

# Periodic solutions of weakly coupled superlinear systems

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## Abstract

By the use of a higher dimensional version of the Poincaré–Birkhoff theorem, we are able to generalize a result of Jacobowitz [15], thus proving the existence of infinitely many periodic solutions for a weakly coupled superlinear system.

## 1 Introduction

In 1976, using the Poincaré–Birkhoff fixed point theorem, Jacobowitz [15] proved the existence of infinitely many periodic solutions for a superlinear second order scalar equation of the type

$$x'' + f(t, x) = 0. \quad (1)$$

His result was refined one year later by Hartman [14], assuming the function  $f : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  to be continuous,  $T$ -periodic in its first variable  $t$ , locally Lipschitz continuous in its second variable  $x$  (so to guarantee the uniqueness of solutions to the associated Cauchy problems), and satisfying the following two hypotheses:

- (i)  $\frac{f(t, x)}{x}$  is bounded near  $x = 0$ , uniformly in  $t \in [0, T]$ ;
- (ii)  $\lim_{|x| \rightarrow \infty} \frac{f(t, x)}{x} = +\infty$ , uniformly in  $t \in [0, T]$ .

Our aim here is to provide a generalization of this result for weakly coupled systems.

In order to simplify the statements of our results, let us consider a slightly less general situation, and write equation (1) as

$$-x'' = x h(t, x),$$

assuming  $h : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  to be continuous. In such a way, condition (i) above is trivially satisfied, while (ii) becomes

$$(ii') \quad \lim_{|x| \rightarrow \infty} h(t, x) = +\infty, \text{ uniformly in } t \in [0, T].$$

We now consider a system of the type

$$(S) \quad \begin{cases} -x_1'' = x_1 [h_1(t, x_1) + p_1(t, x_1, \dots, x_N)] \\ \quad \quad \quad \vdots \\ -x_N'' = x_N [h_N(t, x_N) + p_N(t, x_1, \dots, x_N)]. \end{cases}$$

Here, all the functions  $h_i : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  are continuous,  $T$ -periodic in their first variable  $t$ , and such that

$$\lim_{|x| \rightarrow \infty} h_i(t, x) = +\infty, \text{ uniformly in } t \in [0, T]. \quad (2)$$

The functions  $p_i : \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}$  are continuous,  $T$ -periodic in their first variable  $t$ , and bounded. Hence, there is a constant  $D > 0$  for which

$$|p_i(t, x_1, \dots, x_N)| \leq D, \quad (3)$$

for every  $(t, x_1, \dots, x_N) \in [0, T] \times \mathbb{R}^N$  and  $i = 1, \dots, N$ .

Similarly as in [14, 15], we assume that our nonlinearities are Lipschitz continuous in  $x_1, \dots, x_N$ . Moreover, we require (S) to be a Hamiltonian system: hence, we assume the existence of a function  $\mathcal{U} : \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}$  such that

$$\frac{\partial \mathcal{U}}{\partial x_i}(t, x_1, \dots, x_N) = x_i p_i(t, x_1, \dots, x_N),$$

for every  $(t, x_1, \dots, x_N) \in [0, T] \times \mathbb{R}^N$  and  $i = 1, \dots, N$ .

Simple examples of such functions can be given. For instance, we could have

$$\mathcal{U}(t, x_1, \dots, x_N) = \alpha(t) \prod_{i=1}^n G_i(x_i),$$

where the  $C^2$ -functions  $G_i : \mathbb{R} \rightarrow \mathbb{R}$  are bounded with their derivatives, satisfying  $G_i'(0) = 0$ . Or else, we could choose  $\mathcal{U}(t, x_1, \dots, x_N) = \tilde{\mathcal{U}}(t, r)$ , where  $r = x_1^2 + \dots + x_n^2$ , with a bounded derivative  $\partial \tilde{\mathcal{U}} / \partial r$ .

We will prove the following result.

**Theorem 1** *There exists a positive integer  $K$  such that, for any choice of  $N$  integers  $K_1, \dots, K_N \geq K$ , there are  $N + 1$  solutions of system (S) which are  $T$ -periodic and such that, for every index  $i$ , the component  $x_i$  has exactly  $2K_i$  simple zeros in the interval  $[0, T[$ .*

In the above theorem, we only consider  $T$ -periodic solutions with all nontrivial components. Admitting that the solutions to system (S) can have trivial components, we can state the following.

**Corollary 2** *There exists an integer  $K > 0$  such that, for every choice of  $N$  integers  $K_1, \dots, K_N \geq K$ , there exist  $2^{N-1}(N+2)$  distinct  $T$ -periodic solutions to system (S) such that, for every index  $i$ , the component  $x_i$  is either identically zero, or has exactly  $2K_i$  zeros in the period  $[0, T[$ .*

Indeed, for every fixed  $k = 1, \dots, N$  one has  $\binom{N}{k}$  possibilities of having exactly  $k$  nontrivial components, and for each of these Theorem 1 provides  $k+1$  solutions. The result then follows from the formula

$$\sum_{k=0}^N \binom{N}{k} (k+1) = 2^{N-1}(N+2).$$

Notice that, in the case  $N = 1$ , Corollary 2 gives us the existence of three  $T$ -periodic solutions for equation (1): the equilibrium  $x \equiv 0$  and the two nontrivial solutions having exactly  $2K_1$  zeros in the period interval  $[0, T[$ , given by the Poincaré–Birkhoff theorem.

As a direct corollary of Theorem 1, we also find  $mT$ -periodic solutions, where  $m$  is any positive integer. Notice that, if at least one of the integers  $K_1, \dots, K_N$  is relatively prime with  $m$ , these solutions will not be  $\ell T$ -periodic, for any  $\ell = 1, \dots, m-1$ . Such solutions are sometimes named *subharmonic solutions*. However, their minimal period could be small, as it happens for an autonomous scalar equation like  $x'' + x^3 = 0$ . On the other hand, for some particularly structured equations, the period of the solutions cannot be less than  $T$ : take, for instance, an equation like  $x'' + q(t)x^3 = 0$ , with  $q(t)$  having minimal period  $T$ . We thus see that Theorem 1 can sometimes be used to provide subharmonic solutions having *minimal period*  $mT$ .

The proof of Theorem 1 will be carried out using a higher dimensional version of the Poincaré–Birkhoff theorem for Poincaré maps of Hamiltonian systems, recently obtained by the first author and A. J. Ureña in [12]. Let us mention that, in our assumptions, the solutions to Cauchy problems need not be globally defined. For instance, it was shown in [7] that there are positive continuous functions  $q(t)$  such that the differential equation  $x'' + q(t)x^3 = 0$  has a solution which does not exist on  $[0, T]$ . So, the Poincaré map may not be well defined. To overcome this difficulty, Jacobowitz [15] considered the *successor map*, instead of the Poincaré map. However, in order to have this map well defined, he needed to assume an additional sign condition on the nonlinearity. Our approach will follow the idea of Hartman [14] of modifying the nonlinearities, and making use of some *a priori estimates* for the solutions with a *prescribed number of rotations* in the phase plane (see also [18]).

The global existence problem does not arise when the differential equation has some particular structure, e.g. when  $f(t, x)$  in (1) is of the type  $f(x) + e(t)$ . In this case, the existence of periodic solutions has been proved

in [5, 6, 8, 13, 17] (see also [4]). Let us mention that existence results for systems have also been obtained by the use of variational methods, see e.g. [1, 2, 3, 9, 16, 19, 20]. Usually, the main difficulty encountered in these papers is the verification of the Palais–Smale condition, for which some peculiar hypotheses must be assumed. Our method does not need such technical assumptions: however, we can only apply it to systems which are weakly coupled.

## 2 Proof of Theorem 1

In this section, we provide a proof for Theorem 1, mainly based on the higher dimensional version of the Poincaré–Birkhoff theorem given in [12, Theorem 8.2].

In order to recover the existence of global solutions to the associated Cauchy problems, let us introduce the following truncations of the functions  $h_i$ . For every  $R > 1$ , let

$$h_i^R(t, x_i) = \begin{cases} h_i(t, -R), & \text{if } x_i < -R, \\ h_i(t, x_i), & \text{if } |x_i| \leq R, \\ h_i(t, R), & \text{if } x_i > R. \end{cases}$$

To simplify the notation, we will write the vector in  $\mathbb{R}^{N-1}$ , obtained removing from some  $x \in \mathbb{R}^N$  its  $i$ -th component, as follows:

$$\lambda^i = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_N).$$

We define the functions  $g_i^R : \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{N-1} \rightarrow \mathbb{R}$  by

$$g_i^R(t, x_i, \lambda^i) = x_i [h_i^R(t, x_i) + p_i(t, x_1, \dots, x_N)].$$

These functions are continuous, and satisfy, for every  $i = 1, \dots, N$  and  $R > 1$ ,

$$g_i^R(t, 0, \lambda^i) = 0, \quad \text{for every } t \in [0, T] \text{ and } \lambda^i \in \mathbb{R}^{N-1}. \quad (4)$$

Moreover, it is readily verified that, by continuity, there is a constant  $a > 0$  such that

$$\left| \frac{g_i^R(t, x_i, \lambda^i)}{x_i} \right| \leq a, \quad \text{when } 0 < |x_i| < 1, \quad (5)$$

for every  $t \in [0, T]$  and  $\lambda^i \in \mathbb{R}^{N-1}$ .

Let us now consider the modified system

$$(S_R) \quad \begin{cases} -x_1'' = g_1^R(t, x_1, \lambda^1) \\ \vdots \\ -x_N'' = g_N^R(t, x_N, \lambda^N). \end{cases}$$

Notice that, by construction, the solutions of the Cauchy problems associated to  $(S_R)$  are unique and globally defined.

By (4), a solution  $x = (x_1, \dots, x_N)$  to system  $(S_R)$  may have some trivial components. The uniqueness of the solutions to the associated Cauchy problems allows us to state the following.

**Proposition 3** *If a solution  $x = (x_1, \dots, x_N)$  to system  $(S_R)$  satisfies  $x_i(t_0) = x'_i(t_0) = 0$ , for some  $t_0 \in [0, T]$  and  $i \in \{1, \dots, N\}$ , then the component  $x_i$  must be constantly equal to zero.*

Let us write the  $2N$ -dimensional system

$$(P_R) \quad \begin{cases} x'_i = y_i \\ -y'_i = g_i^R(t, x_i, \lambda^i) \end{cases} \quad i = 1, \dots, N,$$

which is equivalent to  $(S_R)$ . In the following, we will denote by  $y$  the vector  $(y_1, \dots, y_N)$ , and we will write  $z = (x, y) \in \mathbb{R}^{2N}$ . We know by Proposition 3 that, if a solution  $x$  of  $(S_R)$  has a component  $x_i$  which is not identically zero, then  $(x_i(t), y_i(t)) \neq (0, 0)$  for every  $t$ , so that we can parametrize  $z_i(t) = (x_i(t), y_i(t))$  in polar coordinates as

$$x_i(t) = \rho_i(t) \cos(\vartheta_i(t)), \quad y_i(t) = \rho_i(t) \sin(\vartheta_i(t)). \quad (6)$$

In this case, we can define the *rotation number* of the  $i$ -th component of  $z = (x, y)$  as

$$\text{rot}^i(z) = -\frac{1}{2\pi} (\vartheta_i(T) - \vartheta_i(0)).$$

In particular, if  $x$  is a  $T$ -periodic solution of  $(S_R)$  with a nontrivial component  $x_i$ , then  $\text{rot}^i(z)$  is an integer, and  $x_i$  has exactly  $2 \text{rot}^i(z)$  simple zeros in  $[0, T[$ . Indeed,  $\text{rot}^i(z)$  is the number of clockwise rotations performed by  $z_i = (x_i, y_i)$  around the origin, in the time interval  $[0, T]$ .

## 2.1 The a priori bound

The following lemma gives us an a priori bound to the  $T$ -periodic solutions of  $(P_R)$  whose nontrivial components perform at most a fixed number of rotations around the origin.

**Lemma 4 (A priori bound)** *For every positive integer  $K_0$ , there exists a constant  $\mathcal{R} = \mathcal{R}(K_0) > 1$  with the following property: if  $z = (x, y)$  is a  $T$ -periodic solution to  $(P_R)$ , with  $R > \mathcal{R}$ , such that  $\text{rot}^i(z) \leq K_0$  for every nontrivial component  $z_i$ , then*

$$|z_i(t)| < \mathcal{R}, \quad \text{for every } t \in [0, T] \text{ and } i = 1, \dots, N.$$

*Proof.* Given  $K_0$ , fix a constant  $b > 1$  such that,

$$b > 2 \left( \frac{2\pi}{T} K_0 \right)^2.$$

By (2) and (3), there exist two positive constants  $c$  and  $\bar{R}$ , with  $1 < c < \bar{R}$ , such that, for every  $R \geq \bar{R}$  and every index  $i$ ,

$$\frac{g_i^R(t, x_i, \lambda^i)}{x_i} = h_i^R(t, x_i) + p_i(t, x_i, \lambda^i) > h_i^R(t, x_i) - D > b, \quad (7)$$

for every  $x_i$  with  $|x_i| \geq c$ ,  $t \in [0, T]$ , and  $\lambda^i \in \mathbb{R}^{N-1}$ ,

Moreover, by continuity, it is possible to find a constant  $M > 0$  such that, for every index  $i$ ,

$$|g_i^R(t, x_i, \lambda^i)| < M, \quad (8)$$

for every  $x_i$  with  $|x_i| \leq c$ ,  $t \in [0, T]$ , and  $\lambda^i \in \mathbb{R}^{N-1}$ .

Then, choose  $d > 0$  large enough to have  $d^2 > \max\{4Mc, 4c^2b\}$ , and define

$$\mathcal{A}_0 = [-c, c] \times [-d, d].$$

Consider now a  $T$ -periodic solution  $(x, y)$  to  $(P_R)$ , with  $R > \bar{R}$ , such that, for some index  $i_0$ ,

$$(x_{i_0}(t), y_{i_0}(t)) \notin \mathcal{A}_0, \quad \text{for every } t \in [0, T]. \quad (9)$$

We will now show that, in such a case,  $(x_{i_0}, y_{i_0})$  has to perform more than  $K_0$  rotations around the origin in the time interval  $[0, T]$ . Indeed, if  $|x_{i_0}(t)| \leq c$ , then  $|y_{i_0}(t)| \geq d$ , so one has

$$\begin{aligned} -\vartheta'_{i_0}(t) &= \frac{g_{i_0}^R(t, x_{i_0}(t), \lambda^{i_0}(t)) x_{i_0}(t) + y_{i_0}(t)^2}{x_{i_0}(t)^2 + y_{i_0}(t)^2} \\ &\geq \frac{-Mc + y_{i_0}(t)^2}{x_{i_0}(t)^2 + y_{i_0}(t)^2} \\ &> \left(1 - \frac{Mc}{d^2}\right) \sin^2(\vartheta_{i_0}(t)) \\ &> \frac{3}{4} \sin^2(\vartheta_{i_0}(t)) \\ &> b \cos^2(\vartheta_{i_0}(t)) + \frac{1}{2} \sin^2(\vartheta_{i_0}(t)), \end{aligned}$$

and, if  $|x_{i_0}(t)| \geq c$ , one has

$$\begin{aligned} -\vartheta'_{i_0}(t) &= \frac{g_{i_0}^R(t, x_{i_0}(t), \lambda^{i_0}(t)) x_{i_0}(t) + y_{i_0}(t)^2}{x_{i_0}(t)^2 + y_{i_0}(t)^2} \\ &> b \cos^2(\vartheta_{i_0}(t)) + \sin^2(\vartheta_{i_0}(t)) \\ &> b \cos^2(\vartheta_{i_0}(t)) + \frac{1}{2} \sin^2(\vartheta_{i_0}(t)). \end{aligned}$$

Consequently, every solution to  $(P_R)$ , with  $R > \bar{R}$ , satisfying (9) is such that the component  $(x_{i_0}, y_{i_0})$  takes a time  $\tau$  to perform a rotation around the origin estimated by

$$\tau < \int_0^{2\pi} \frac{d\vartheta}{b \cos^2 \vartheta + \frac{1}{2} \sin^2 \vartheta} = \frac{2\pi}{\sqrt{b/2}} < \frac{T}{K_0}.$$

Hence, every  $T$ -periodic solution  $z = (x, y)$  to  $(P_R)$ , with  $R > \bar{R}$ , satisfying (9) is such that

$$\text{rot}^{i_0}(z) > K_0.$$

Now, in order to find the needed a priori bound, we will use the so-called *admissible spiral* method, a tool introduced by the authors in [10, 11]. We recall that, roughly speaking, an admissible spiral for a system in  $\mathbb{R}^2$  is a curve which guides the solutions in the phase plane, forcing them to rotate around the origin when they increase in norm. The generalization to systems in  $\mathbb{R}^{2N}$  of the type

$$\begin{cases} z'_1 = F_1(t, z_1, \dots, z_N) \\ \vdots \\ z'_N = F_N(t, z_1, \dots, z_N), \end{cases}$$

with  $z_i \in \mathbb{R}^2$ , has been explained in [11], and is depicted in Figure 1.

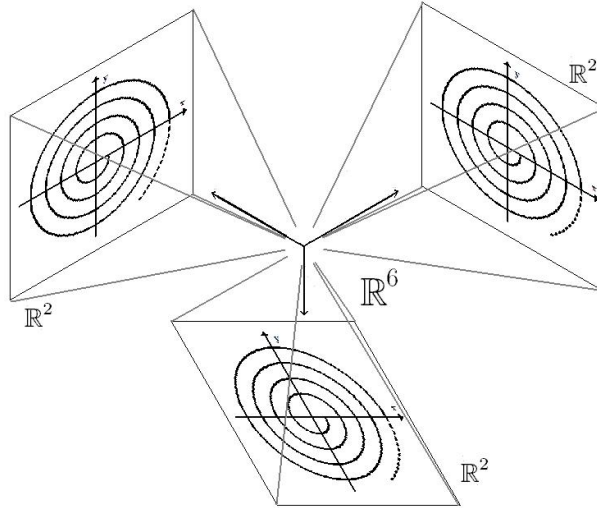


Figure 1: *An artistic view of the admissible spirals when  $N = 3$ .*

Precisely, for each index  $i$ , an  $i$ -admissible spiral is a continuous and injective curve  $\gamma^i : [0, +\infty[ \rightarrow \mathbb{R}^2$ , satisfying the following properties:

1. There exists an unlimited strictly increasing sequence

$$0 = \sigma_0 < \sigma_1 < \sigma_2 < \dots < \sigma_k < \sigma_{k+1} < \dots$$

such that the restriction of  $\gamma^i$  to every closed interval  $[\sigma_k, \sigma_{k+1}]$  is continuously differentiable, and

$$\langle J\dot{\gamma}^i(s), \gamma^i(s) \rangle > 0, \quad \text{for every } s \in [\sigma_k, \sigma_{k+1}].$$

Here,  $J$  denotes the standard symplectic  $2 \times 2$  matrix:

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

2. The curve grows in norm to infinity:

$$\lim_{s \rightarrow +\infty} |\gamma^i(s)| = +\infty.$$

3. The curve rotates clockwise infinitely many times:

$$\int_0^{+\infty} \frac{\langle J\dot{\gamma}^i(s), \gamma^i(s) \rangle}{|\gamma^i(s)|^2} ds = +\infty.$$

4. When restricted to any subinterval  $[\sigma_k, \sigma_{k+1}]$ , it satisfies

$$\langle J\dot{\gamma}^i(s), F_i(t, z_1, \dots, z_{i-1}, \gamma^i(s), z_{i+1}, \dots, z_N) \rangle < 0,$$

for every  $t \in [0, T]$ ,  $s \in [\sigma_k, \sigma_{k+1}]$ , and  $z_j \in \mathbb{R}^2$  with  $j \neq i$ .

So, we need to construct, for every index  $i = 1, \dots, N$ , an  $i$ -admissible spiral  $\gamma^i$ . The construction will be essentially the same for every index  $i$ , and we will briefly follow the ideas explained in [10, 11] (see, in particular, [10, Theorem 3.10] and [11, Theorem 4.1]). Let  $\eta \in ]0, 1[$  be a small constant. The spirals will be defined piecewise in the following regions (see Figure 2).

$$\begin{aligned} \mathcal{A}_1 &= [c, +\infty[ \times ]-\eta, \eta], \\ \mathcal{A}_2 &= [c, +\infty[ \times ]-\infty, -\eta], \\ \mathcal{A}_3 &= [-c, c] \times ]-\infty, -d], \\ \mathcal{A}_4 &= ]-\infty, -c] \times ]-\infty, -\eta], \\ \mathcal{A}_5 &= ]-\infty, -c] \times ]-\eta, \eta], \\ \mathcal{A}_6 &= ]-\infty, -c] \times ]\eta, +\infty[, \\ \mathcal{A}_7 &= [-c, c] \times [d, +\infty[, \\ \mathcal{A}_8 &= [c, +\infty[ \times ]\eta, +\infty[. \end{aligned}$$

The construction in the regions with an even index makes use of some energy estimates. Recalling (7), notice that, for every index  $i$ , it is possible to find two continuous functions  $g_{i,j} : \mathbb{R} \rightarrow \mathbb{R}$ , with  $j = 1, 2$ , such that, for every sufficiently large positive  $R$ ,

$$g_{i,1}(x_i) < g_i^R(t, x_i, \lambda^i) < g_{i,2}(x_i) = -1, \quad \text{when } x_i \leq -c,$$



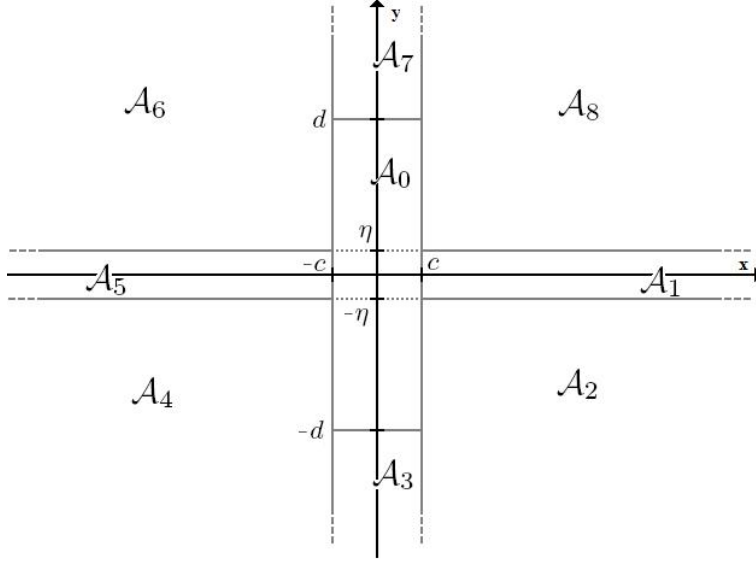


Figure 2: *The nine considered regions.*

and

$$1 = g_{i,1}(x_i) < g_i^R(t, x_i, \lambda^i) < g_{i,2}(x_i), \quad \text{when } x_i \geq c.$$

Consider then their primitives

$$G_{i,j}(\xi) = \int_0^\xi g_{i,j}(\sigma) d\sigma,$$

and the associated energy functions  $E_{i,j}(\xi, v) = \frac{1}{2}v^2 + G_{i,j}(\xi)$ . It is easy to see that, for every solution  $z = (x, y)$  of  $(P_R)$ , with  $x = (x_1, \dots, x_N)$  and  $y = (y_1, \dots, y_N)$ ,

$$\frac{d}{dt}E_{i,1}(x_i(t), y_i(t)) < 0, \quad \text{if } (x_i(t), y_i(t)) \in \mathcal{A}_4 \cup \mathcal{A}_8,$$

and

$$\frac{d}{dt}E_{i,2}(x_i(t), y_i(t)) < 0, \quad \text{if } (x_i(t), y_i(t)) \in \mathcal{A}_2 \cup \mathcal{A}_6.$$

So, the spiral  $\gamma^i$  will be chosen as a level curve of  $E_{i,1}$  in  $\mathcal{A}_4$  and  $\mathcal{A}_8$ , and as a level curve of  $E_{i,2}$  in  $\mathcal{A}_2$  and  $\mathcal{A}_6$ .

The branches of  $\gamma^i$  in  $\mathcal{A}_3$  and  $\mathcal{A}_7$  will be built following the idea in [11, Lemma 4.2]. In fact, by (8), in these regions, the superlinearity property of  $g_i$  does not affect the estimate of the radial velocity of each component of the solutions and, being the angular velocity positively bounded from below, it is possible to have a limitation to the radial velocity with respect to the spanned angle (see the proof of [11, Lemma 4.2]). At last, in regions  $\mathcal{A}_1$  and

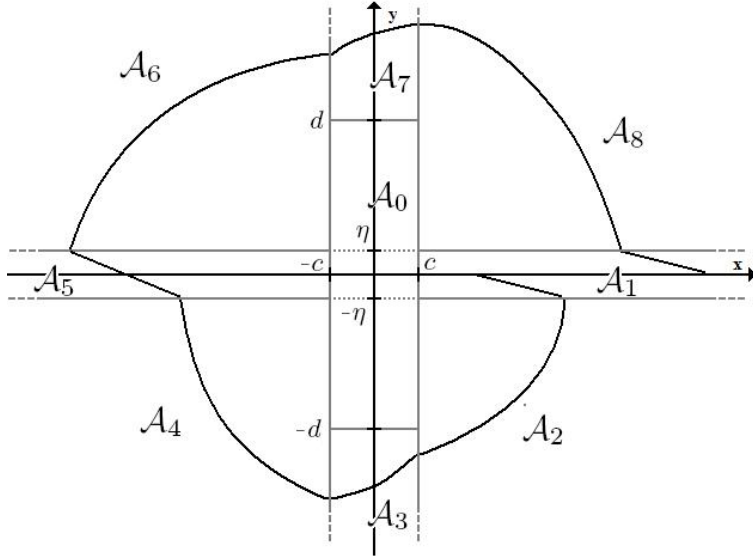


Figure 3: *The construction of the first lap of admissible spiral.*

$\mathcal{A}_5$  the spiral  $\gamma^i$  consists of lines with sufficiently small negative slope  $-\chi$ . See Figure 3 for an overall view of the first lap of the spiral thus constructed.

Once clarified how every branch must be built, we can choose a starting point  $P = (r_0, 0)$  lying in  $\mathcal{A}_1$  and draw the first lap of the spirals  $\gamma^i$  passing in the eight regions. Iterating this procedure, we can construct an infinite number of laps for each of this curves. We take  $r_0$  large enough, so that the spirals do not intersect  $\mathcal{A}_0$ . Moreover, choosing the slope  $\chi$  of the lines in  $\mathcal{A}_1$  and  $\mathcal{A}_5$  small enough at each lap, we can ensure that all the spirals are injective and grow to infinity in norm.

Let  $\mathcal{R} > \max\{\sqrt{c^2 + d^2}, \bar{R}\}$  be such that the open ball  $B_{\mathcal{R}}$  fully contains all the first  $K_0 + 1$  laps of the spirals  $\gamma^i$ . This radius represents the a priori bound we are looking for. In fact, assume by contradiction that there exists a  $T$ -periodic solution  $z = (x, y)$  to system  $(P_{\mathcal{R}})$ , with  $R > \mathcal{R}$ , satisfying  $\text{rot}^i(z) \leq K_0$  for every  $i$ , and that there also exist an index  $i_0$  and an instant  $t_1$  for which  $|z_{i_0}(t_1)| \geq \mathcal{R}$ . We know that if such a solution satisfies (9), then  $\text{rot}^{i_0}(z) > K_0$ . Hence, there must exist an instant  $t_2 \in [t_1, t_1 + T]$  at which  $z_{i_0}(t_2) \in \mathcal{A}_0$ . Therefore, the  $i_0$ -th component of  $z$  must go from  $\mathcal{A}_0$  to  $\mathbb{R}^2 \setminus B_{\mathcal{R}}$  in the time-interval  $[t_2, t_1 + T]$ , guided by the spiral  $\gamma^{i_0}$ , thus performing at least  $K_0 + 1$  rotations around the origin in this time interval  $[t_2, t_1 + T]$ . On the other hand, since  $x'_{i_0} = y_{i_0}$ , the  $i_0$ -th component of the solutions can never perform counterclockwise rotations. So, the  $i_0$ -th component has to perform more than  $K_0$  rotations around the origin in the time interval  $[0, T]$ , a contradiction which ends the proof of the lemma.

## 2.2 The Poincaré–Birkhoff geometrical setting

To complete the proof of Theorem 1, we will apply a higher dimensional version of the Poincaré–Birkhoff theorem for Poincaré maps of Hamiltonian systems recently obtained in [12]. Precisely, let  $0 < R_1 < R_2$  and consider the  $N$ -annulus in  $\mathbb{R}^{2N}$  defined as

$$\Omega = [B_{R_2} \setminus \overline{B_{R_1}}]^N = (B_{R_2} \setminus \overline{B_{R_1}}) \times \cdots \times (B_{R_2} \setminus \overline{B_{R_1}}) .$$

As a direct consequence of [12, Theorem 8.2], the following statement holds.

**Theorem 5** *Assume that every solution  $z$  of  $(P_R)$ , departing from  $z(0) \in \partial\Omega$ , is defined on  $[0, T]$  and satisfies*

$$z_i(t) \neq (0, 0), \quad \text{for every } t \in [0, T] \text{ and } i = 1, \dots, N .$$

*Assume moreover that there are positive integers  $K_1, \dots, K_N$  such that, for each  $i = 1, \dots, N$*

$$\text{rot}^i(z) < K_i \text{ if } |z_i(0)| = R_1, \quad \text{and} \quad \text{rot}^i(z) > K_i \text{ if } |z_i(0)| = R_2 .$$

*Then, the Hamiltonian system has at least  $N+1$  distinct  $T$ -periodic solutions  $z$ , with  $z(0) \in \Omega$ , such that  $\text{rot}^i(z) = K_i$ , for every  $i = 1, \dots, N$ .*

The first ingredient for completing the proof of Theorem 1 is an estimate of the rotation number for solutions having a *small* component.

**Lemma 6** *There exist a positive integer  $K$  and a positive constant  $\delta$  such that, if a solution  $z = (x, y)$  to system  $(P_R)$  satisfies  $0 < |z_i(0)| < \delta$  for a certain index  $i$ , then  $\text{rot}^i(z) < K$ .*

*Proof.* Assume  $0 < |z_i(0)| < 1$ . After parametrizing  $z_i = (x_i, y_i)$  in polar coordinates (6), one has, by (5), as long as  $\rho_i(t) = |z_i(t)| < 1$ ,

$$\begin{aligned} \rho_i'(t) &= \frac{x_i'(t)x_i(t) + y_i'(t)y_i(t)}{\sqrt{x_i(t)^2 + y_i(t)^2}} \\ &= \frac{x_i(t)y_i(t) \left( 1 - \frac{g_i^R(t, x_i(t), \lambda^i(t))}{x_i(t)} \right)}{\sqrt{x_i(t)^2 + y_i(t)^2}} \\ &\leq (a+1)\rho_i(t) . \end{aligned}$$

By Gronwall Lemma, choosing  $\delta < e^{-(a+1)T}$ , we see that, if  $\rho_i(0) < \delta$ , then

$$\rho_i(t) \leq \rho_i(0)e^{(a+1)t} < e^{(a+1)(t-T)},$$

so that necessarily  $\rho_i(t) < 1$  for every  $t \in [0, T]$ . Now, recalling (5), the following estimate holds for the angular velocity of  $z_i = (x_i, y_i)$ :

$$\begin{aligned} -\vartheta'_i(t) &= \frac{g_i^R(t, x_i(t), \lambda^i(t)) x_i(t) + y_i(t)^2}{x_i(t)^2 + y_i(t)^2} \\ &< a \cos^2(\vartheta_i(t)) + \sin^2(\vartheta_i(t)). \end{aligned}$$

Thus, the time needed by  $z_i$  to perform a complete rotation around the origin must be greater than  $2\pi/\sqrt{a}$ . Hence, choosing the integer  $K$  satisfying

$$a < \left( \frac{2\pi}{T} K \right)^2,$$

one has that  $\text{rot}^i(z) < K$ , thus completing the proof of the lemma.  $\blacksquare$

We now estimate the number of rotations performed by a solution with a component which starts *sufficiently far* from the origin.

**Lemma 7** *Given a positive integer  $K_0$ , let  $\mathcal{R} = \mathcal{R}(K_0)$  be the constant provided by Lemma 4. If  $z = (x, y)$  is any solution to  $(P_R)$ , with  $R > \mathcal{R}$ , such that  $|z_i(0)| = \mathcal{R}$  for some index  $i$ , then  $\text{rot}^i(z) > K_0$ .*

*Proof.* Following the proof of Lemma 4, we only need to consider the case of a solution, starting with  $|z_i(0)| = \mathcal{R}$  for some index  $i$ , such that  $z_i = (x_i, y_i)$  enters the region  $\mathcal{A}_0$  at a certain time  $t_0 \in ]0, T]$ . If we change the time variable  $t$  into  $s = t_0 - t$ , we obtain a solution  $\tilde{z} = (\tilde{x}, \tilde{y})$  to the system

$$\begin{cases} \tilde{x}'_i = \tilde{y}_i \\ -\tilde{y}'_i = g_i^R(t_0 - s, \tilde{x}_1, \dots, \tilde{x}_N) \end{cases} \quad i = 1, \dots, N,$$

such that  $\tilde{z}_i(0) \in \mathcal{A}_0$  and  $|\tilde{z}_i(t_0)| = \mathcal{R}$ . Clearly, the admissible spiral  $\gamma^i$  still guides also the solution  $\tilde{z}$  of this system, forcing the component  $\tilde{z}_i = (\tilde{x}_i, \tilde{y}_i)$  to rotate clockwise at least  $K_0 + 1$  times around the origin in the time interval  $[0, t_0]$ . By the same argument in the proof of Lemma 4, we then have that  $\text{rot}^i(\tilde{z}) > K_0$ . The proof is thus completed, since  $\text{rot}^i(z) = \text{rot}^i(\tilde{z})$ .  $\blacksquare$

We are now ready to conclude the proof of Theorem 1. Using Lemma 6, we find the integer  $K$ . Then, once the integers  $K_1, \dots, K_N \geq K$  are given, we can define  $K_0 = \max\{K_1, \dots, K_N\}$  and apply Lemma 4 so to find the constant  $\mathcal{R} = \mathcal{R}(K_0)$ . Choose  $R > \mathcal{R}$  and consider the Poincaré map associated to the Hamiltonian system  $(P_R)$ . By Lemmas 6 and 7, choosing  $R_1 = \delta$  and  $R_2 = \mathcal{R}$ , we can apply Theorem 5 above so to find  $N + 1$  distinct  $T$ -periodic solutions whose  $i$ -th component performs exactly  $K_i$  clockwise rotations around the origin in the time interval  $[0, T]$ . Lemma 4 guarantees us that all these solutions are also solutions to system  $(S)$ , since  $g_i(t, x_i, \lambda^i) = g_i^R(t, x_i, \lambda^i)$  when  $|x_i| < \mathcal{R}$ . The proof of Theorem 1 is thus completed.

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