From isochronous potentials to isochronous systems

Andrea Sfecci

Abstract

There is a wide literature involving the study of isochronous equations of the type

$$x''(t) + \frac{\partial}{\partial x} V\big(x(t)\big) = 0\,,$$

where V is a C^2-function. In this paper we show how the kinetic energy $T(y) = \frac{1}{2}y^2$ can be modified still preserving the isochronicity property of the corresponding system. More generally we provide estimates for the periods, and show an application to the Steen's equation and other systems related to the anharmonic potential $V(x) = ax^2 + bx^{-2}$.

1 Introduction

The study of *isochronous centers* for differential systems of the type

$$\begin{cases} x' = \mathcal{P}(x,y) \\ y' = \mathcal{Q}(x,y)\,, \end{cases} \tag{1}$$

presents a wide literature, in particular when the functions $\mathcal{P}$ and $\mathcal{Q}$ are assumed to be polynomials (see [4, 5, 13, 14] and the references therein). In this paper we are interested in systems of the type $\mathcal{P}(x,y) = \mathcal{P}(y)$ and $\mathcal{Q}(x,y) = \mathcal{Q}(x)$, where the functions are assumed to be C^1-functions. The simplest case $\mathcal{P}(y) = y$ is related to the differential equation

$$x''(t) + \frac{\partial}{\partial x} V\big(x(t)\big) = 0\,, \tag{2}$$

where V is a C^2-function. This equation has already been studied widely: the solutions have constant energy

$$E(x(t), x'(t)) = \frac{1}{2}(x'(t))^2 + V(x(t))\,,$$

and the periodic solutions have period

$$\tau_0 = \sqrt{2}\int_{x_1}^{x_2} \frac{dx}{\sqrt{E - V(x)}}\,, \tag{3}$$

where $x_1 < x_2$ are the extremals of the orbit. The study of functions V providing a constant period function τ leads to the following definition.

Definition 1.1 (Isochronous potential) *A real valued C^2-function $A = A(u)$, defined on an open interval I, is said to be an isochronous potential if the equation*

$$u''(t) + \frac{\partial}{\partial u} A\big(u(t)\big) = 0 \tag{4}$$

has a unique equilibrium point and all the other solutions such that $u(t) \in I$ for every $t \in \mathbb{R}$, are periodic of the same period.

Notice that the elastic potential $V(x) = \frac{1}{2}kx^2$, which is known to be isochronous, has the same form of the kinetic energy function $T(y) = \frac{1}{2}my^2$. Hence, also the kinetic energy function is an isochronous potential in the sense of Definition 1.1. There are several well-known isochronous potentials $A(u)$, for instance, besides the harmonic one u^2, we have the anharmonic one $u^2 + u^{-2}$ with $u > 0$ and Urabe's one $1 + u - \sqrt{1 + 2u}$ for $u > -1/2$ (see [13]). Other examples can be given where the isochronicity property only holds locally, i.e. in a neighborhood of an equilibrium point. The results established below for the global case could be extended to the local case, as well. For briefness, we will not enter into such details.

In 2003, Bolotin and MacKay in [3], proved, quoting their paper, that

> *"a C^2-potential is isochronous if and only if its graph arises by horizontally shearing the graph of a parabola, by a shear which preserves monotonicity of the two sides"*

(see also [1, 2]). In this paper we consider systems of the type

$$\begin{cases} -y' = \dfrac{\partial}{\partial x} V(x) \\ x' = \dfrac{\partial}{\partial y} T(y)\,, \end{cases} \tag{5}$$

where T and V are C^2-functions. Our aim is to evaluate the periods of the periodic solutions in relation with the value of the energy function

$$E(x, y) = T(y) + V(x)\,, \tag{6}$$

which is preserved by the solutions. We also want to study how the kinetic energy function T can be modified in order to obtain a system having periodic solutions of the same period, that is an *isochronous system*. We will find, in particular, the estimates of the period of all the periodic orbits of system (5), when T is assumed to be an isochronous potential. We will show that there is a constant $\mathrm{C}_T > 0$, determined by T, for which all the periodic solutions to system (5) have period

$$\tau = \mathrm{C}_T \sqrt{2} \int_{x_1}^{x_2} \frac{dx}{\sqrt{E - V(x)}}\,,$$

where E is the constant energy of the orbit, and $x_1 < x_2$ are the extremals of the orbit. In the classical case $T(y) = \frac{1}{2}y^2$ we will consistently obtain $\mathrm{C}_T = 1$. (The above formula holds requiring that the minimum of the energy function (6) is zero; such a condition will be always assumed implicitly throughout this paper.) As a consequence we will obtain, in Theorem 2.5, that system (5) is isochronous when T and V are isochronous potentials. This result generalizes [15, Theorem 1.3], where these potentials are required to be analytical functions.

The paper is organized as follows: in Section 2 we will prove our main results. In Section 3 we provide an example of application: from the estimate of the period of the isochronous system related to the *Steen's equation*

$$x'' - \frac{c}{x^3} + \mu x = 0\,, \qquad \text{where } x > 0\,,$$

we prove that the system (all the constants are assumed to be positive)

$$\begin{cases} -y' = -\dfrac{c}{x^3} + \mu x \\ x' = -\dfrac{d}{y^3} + \nu y \\ x > 0\,, \quad y > 0\,, \end{cases}$$

which we denominate *Steen's planar system*, is an isochronous system.

2 Main results

Let us first introduce some notations and results referring to [3].

Definition 2.1 (Normal potential) *A real valued C^2-function $A = A(u)$, defined in (a, b) with $-\infty \le a < 0 < b \le +\infty$, is said to be a normal potential if it satisfies the following properties*

- $A(0) = \dfrac{\partial}{\partial u}A(0) = 0$, $\dfrac{\partial^2}{\partial u^2}A(0) = 2$,
- $\dfrac{\partial}{\partial u}A(u) > 0$ *when* $u > 0$ *and* $\dfrac{\partial}{\partial u}A(u) < 0$ *when* $u < 0$,
- $\lim\limits_{u\to a} A(u) = \lim\limits_{u\to b} A(u) = \ell \in (0, +\infty]$.

Given an isochronous potential, it is possible to obtain, by shifts of coordinates and scaling time, a normal potential. Let us consider a normal potential A. It is possible to find two *local inverse* functions $A_+^{-1} : [0, \ell\,) \to [0, b)$ and $A_-^{-1} : [0, \ell\,) \to (a, 0]$ such that $A(A_\pm^{-1}(\xi)) = \xi$ for every $\xi \in [0, \ell\,)$.

Defining

$$\Delta_A(\xi) = A_+^{-1}(\xi) - A_-^{-1}(\xi)\,, \tag{7}$$

Bolotin and MacKay proved in [3] the following proposition.

Proposition 2.2 *A normal potential is isochronous if and only if*

$$\Delta_A(\xi) = 2\sqrt{\xi}\,.$$

This result describes the property that an isochronous potential is obtained from a parabola by shearing, as quoted in the introduction (see also [1, 2]). There are normal isochronous potentials such that the interval (a,b) is bounded and $\ell < \infty$: for example Urabe's normal potential $A(u) = 2\left[1+u-\sqrt{1+2u}\right]$, defined in $(-1/2\,,\,3/2)$, with $\ell = 1$. See [3] for other examples.

We are now ready to state our main result for system (5).

Theorem 2.3 *Suppose that T is a normal isochronous potential. Then every non-trivial closed orbit of* (5) *corresponds to a periodic solution with period*

$$\tau = \int_{x_1}^{x_2} \frac{dx}{\sqrt{E_0 - V(x)}}$$

where E_0 is the energy of the solutions and $x_1 < x_2$ are its horizontal extremals, as in Figure 1.

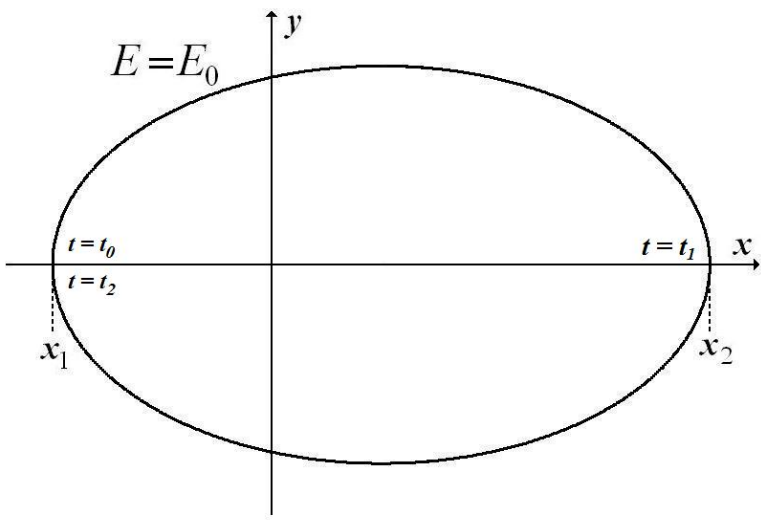

Figure 1: An example of the orbit of energy level $E = E_0$.

Proof. Consider a non-trivial closed orbit of energy E_0 as in the assumption of the theorem, with horizontal extremals $x_1 < x_2$. Then the orbit is the image of a solution $(x(t), y(t))$ of (5) which reaches the point $(x_1, 0)$ such that $V(x_1) = E_0$ at a certain time t_0. Denote with $t_1 > t_0$ the smallest time in which the solution reaches the point $(x_2, 0)$, and by $t_2 > t_1$ the smallest time in which the solution reaches again $(x_1, 0)$. See Figure 1. Hence, $\tau = t_2 - t_0$. By the second equation in (5) one has, using the previously introduced notation, and setting $\phi(x(t)) = E_0 - V(x(t))$,

$$x'(t) = \frac{\partial}{\partial y} T(y(t)) = \begin{cases} \frac{\partial}{\partial y} T\big(T_+^{-1}(\phi(x(t)))\big) & \text{if } y(t) > 0 \\ \frac{\partial}{\partial y} T\big(T_-^{-1}(\phi(x(t)))\big) & \text{if } y(t) < 0 \end{cases}$$
$$= \begin{cases} \big[D_\xi\big(T_+^{-1}\big)\big(\phi(x(t))\big)\big]^{-1} & \text{if } y(t) > 0 \\ \big[D_\xi\big(T_-^{-1}\big)\big(\phi(x(t))\big)\big]^{-1} & \text{if } y(t) < 0\,, \end{cases}$$

where $D_\xi(T_\pm^{-1})$ indicates the derivative of the function $T_\pm^{-1} = T_\pm^{-1}(\xi)$. Hence, we have the estimate of the period

$$\begin{aligned} \tau &= \int_{t_0}^{t_1} x'(t) D_\xi\big(T_+^{-1}\big)\big(\phi(x(t))\big) dt + \int_{t_1}^{t_2} x'(t) D_\xi\big(T_-^{-1}\big)\big(\phi(x(t))\big) dt \\ &= \int_{x_1}^{x_2} \big[D_\xi\big(T_+^{-1}\big)\big(\phi(x)\big) - D_\xi\big(T_-^{-1}\big)\big(\phi(x)\big)\big]\, dx \\ &= \int_{x_1}^{x_2} \big[D_\xi\big(T_+^{-1} - T_-^{-1}\big)\big(\phi(x)\big)\big]\, dx\,. \end{aligned}$$

Using the notation in (7), we have by Proposition 2.2,

$$T_+^{-1}(\xi) - T_-^{-1}(\xi) = \Delta_T(\xi) = 2\sqrt{\xi}\,,$$

with derivative (when $\xi > 0$)

$$D_\xi(T_+^{-1} - T_-^{-1})(\xi) = \frac{1}{\sqrt{\xi}}\,,$$

thus giving us

$$\tau = \int_{x_1}^{x_2} \frac{dx}{\sqrt{\phi(x)}} = \int_{x_1}^{x_2} \frac{dx}{\sqrt{E_0 - V(x)}}\,.$$

■

Let us consider now a system of the type (5) where the function T is an isochronous potential which is not normal. Nevertheless T has a unique non-degenerate minimum $T_0 = T(y_0)$. We assume, without loss of generality, $T_0 = 0$. Then, with the change of variable $w = y - y_0$, scaling the time variable with $\tilde{t} = kt$ and indicating with † the derivative with respect $\tilde{t}$, we obtain

$$\begin{cases} -w^\dagger = \dfrac{\partial}{\partial x}\widetilde{V}(x) \\ x^\dagger = \dfrac{\partial}{\partial w}\widetilde{T}(w)\,. \end{cases} \tag{8}$$

where

$$\widetilde{V}(x) = \frac{1}{k}V(x) \qquad \text{and} \qquad \widetilde{T}(w) = \frac{1}{k}T(y_0 + w)\,,$$

with

$$k = \frac{1}{2}\frac{\partial^2}{\partial y^2}T(y_0)\,.$$

With such a procedure we have a new potential $\widetilde{T}$ which is normal isochronous, thus permitting us to apply Theorem 2.3, obtaining the estimate of the period $\widetilde{\tau}$ of a periodic orbit of system (8), where the associated energy function is

$$\widetilde{E}(x, w) = \widetilde{T}(w) + \widetilde{V}(x) = \frac{1}{k}E(x, y_0 + w)\,.$$

Hence, the period τ of periodic orbits of system (5) is

$$\begin{aligned} \tau = \frac{1}{k}\,\widetilde{\tau} = \frac{1}{k}\int_{x_1}^{x_2} \frac{dx}{\sqrt{\widetilde{E}_0 - \widetilde{V}(x)}} \\ = \frac{1}{\sqrt{k}}\int_{x_1}^{x_2} \frac{dx}{\sqrt{E_0 - V(x)}}\,. \end{aligned}$$

We have just proved the following corollary of Theorem 2.3.

Corollary 2.4 *Suppose that T is an isochronous potential with minimum in y_0. Then every non-trivial closed orbit of system* (5) *corresponds to a periodic solution with period*

$$\tau = \mathrm{C}_T\,\sqrt{2}\int_{x_1}^{x_2} \frac{dx}{\sqrt{E_0 - V(x)}}\,,$$

where E_0 is the energy of the solution, $x_1 < x_2$ are its horizontal extremals (see Figure 2), and

$$\mathrm{C}_T = \left[\frac{\partial^2}{\partial y^2}T(y_0)\right]^{-1/2}\,.$$

In the next result we focus our attention on systems where both the functions T and V are isochronous potentials. We thus obtain a generalization of [15, Theorem 1.3], where the potentials were required to be analytic.

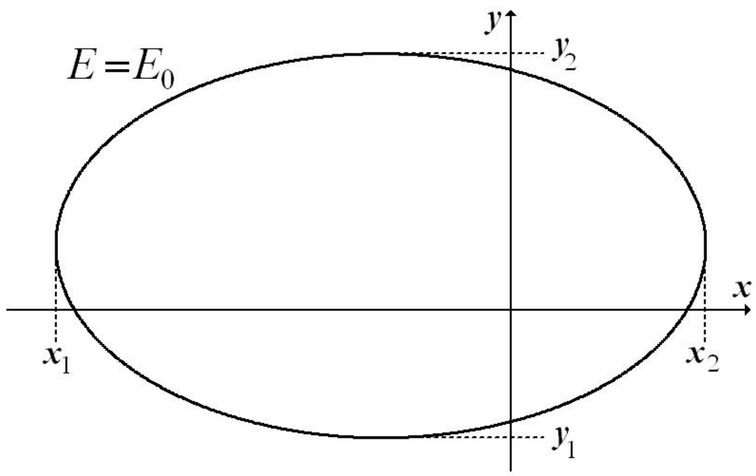

Figure 2: An example of the orbit of energy level $E = E_0$, when the potentials T and V are not normal potentials.

Theorem 2.5 *Suppose that T is an isochronous potential. Then system* (5) *is isochronous if and only if V is an isochronous potential. In such a case, the period is*

$$\tau = 2\pi \, \mathrm{C}_T \, \mathrm{C}_V \,,$$

with

$$\mathrm{C}_T = \left[\frac{\partial^2}{\partial y^2} T(y_0)\right]^{-1/2} \qquad \textit{and} \qquad \mathrm{C}_V = \left[\frac{\partial^2}{\partial x^2} V(x_0)\right]^{-1/2} ,$$

where y_0 and x_0 are the minimum points of T and V, respectively.

Proof. Applying Corollary 2.4, we obtain

$$\tau = \mathrm{C}_T \, \tau_0 \,,$$

with τ_0 as in (3). On the other hand, it is well-known that the period of all the periodic solutions of equation (2) is constant if and only if V is an isochronous potential, having

$$\tau_0 = 2\pi \left[\frac{\partial^2}{\partial x^2} V(x_0)\right]^{-1/2} = 2\pi \, \mathrm{C}_V \,.$$

Therefore,

$$\tau = \mathrm{C}_T \, \tau_0 = 2\pi \, \mathrm{C}_T \, \mathrm{C}_V \,.$$

■

The previous theorem leaves open the problem of characterizing isochronous systems of the type (5), when neither T nor V are isochronous potentials.

Let us finally state the following immediate consequence of Theorem 2.5, in the spirit of Bolotin and MacKay's paper [3].

Corollary 2.6 *If T and V are C^2-potentials obtained by horizontally shearing the graph of a parabola, by a shear which preserves monotonicity of the two sides, then system* (5) *is isochronous.*

3 An example of application

In [3], Bolotin and MacKay present a method to create different isochronous potentials starting from suitable C^2-functions (see also [8]). In this section we focus our attention to systems where one or both the potentials are anharmonic.

Consider the *Steen's equation*

$$x'' - \frac{c}{x^3} + \mu x = 0\,, \qquad \text{where } x > 0\,, \tag{9}$$

with $\mu > 0$ and $c > 0$, and the associated planar system of the form (5), with

$$V(x) = \frac{1}{2}\left(cx^{-2} + \mu x^2\right) - \sqrt{\mu c} \qquad \text{and} \qquad T(y) = \frac{1}{2}y^2\,.$$

It is well-known that V is an isochronous potential (see e.g. [11], and [6, 7, 10, 12] for related problems), having minimum at $x_0 = (c/\mu)^{1/4}$ with $\frac{\partial^2}{\partial x^2}V(x_0) = 4\mu$. Hence, the period of all the periodic solutions is $\tau = \pi/\sqrt{\mu}$, independently of the choice of c. A natural generalization of Steen's equation (9), is what we call *Steen's planar system*

$$\begin{cases} -y' = -\dfrac{c}{x^3} + \mu x \\ x' = -\dfrac{d}{y^3} + \nu y \\ x > 0\,, \quad y > 0\,, \end{cases} \tag{10}$$

where again all the constants are assumed to be positive. The associated potentials

$$V(x) = \frac{1}{2}\left(cx^{-2} + \mu x^2\right) - \sqrt{\mu c} \qquad \text{and} \qquad T(y) = \frac{1}{2}\left(dy^{-2} + \nu y^2\right) - \sqrt{\nu d}$$

are isochronous. Hence, applying Theorem 2.5, we obtain the next result.

Proposition 3.1 *The Steen's planar system* (10) *is an isochronous system of period*

$$\tau = \frac{\pi}{2\sqrt{\mu\nu}}\,,$$

independently of the constants c and d.

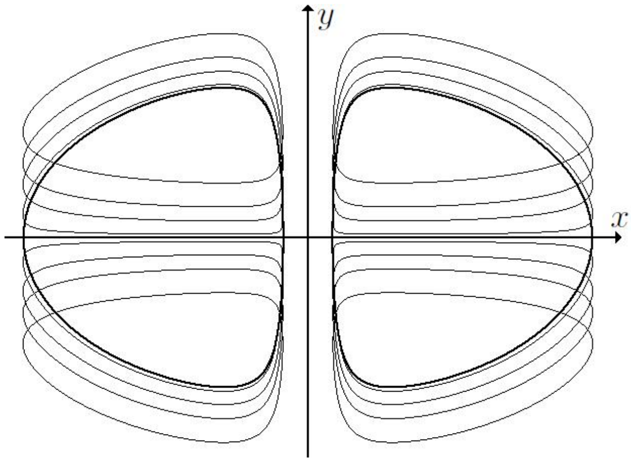

Figure 3: Level curves of value $\widehat{E}_2 = 2$, with $c = 1/2$, $\mu = 1/3$, and $d \in \{4, 1, 0.25, 0.01\}$. When d tends to zero the branches of level curves go to glue together forming the level curve $\widehat{E}_1 = 2$, with $c = 1/2$, $\mu = 1/3$.

In what follows we want to briefly study the relations between the Steen's equation (9), the Steen's planar system (10), assuming for simplicity $\nu = 1$, and the second order differential equation $x'' + \mu x = 0$. These are three examples of isochronous systems with periods, respectively, $\widehat{\tau}_2 = \pi/\sqrt{4\mu}$, $\widehat{\tau}_1 = \pi/\sqrt{\mu}$ and $\widehat{\tau}_0 = 2\pi/\sqrt{\mu}$. Let us consider the respective energy functions

$$
\begin{aligned}
\widehat{E}_2 &= \frac{1}{2}\left(dy^{-2} + y^2\right) + \frac{1}{2}\left(cx^{-2} + \mu x^2\right) - \sqrt{d} - \sqrt{\mu c}\,, \\
\widehat{E}_1 &= \frac{1}{2}y^2 + \frac{1}{2}\left(cx^{-2} + \mu x^2\right) - \sqrt{\mu c}\,, \\
\widehat{E}_0 &= \frac{1}{2}y^2 + \frac{\mu}{2}x^2\,,
\end{aligned}
$$

as defined on their full domain. Figure 3 shows how the branches of level curves of $\widehat{E}_2$ tend to the x-axis when the constant d goes to zero, and seem to glue together forming the branches of level curves of $\widehat{E}_1$. Similarly, in Figure 4, the branches of the level curves of $\widehat{E}_2$ tends to the axes when both the constants c and d go to zero, and seem to glue together forming the level curves of $\widehat{E}_0$. It has been proved (cf. [7, 12]) that the speed of a solution to (9), or resp. to (10), increase to infinity when it approaches the y-axis, or resp. one of the axes. Hence, the largest part of the period of a solution is spent when it is far from the y-axis, or resp. the axes. Consistently, we have $\widehat{\tau}_0 = 4\widehat{\tau}_2$ and $\widehat{\tau}_1 = 2\widehat{\tau}_2$.

Acknowledgement

I express my gratitude to Alessandro Fonda for having presented me the paper by Bolotin and MacKay, and for useful discussions.

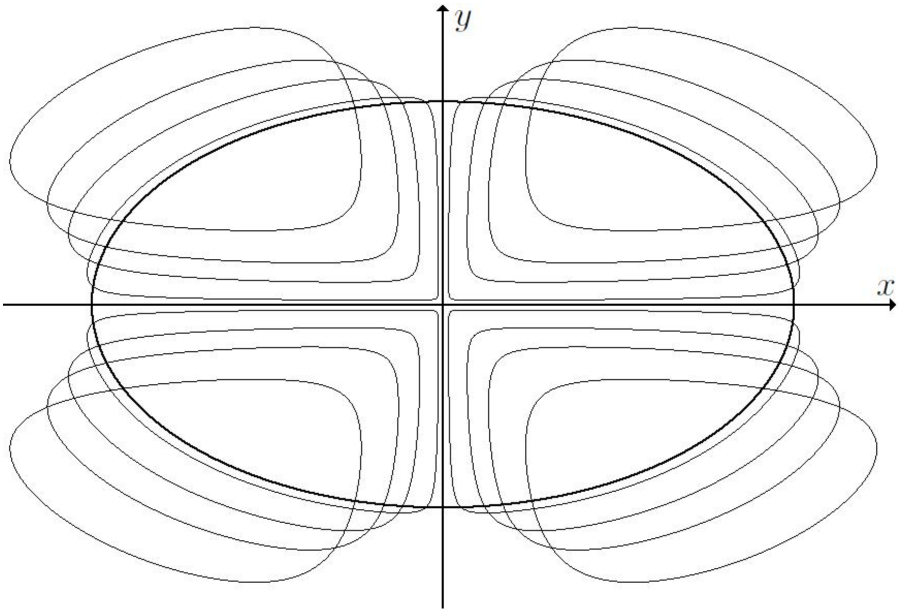

Figure 4: Level curves of value $\widehat{E}_2 = 2$ with $\mu = 1/3$, and $c = d \in \{4, 1, 0.25, 0.01\}$. When both c and d tends to zero the branches of level curves go to glue together forming the level curve $\widehat{E}_0 = 2$, with $\mu = 1/3$.

References

[1] C. Antón and J.L. Brun, *Isochronous oscillations: potentials derived from a parabola by shearing*, Amer. J. Phys. **76** (2008), 537–540.

[2] D. Bonheure, C. Fabry and D. Smets, *Periodic solutions of forced isochronous oscillators at resonance*, Discrete Contin. Dyn. Syst. **8** (2002), 907–930.

[3] S. Bolotin and R.S. MacKay, *Isochronous potentials*, in *Proceedings of the third conference: Localization and Energy Transfer in Nonlinear Systems*, World Scientific, London 2003, pp. 217–224.

[4] J. Chavarriga and M. Sabatini, *A survey of isochronous centers*, Qual. Theory Dyn. Syst. **1** (1999), 1–70.

[5] A. Cima, F. Mañonas and J. Villadelprat, *Isochronicity for several classes of Hamiltonian systems*, J. Differential Equations **157** (1999), 373–413.

[6] A. Fonda and M. Garrione, *A Landesman-Lazer type condition for asymptotically linear second order equations with a singulrity*, Proc. Roy. Soc. Edinburgh Sect. A **142** (2012), 1263–1277.

[7] A. Fonda and R. Toader, *Periodic orbits of radially symmetric systems with a singularity: the repulsive case*, Adv. Nonlinear Stud. **11** (2011), 853–874.

[8] G. Gorni and G. Zampieri, *Global isochronous potentials*, preprint.

[9] R. Ortega, *Periodic perturbations of an isochronous center*, Qual. Theory Dyn. Syst. **3** (2002), 83–91.

[10] E. Pinney, *The nonlinear differential equation* $y'' + p(x)y + cy^{-3} = 0$, Proc. Amer. Math. Soc **1** (1950), 681.

[11] R. Redheffer, *Steen's equation and its generalizations*, Aequationes Math. **58** (1999), 60–72.

[12] A. Sfecci, *Positive periodic solutions for planar differential systems with repulsive singularities on the axes*, to appear.

[13] M. Urabe, *Potential forces which yield periodic motions of a fixed period*, J. Math. Mech. **10** (1961), 569–578.

[14] M. Urabe, *The potential force yielding a periodic motion whose period is an arbitrary continuously differentiable function of the amplitude*, J. Sci. Hiroshima Univ. **26** (1962), 93–109.

[15] L. Yang, *Isochronous centers and isochronous functions*, Acta Math. Appl. Sin. Engl. Ser. **18** (2002), 315-324.

Author's address:

Andrea Sfecci
Università degli Studi di Trieste
Dipartimento di Matematica e Geoscienze
Via Valerio 12/1, I-34127 Trieste, Italy
e-mail: asfecci@units.it