and uniqueness of solutions of (8) follow by observing that, for any $\delta > 0$ small enough, the operator $\mathcal{S}$, defined by

$$(\mathcal{S}v)(r) = d + \int_0^r \varphi^{-1} \left(\int_0^s \left(\frac{t}{s} \right)^{N-1} \left(av(t) - \frac{b}{\sqrt{1+v'(t)^2}} \right) dt \right) ds,$$

is a contraction in the space $C^1([0,\delta])$, endowed with the usual norm; here the global Lipschitz continuity of φ^{-1} is in particular exploited. The continuous dependence of local solutions on the initial datum d follows from the continuous dependence of the fixed points of $\mathcal{S}$ on the parameter d.

Let us now denote by $g :]0,+\infty[\times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, defined by

$$g(r,s,\xi) = \left(as - \frac{b}{\sqrt{1+\xi^2}} \right) \frac{1}{\varphi'(\xi)} - \frac{N-1}{r} \frac{\varphi(\xi)}{\varphi'(\xi)},$$

the function which appears at the right-hand side of the equation in (9). Since g is locally Lipschitz continuous in $[\delta,R] \times \mathbb{R} \times \mathbb{R}$ and grows linearly in $(s,\xi) \in \mathbb{R} \times \mathbb{R}$ uniformly in $r \in [\delta,R]$, any local solution of (9) can be uniquely continued to $[0,R]$.

Finally, the continuous dependence of these global solutions on the initial datum d is a standard consequence of the uniqueness.

Step 3: Qualitative properties. We shall show that, for any given $d < b/a$, the solution v of (9), defined on $[0,R]$, satisfies conditions (ii) and (iii).

Let us prove (ii). As $v'(0) = 0$ and $v''(0) < 0$, there exists $\delta_1 > 0$ such that $v'(r) < 0$ in $]0,\delta_1[$. Assume by contradiction that there exists $r_0 \in]0,R]$ such that

$v'(r_0) \geq 0$. We can suppose that $v'(r_0) = 0$ and $v'(r) < 0$ in $]0, r_0[$. This yields in particular $v(r_0) < b/a$. As from the equation in (9) we have

$$v''(r_0) = a\, v(r_0) - b < 0,$$

there exists $\delta_2 > 0$ such that $v'(r) > 0$ in $]r_0 - \delta_2, r_0[$, which is a contradiction.

Let us prove (iii). Assume by contradiction that there exists $\overline{r} \in]0, R]$ such that $v''(\overline{r}) \geq 0$. As $v''(0) < 0$, we can suppose that there exists $r_0 \in]0, \overline{r}[$ such that $v''(r_0) = 0$ and $v''(r) < 0$ in $[0, r_0[$. Define, for $r \in [0, R]$,

$$\psi(r) = v''(r)\varphi'(v'(r)) = av(r) - \frac{b}{\sqrt{1 + v'(r)^2}} - \frac{N-1}{r}\varphi(v'(r)).$$

We have $\psi(r_0) = 0$ and, by (ii),

$$\begin{aligned} \psi'(r_0) &= av'(r_0) + b(1 + v'(r_0)^2)^{-3/2} v'(r_0) v''(r_0) \\ &\qquad + \frac{N-1}{r_0^2}\varphi(v'(r_0)) - \frac{N-1}{r_0}\varphi'(v'(r_0))v''(r_0), \\ &= av'(r_0) + \frac{N-1}{r_0^2}\varphi(v'(r_0)) < 0. \end{aligned}$$

This implies the existence of $\delta_3 > 0$ such that $\psi(r) > 0$ on $]r_0 - \delta_3, r_0[$ and in particular $v''(r) > 0$ in $]r_0 - \delta_3, r_0[$, which is a contradiction.

Step 4: Solvability. The map $\mathcal{T} : [0, b/a] \to \mathbb{R}$ defined by $\mathcal{T}(d) = v(R)$, where v is the solution of (9), is continuous and satisfies, according to condition (ii), $\mathcal{T}(0) < 0$ and $\mathcal{T}(b/a) = b/a > 0$. Then there exists $d \in]0, b/a[$ such that $\mathcal{T}(d) = 0$. The corresponding solution $v \in C^2([0, R])$ satisfies $v'(0) = 0 = v(R)$, as well as, by Step 3, conditions $(i), (ii), (iii)$.

In order to show that v is the desired solution of (3), we prove that v also satisfies

$$\|v'\|_\infty \leq \sqrt{\exp(2b^2/a) - 1} = c,$$

or equivalently, by (iii),

$$v'(R) \geq -c.$$

From the equation in (9) we easily get

$$v''(r) \geq \frac{-b}{\varphi'(v'(r))\sqrt{1 + v'(r)^2}} \geq -b(1 + v'(r)^2),$$

and hence

$$\frac{v'(r)\, v''(r)}{1 + v'(r)^2} \leq -b\, v'(r).$$

Integrating this inequality over $[0, R]$, we obtain

$$\frac{1}{2}\ln(1+v'(R)^2) \le b\,v(0) < \frac{b^2}{a},$$

that is

$$|v'(R)| < \sqrt{\exp(2b^2/a)-1} = c.$$

This concludes the proof. $\square$

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