Positive periodic solutions for planar differential systems with repulsive singularities on the axes

Andrea Sfecci

Abstract

In this paper we provide a result of existence of periodic solutions to planar systems having a repulsive singularity on both the axes. We require a nonresonance assumption, related to the spectrum of the periodic problem associated to the *Steen's planar system*

$$\begin{cases} -y' = -\dfrac{c}{x^3} + \mu x \\ x' = -\dfrac{d}{y^3} + \nu y \\ x > 0\,, \quad y > 0\,, \end{cases}$$

which is isochronous (see [12]). As a particular case we obtain the existence of periodic solutions to perturbed Steen's planar system.

1 Introduction and main result

The Steen's equation

$$x'' - \frac{1}{x^3} + \beta x = 0\,,$$

(cf. [10, 11]), has an isochronous center of period $\pi/\sqrt{\beta}$ (see [1]). Such a property leads to the study of the resonance phenomenon to the perturbed equation

$$x'' - \frac{c}{x^3} + \mu x = e(t)\,,$$

where e is a continuous periodic function and μ and c are positive constants. The existence of periodic solutions have been studied under nonresonance and resonance hypotheses (see e.g. [2, 7]), while in [9] the possibility of unbounded solutions is treated.

It has been recently proved by the author (see [12]), that the *Steen's planar system*

$$\begin{cases} -y' = -\dfrac{c}{x^3} + \mu x \\ x' = -\dfrac{d}{y^3} + \nu y \\ x > 0\,, \quad y > 0 \end{cases}$$

(all the constants are assumed to be positive) is isochronous of period

$$T_S = \frac{\pi}{2\sqrt{\mu\nu}}\,.$$

Hence, as follows from [9], one has the existence of unbounded solutions to the perturbed system

$$\begin{cases} -y' = -\dfrac{c}{x^3} + \mu x + e(t) \\ x' = -\dfrac{d}{y^3} + \nu y + f(t) \\ x > 0\,, \quad y > 0\,, \end{cases}$$

when e and f are suitable T_S-periodic continuous functions. We expect, instead, to obtain the existence of periodic solutions assuming these function to be T-periodic with $T \neq T_S$.

In this paper we consider a general system of the type

$$\begin{cases} -y' = f_1(t,x,y) \\ x' = f_2(t,x,y) \\ x > 0\,, \quad y > 0\,, \end{cases} \tag{1}$$

where $f_1 : \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+_0 \to \mathbb{R}$ and $f_2 : \mathbb{R} \times \mathbb{R}^+_0 \times \mathbb{R}^+ \to \mathbb{R}$ are two continuous functions which are T-periodic in the first variable, having a repulsive singularity respectively in the second and third variable. We use the notation: $\mathbb{R}^+ = (0, +\infty)$ and $\mathbb{R}^+_0 = [0, +\infty)$. As a particular example we could have

$$\begin{cases} -y' = -\dfrac{c}{x^{\alpha_1}} + \mu x + \beta y + e(t) \\ x' = -\dfrac{d}{y^{\alpha_2}} + \nu y + \beta x + f(t) \\ x > 0\,, \quad y > 0\,, \end{cases}$$

where e and f are continuous functions, $\alpha_1, \alpha_2 > 1$, the constants c, d, μ, ν are positive and β of arbitrary sign.

For simplicity, we assume that, for any given initial datum, the Cauchy problems associated to system (1) have a unique solution. This condition is not restrictive, since the general case can be obtained by a limit procedure using some approximations of the functions f_1 and f_2, by functions which are locally-Lipschitz continuous in the second and the third variables.

Let us introduce the following assumptions on functions f_1 and f_2: the first one provides the repulsivity conditions for the singularities, the second introduces an asymptotically linear growth condition at infinity.

Hypothesis 1.1 (Condition at zero) *Suppose that there exist $\delta > 0$ and two functions $\hat{g}_1, \hat{g}_2 : (0, \delta) \to \mathbb{R}$ such that*

$$f_1(t,x,y) < \hat{g}_1(x) < 0, \qquad \text{for every } t \in \mathbb{R},\ x \in (0,\delta) \text{ and } y \in \mathbb{R}_0^+ ,$$

$$f_2(t,x,y) < \hat{g}_2(y) < 0, \qquad \text{for every } t \in \mathbb{R},\ x \in \mathbb{R}_0^+ \text{ and } y \in (0,\delta) .$$

Moreover we require that

$$\lim_{\xi \to 0^+} \hat{g}_i(\xi) = -\infty \qquad \text{and} \qquad \int_0^\delta \hat{g}_i(\xi)\, d\xi = -\infty , \qquad i = 1, 2 .$$

Hypothesis 1.2 (Condition at infinity) *Suppose that there exist some positive constants R, $\check{C}$, $\check{D}$, $\hat{C}$, $\hat{D}$, $\check{\mu} < \hat{\mu}$, $\check{\nu} < \hat{\nu}$, and other two constants $\check{\beta} < \hat{\beta}$ of arbitrary sign, such that*

$$\check{\mu}x + \check{\beta}y - \check{C} < f_1(t,x,y) < \hat{\mu}x + \hat{\beta}y + \hat{C} , \tag{2}$$

for every $t \in \mathbb{R}$, and (x, y) satisfying both $x \geq \delta$ and $x^2 + y^2 > R^2$, and also such that

$$\check{\beta}x + \check{\nu}y - \check{D} < f_2(t,x,y) < \hat{\beta}x + \hat{\nu}y + \hat{D} , \tag{3}$$

for every $t \in \mathbb{R}$ and (x, y) satisfying both $y \geq \delta$ and $x^2 + y^2 > R^2$. Moreover we assume that

$$\check{\beta}^2 < \check{\mu}\check{\nu} \qquad \text{and} \qquad \hat{\beta}^2 < \hat{\mu}\hat{\nu} .$$

Using, for $mn > b^2$, the notation

$$\tau(m,n,b) = \frac{1}{\sqrt{mn - b^2}} \left[\frac{\pi}{2} - \arctan\left(\frac{b}{\sqrt{mn - b^2}} \right) \right] ,$$

we can state our main result.

Theorem 1.3 *Suppose that Hypotheses 1.1 and 1.2 are satisfied. Assume that*

$$\frac{T}{N+1} < \tau(\hat{\mu}, \hat{\nu}, \hat{\beta}) < \tau(\check{\mu}, \check{\nu}, \check{\beta}) < \frac{T}{N} , \tag{4}$$

for a suitable positive integer N. Then, there exists at least one T-periodic solution to system (1).

This result follows as an application of the degree theory. In particular we use a method which has been introduced by A. Fonda and the author in [4] in the study of planar systems. In that paper a *guiding curve*, having the shape of a spiral, is built in order to control the behavior of those solution which are large in norm. If an estimate of the time needed by such solutions to perform a complete rotation in the plane is given, then it is possible to find a fixed point of the Poincaré map associated to the system. In the next section we provide the proof of our main theorem showing how to adapt this technique to our situation.

2 Proof

In this section we provide the proof of Theorem 1.3. The next claim will simplify it.

Claim 2.1 *In Hypotheses 1.1, it is not restrictive to assume $\delta = 1$.*

Proof. Using the change of variable $x = \delta X$, $y = \delta Y$ one obtains the system

$$\begin{cases} -Y' = \phi_1(t, X, Y) \\ X' = \phi_2(t, X, Y) \\ X > 0\,, \quad Y > 0\,, \end{cases} \tag{5}$$

where the functions $\phi_i(t, X, Y) = \frac{1}{\delta} f_i(t, \delta X, \delta Y)$ $(i = 1, 2)$ satisfy Hypotheses 1.1 and 1.2 using the functions $\hat{g}_i^{new}(\zeta) = \frac{1}{\delta}\hat{g}_i(\delta\zeta)$, introducing $R^{new} = R/\delta$, and similarly modifying the values $\check{C}$, $\hat{C}$, $\check{D}$, $\hat{D}$. We emphasize that the other constants, also appearing in (4), do not change. ∎

By the claim, in what follows, we assume $\delta = 1$. Define $\mathcal{Q}_1 = \mathbb{R}^+ \times \mathbb{R}^+$. Following an idea introduced in [4], we will prove this theorem constructing a *guiding curve* $\gamma : [0, +\infty) \to \mathcal{Q}_1$ having the shape of a *spiral*, rotating clockwise around the point $(1, 1)$, and approaching to the boundary of $\mathcal{Q}_1$ in the following sense: once defined the function $\mathcal{R} : \mathcal{Q}_1 \to \mathbb{R}$ such that

$$\mathcal{R}(x, y) = \frac{1}{x^2} + \frac{1}{y^2} + x^2 + y^2\,,$$

one has

$$\lim_{s\to+\infty} \mathcal{R}(\gamma(s)) = +\infty\,. \tag{6}$$

Once such a curve will be constructed, we will consider its parametrization in the following polar coordinates

$$\gamma(s) = (1 + r(s)\cos s, 1 - r(s)\sin s) \quad \text{with}\, r(s) = |\gamma(s)| \text{ and } s \in [0, +\infty)\,.$$

We denote with $|\cdot|$ the euclidian norm in the plane. The choice of $(1+\rho(0), 1)$ as starting point is not restrictive. It will be useful to consider the bounded region delimited by the k-th lap of the spiral. More precisely, we define

$$\Gamma_k = \big\{\gamma(s) \,:\, s \in [2\pi k, 2\pi(k + 1)]\big\} \cup \big[\gamma(2\pi k), \gamma(2\pi(k + 1))\big]\,, \tag{7}$$

where $[E_1, E_2]$ denotes the closed segment of extremals E_1 and E_2. Let Ω_k be the open bounded region delimited by Γ_k. The spiral γ will be constructed in order to have that every solution $(x(t), y(t))$ to system (1), crossing the curve at a certain point of Γ_k, must enter in the region Ω_k. In this way a solution approaching to the boundary of $\mathcal{Q}_1$ - such that $\mathcal{R}(x(t), y(t))$ is increasing to infinity - must rotate around the point $(1, 1)$ guided by the spiral γ. Figure 1 clarifies this situation.

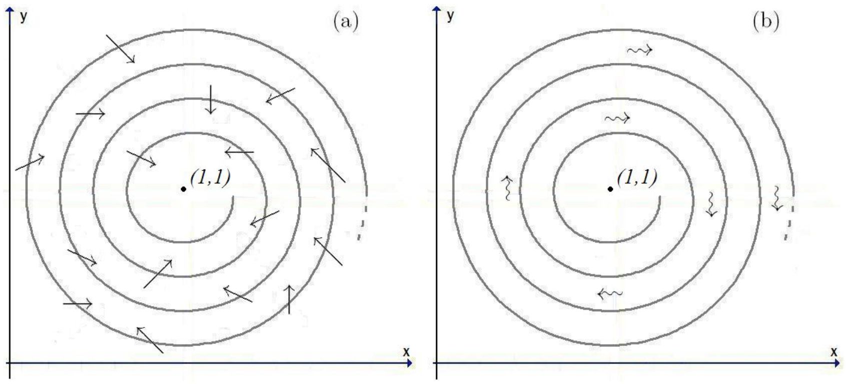

Figure 1: The property of the spiral γ: all the solutions cross the curve only from the outer part to the inner part, as in (a). So, any solution approaching to the boundary of $\mathcal{Q}_1$ must rotate clockwise infinitely many times, as in (b).

The proof of Theorem 1.3 will arise by the application of the Poincaré-Bohl Theorem to the Poincaré map associated to system (1) (it will be shown that this map is well defined). In order to apply it we need some estimates on the time spent by a solution to perform a complete rotation around the origin. To this aim, we will consider sometimes the next parametrization for the solution of system (1)

$$\begin{cases} x(t) = 1 + \rho(t)\cos(\vartheta(t)) \\ y(t) = 1 + \rho(t)\sin(\vartheta(t))\,. \end{cases} \tag{8}$$

Thus, the angular velocity of a solution with respect the point $(1,1)$ is expressed by the following formula,

$$\begin{aligned} -\vartheta'(t) &= \frac{x'(t)(y(t)-1) - y'(t)(x(t)-1)}{(x(t)-1)^2 + (y(t)-1)^2} \\ &= \frac{f_2(t,x,y)(y(t)-1) + f_1(t,x,y)(x(t)-1)}{(x(t)-1)^2 + (y(t)-1)^2}\,, \end{aligned} \tag{9}$$

and the radial velocity by

$$\begin{aligned} \rho'(t) &= \frac{x'(t)(x(t)-1) + y'(t)(y(t)-1)}{[(x(t)-1)^2 + (y(t)-1)^2]^{1/2}} \\ &= \frac{f_2(t,x,y)(x(t)-1) - f_1(t,x,y)(y(t)-1)}{[(x(t)-1)^2 + (y(t)-1)^2]^{1/2}}\,. \end{aligned} \tag{10}$$

Before explaining how to get the spiral, we need to introduce some additional tools. In this way, consider the following subsets of $\mathcal{Q}_1$ (see Figure 2).

$$\begin{aligned}
&\mathcal{A}_1 = [R, +\infty) \times (0, 1]\,,\\
&\mathcal{A}_2 = [1, R] \times (0, 1]\,,\\
&\mathcal{A}_3 = (0, 1] \times (0, 1]\,,\\
&\mathcal{A}_4 = (0, 1] \times [1, R]\,,\\
&\mathcal{A}_5 = (0, 1] \times [R, +\infty)\,,\\
&\mathcal{A}_6 = \{(x, y) \,:\, x^2 + y^2 \geq R^2 \text{ and } x \geq 1,\, y \geq 1\}\,.
\end{aligned}$$

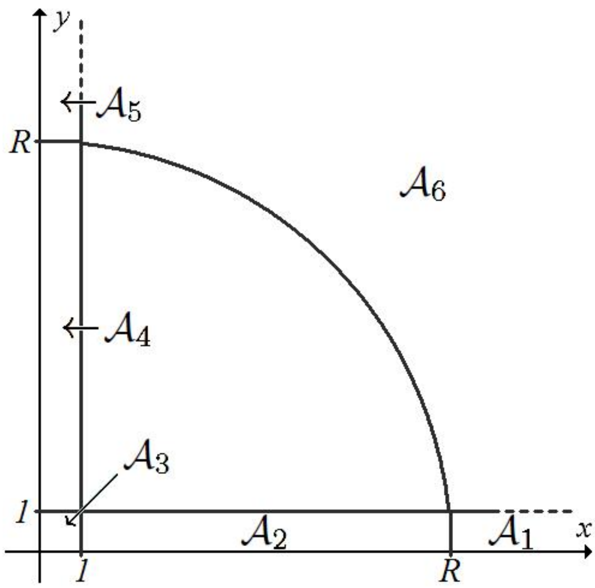

Figure 2: The six regions $\mathcal{A}_1, \ldots, \mathcal{A}_6$.

It is not restrictive, under Hypothesis 1.2, to assume that R is large enough to have

$$f_1(t, x, y) > 0 \qquad \text{in}\, \mathbb{R} \times \mathcal{A}_1\,, \tag{11}$$

$$f_2(t, x, y) > 0 \qquad \text{in}\, \mathbb{R} \times \mathcal{A}_5\,. \tag{12}$$

Moreover it is possible to find two functions $\check{g}_1$ and $\check{g}_2$ such that

$$\check{g}_1(x) < f_1(t, x, y) < \hat{g}_1(x) < 0, \text{ for every } t \in \mathbb{R},\, x \in (0, 1)\,, y \in \mathbb{R}_0^+\,, \tag{13}$$

$$\check{g}_2(y) < f_2(t, x, y) < \hat{g}_2(y) < 0, \text{ for every } t \in \mathbb{R},\, x \in \mathbb{R}_0^+\,, y \in (0, 1)\,. \tag{14}$$

Clearly, also these functions have the following properties

$$\lim_{\xi \to 0^+} \check{g}_i(\xi) = -\infty \qquad \text{and} \qquad \int_0^1 \check{g}_i(\xi)\, d\xi = -\infty\,, \qquad i = 1, 2\,.$$

We define the primitive of the functions $\check{g}_i$ and $\hat{g}_i$ as

$$\check{G}_i(\xi) = \int_1^\xi \check{g}_i(\sigma)\, d\sigma \qquad \text{and} \qquad \hat{G}_i(\xi) = \int_1^\xi \hat{g}_i(\sigma)\, d\sigma\,, \qquad i = 1, 2\,.$$

It will be useful to consider the following limitations on the functions f_i, for a suitable positive constant M,

$$|f_1(t, x, y)| < M \quad \text{in}\, \mathbb{R} \times \mathcal{A}_2\,, \tag{15}$$

$$|f_2(t, x, y)| < M \quad \text{in}\, \mathbb{R} \times \mathcal{A}_4\,. \tag{16}$$

We are now ready to start the construction of the spiral γ gluing together different branches of some *energy functions* Φ_j, defined respectively in the region $\mathcal{A}_j$, having the property that every solution $u(t) = \big(x(t), y(t)\big)$ to system (1) has decreasing energy Φ_j when $u(t) \in \mathcal{A}_j$. Moreover the level curves of Φ_j approach to the boundary of $\mathcal{Q}_1$ in the following sense

$$\lim_{c \to +\infty} \inf\{\mathcal{R}(x, y) \,:\, (x, y) \in A_j \text{ and } \Phi_j(x, y) = c\} = +\infty\,,$$

for every index j. We will show in details how to construct the first lap of the spiral γ, the other ones can be obtained iterating the procedure.

The energy functions are:

$$\begin{aligned}
\Phi_1(x, y) &= \hat{G}_2(y) + \frac{1}{2}\hat{\mu}x^2 + q_1 x\,,\\
\Phi_2(x, y) &= \hat{G}_2(y)\, x\,,\\
\Phi_3(x, y) &= \hat{G}_1(x) + \check{G}_2(y)\,,\\
\Phi_4(x, y) &= \hat{G}_1(x) - My\,,\\
\Phi_5(x, y) &= \check{G}_1(x) + \frac{1}{2}\check{\nu}y^2 - q_2 y\,,
\end{aligned}$$

where $q_1 = \hat{C} + \max\{0, \hat{\beta}\}$ and $q_2 = \check{D} + \max\{0, -\check{\beta}\}$. The spiral in $\mathcal{A}_6$ will be found using a different technique.

Let us here verify that if $\big(x(t), y(t)\big)$ is a solution of system (1), then, for $j \in \{1, \ldots, 5\}$,

$$\frac{d}{dt}\Phi_j\big(x(t), y(t)\big) < 0 \quad \text{when } \big(x(t), y(t)\big) \in \mathcal{A}_j\,. \tag{17}$$

For every $\big(x(t), y(t)\big) \in \mathcal{A}_1$

$$\begin{aligned}
\frac{d}{dt}\Phi_1\big(x(t), y(t)\big) &= -\hat{g}_2(y) f_1(t, x, y) + f_2(t, x, y)\big(\hat{\mu}x + q_1\big)\\
&= f_1(t, x, y)\big(f_2(t, x, y) - \hat{g}_2(y)\big)\\
&\qquad + f_2(t, x, y)\big(\hat{\mu}x + q_1 - f_1(t, x, y)\big) < 0\,,
\end{aligned}$$

by (2), (11) and (14). For every $\big(x(t),y(t)\big) \in \mathcal{A}_3$,

$$\frac{d}{dt}\Phi_3\big(x(t),y(t)\big) = \hat{g}_1(x) f_2(t,x,y) - \check{g}_2(y) f_1(t,x,y)$$
$$= \hat{g}_1(x)\big(f_2(t,x,y) - \check{g}_2(y)\big) + \check{g}_2(y)\big(\hat{g}_1(x) - f_1(t,x,y)\big) < 0\,,$$

by (13) and (14). For every $\big(x(t),y(t)\big) \in \mathcal{A}_4$,

$$\frac{d}{dt}\Phi_4\big(x(t),y(t)\big) = \hat{g}_1(x) f_2(t,x,y) + M f_1(t,x,y)$$
$$< \hat{g}_1(x)\big(f_2(t,x,y) + M\big) < 0\,,$$

by (13) and (16). For every $\big(x(t),y(t)\big) \in \mathcal{A}_5$,

$$\begin{aligned}\frac{d}{dt}\Phi_5\big(x(t),y(t)\big) &= \check{g}_1(x) f_2(t,x,y) - \big(\check{\nu} y - q_2\big) f_1(t,x,y)\\ &= f_2(t,x,y)\big(\check{g}_1(x) - f_1(t,x,y)\big)\\ &\qquad + f_1(t,x,y)\big(f_2(t,x,y) - \check{\nu} y + q_2\big) < 0\,,\end{aligned}$$

by (3), (12) and (13).

Property (17) for the energy function Φ_2 holds only when y is small enough. By the properties of $\hat{G}_2$, it is possible to find a value $y_c < 1$ such that $\hat{G}_2(y) > R\,M$ for every $y \le y_c$. The following estimates, obtained using (14) and (15),

$$\begin{aligned}\frac{d}{dt}\Phi_2\big(x(t),y(t)\big) &= -\hat{g}_2(y)\, x\, f_1(t,x,y) + \hat{G}_2(y)\, f_2(t,x,y)\\ &< -\hat{g}_2(y)\, R\,M + \hat{G}_2(y)\,\hat{g}_2(y)\\ &= \hat{g}_2(y)\big(\hat{G}_2(y) - R\,M\big) < 0\,,\end{aligned}$$

holds in $\mathcal{A}_2$ requiring $y(t) \le y_c$.

Let us now show how the first lap of the spiral γ is built (see Figure 3). We fix the starting point of the spiral as $P_0 = (x_0, 1)$, where the choice of $x_0 > R$ will be clarified soon. The first branch of γ (contained in $\mathcal{A}_1$) consists of the level curve $\Phi_1(x,y) = \Phi_1(x_0,1)$ and link P_0 to a point $P_1 = (R, y_1)$ with $y_1 < 1$. We must choose x_0 large enough to have that y_1 is less than the previously introduced value y_c. Call $e_0 = \Phi_2(R, y_1)$. It is easy to see that every point $(x,y) \in \mathcal{A}_2$ such that $\Phi_2(x,y) = e_0$ is such that $\frac{d}{dt}\Phi_2\big(x(t),y(t)\big) < 0$. So, the second branch of the spiral γ can be chosen as a subset of the e_0 level curve of function Φ_2, and links P_1 to a point $P_2 = (1, y_2)$, where $y_2 < y_1 < 1$. Notice that this branch is fully contained in $\mathcal{A}_2$. With a similar procedure it is possible to take the branch of the level curve of Φ_3 linking the point P_2 to point $P_3 = (x_3, 1)$ with $0 < x_3 < 1$. Then, using Φ_4 and Φ_5 we can find $P_4 = (x_4, R)$ with $0 < x_4 < x_3$ and $P_5 = (1, y_5)$ with $y_5 > R$.

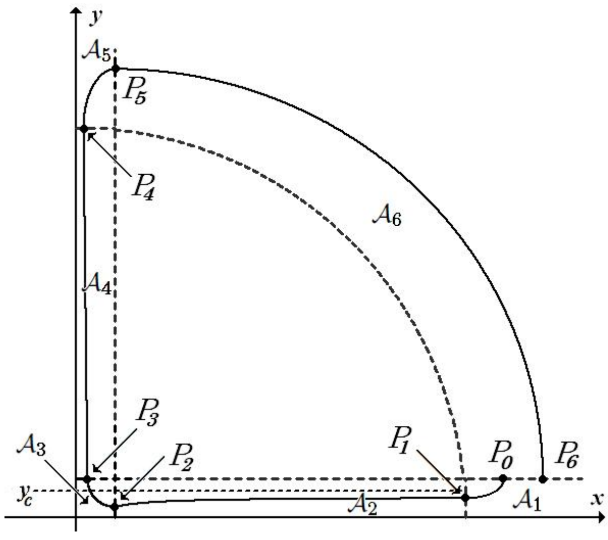

Figure 3: The construction of the first lap of the spiral

The branch of γ contained in $\mathcal{A}_6$ can be constructed adapting [5, Lemma 4.2] to our situation (see also [4, Proposition 2.5]). Let us briefly explain how this lemma works in this case. Consider a solution $\big(x(t), y(t)\big)$ of system (1) and write it in polar coordinates with respect to the point $(1, 1)$ as in (8). By Hypothesis 1.2, we have the next estimates for the angular velocity $-\vartheta'(t)$ and the radial velocity $\rho'(t)$, for some suitable constants:

$$-\vartheta'(t) \geq \omega_0 \,, \quad \rho'(t) \leq \alpha_0 \, \rho(t) \,, \quad \text{when } \big(x(t), y(t)\big) \in \mathcal{A}_6 \,,$$

enlarging, if necessary, the value R. So, when the orbit of a solution is contained in the region $\mathcal{A}_6$, we can express the radial coordinate in terms of the angular one, thus obtaining

$$\frac{d\rho}{d\vartheta} \leq \frac{\alpha_0}{\omega_0} \, \rho \,.$$

Hence, we can define the branch of the curve γ as

$$r(\theta)(\sin\theta, \cos\theta) + (1, 1) \,, \qquad \begin{array}{l} \text{with } r(\theta) = (y_5 - 1)\exp(2\alpha_0\,\theta/\omega_0) \\ \text{and } \theta \in [0, \pi/2] \,, \end{array}$$

so that, any solution to (1) satisfies the required control property. Call $P_6 = (x_6, y_6) = (r(\pi/2) + 1, 1)$, where $x_6 > R$.

We expect, by the construction, that $x_6 > x_0$. The case $x_6 \leq x_0$, gives us immediately the existence of a T-periodic solution to (1) by the application of the Brouwer Fixed Point Theorem, being $y'(t) < 0$ when $x \geq R$. Hence, it is not restrictive to assume that iterating this procedure we can obtain a *spiral* γ which is injective and make an infinite number of rotations around the point $(1,1)$. Moreover, once a parametrization of the curve is given, γ satisfies (6). By construction, any solution of system (1) such that $\lim_{t\to+\infty}\mathcal{R}(x(t),y(t))=+\infty$ must rotate around the point $(1,1)$ guided by the spiral γ, as represented in Figure 1. The succeeding computation will give us some estimates on the time that a solution spend to make a complete rotation around this point.

In what follows we will use Hypotheses 1.2 and 1.1 in order to find some estimates on the angular velocity of a solution to (1) with respect to the point $(1,1)$. By the strict inequalities in (4), there exists a $\delta>0$ such that

$$\frac{T}{N+1}<\tau(\hat\mu,\hat\nu,\hat\beta)-\delta<\tau(\check\mu,\check\nu,\check\beta)+7\delta<\frac{T}{N}\,.$$

Once such a value is fixed, consider a solution $u(t)=\big(x(t),y(t)\big)$ to (1) defined in an interval $[t_a,t_b]$ such that for every t in this interval $u(t)\in\mathcal{A}_6$, and such that $u(t_a)=(1,y_a)$ and $u(t_b)=(x_b,1)$. By (9), using polar coordinates in (8), and the estimate in (2) and (3), one can find that

$$\begin{aligned}&\check\mu\cos^2(\vartheta(t))+2\check\beta\sin(\vartheta(t))\cos(\vartheta(t))+\check\nu\sin^2(\vartheta(t))\\&\qquad-\frac{1}{\rho(t)}\left[\check C\cos(\vartheta(t))+\check D\sin(\vartheta(t))\right]\\&\quad\le-\vartheta'(t)\\&\qquad\le\hat\mu\cos^2(\vartheta(t))+2\hat\beta\sin(\vartheta(t))\cos(\vartheta(t))+\hat\nu\sin^2(\vartheta(t))\\&\qquad\qquad+\frac{1}{\rho(t)}\left[\hat C\cos(\vartheta(t))+\hat D\sin(\vartheta(t))\right].\end{aligned}$$

So, recalling that

$$\int_0^{\pi/2}\frac{d\vartheta}{m\cos^2\vartheta+2b\sin\vartheta\cos\vartheta+n\sin^2\vartheta}=\tau(m,n,b)\,,$$

we get the existence of $R'>R$ large enough to have

$$\tau(\hat\mu,\hat\nu,\hat\beta)-\delta\le t_b-t_a\le\tau(\check\mu,\check\nu,\check\beta)+\delta\,,\tag{18}$$

if $|u(t)|>R'$ for every $t\in[t_a,t_b]$. Moreover assume R' large enough to have also $f_1>1/\delta$ and $f_2>1/\delta$ respectively in $\mathbb{R}\times\mathcal{A}_1$ and $\mathbb{R}\times\mathcal{A}_5$. For simplicity of notation, let us call R' as R. Now, we can find a $c_0\in(0,1)$ such that $f_1<-R/\delta$ and $f_2<-R/\delta$ respectively in $\mathbb{R}\times(0,c_0]\times(0,R]$ and $\mathbb{R}\times(0,R]\times(0,c_0]$. Define the set

$$\mathcal{B}=\{(x,y)\,:\,x^2+y^2\le R^2\text{ and }x\ge c_0\,,y\ge c_0\}\,.$$

By (6), we can find k_0 such that $\gamma(s)\notin\mathcal{B}$ for every $s\ge 2\pi k_0$. We are now ready to estimate the time that a solution needs to perform a rotation around the point $(1,1)$ when it does not enter the region $\mathcal{B}$. So, consider a solution u to (1) performing a rotation around the point $(1,1)$ in a certain interval $[t_1,t_6]$, such that $u(t)\notin\mathcal{B}$ for every $t\in[t_1,t_6]$. Suppose moreover that

$$u(t_1)=(x_1,1) \text{ and } u(t_6)=(x_6,1) \text{ with } x_1,x_6>R\,, \tag{19}$$

and find the instants $t_2,\ldots,t_5$ in the interval $[t_1,t_6]$ such that

$$\begin{aligned} u(t_2)&=(R,y_2)\ \text{with}\ y_2<c_0\,,\\ u(t_3)&=(x_3,y_3)\ \text{with}\ x_3=y_3<c_0\,,\\ u(t_4)&=(x_4,R)\ \text{with}\ x_4<c_0\,,\\ u(t_5)&=(1,y_5)\ \ \text{with}\ y_5>R\,. \end{aligned}$$

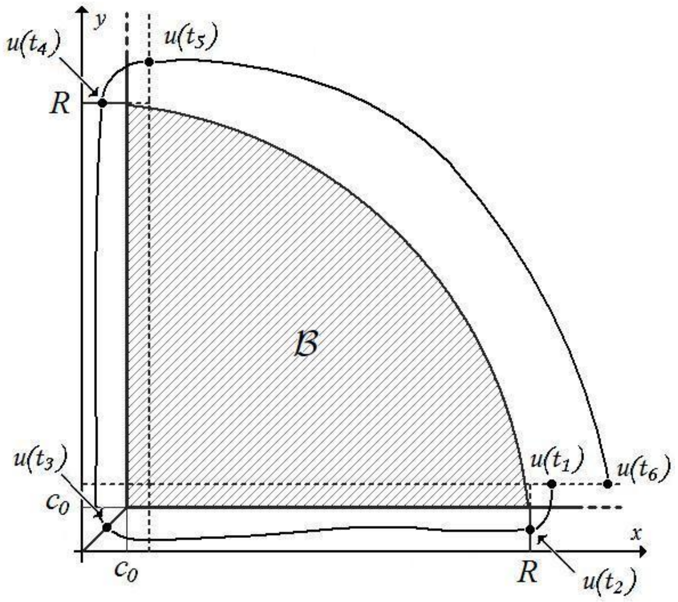

Figure 4: The computation of the time needed by a solution to perform a rotation under the restrictive assumption (19).

It is easy to see now that $t_{i+1}-t_i<\delta$ for every $i\in\{1,2,3,4\}$:

$$\begin{aligned} 1>y(t_1)-y(t_2)&=\textstyle\int_{t_1}^{t_2}-y'(t)\,dt>(t_2-t_1)/\delta\,,\\ R>x(t_2)-x(t_3)&=\textstyle\int_{t_2}^{t_3}-x'(t)\,dt>R(t_3-t_2)/\delta\,,\\ R>y(t_4)-y(t_3)&=\textstyle\int_{t_3}^{t_4}y'(t)\,dt>R(t_4-t_3)/\delta\,,\\ 1>x(t_5)-x(t_4)&=\textstyle\int_{t_4}^{t_5}x'(t)\,dt>(t_5-t_4)/\delta\,. \end{aligned}$$

The estimate for t_6-t_5 has been already made in (18). Hence, summing up we obtain that

$$\tau(\hat\mu,\hat\nu,\hat\beta)-\delta\le t_6-t_1\le\tau(\check\mu,\check\nu,\check\beta)+5\delta\,.$$

The previous computation considers a particular case: assuming (19), we have fixed the initial point and the final point laying in the same radius starting from $(1,1)$. We have to take into account also the other situations which may occur, when the radius is varying, see Figure 5, but with a similar computation one can verify that in general we have

$$\tau(\hat{\mu},\hat{\nu},\hat{\beta}) - \delta \leq t_6 - t_1 \leq \tau(\check{\mu},\check{\nu},\check{\beta}) + 7\delta\,,$$

the solution, in fact, may pass twice through one or two regions. Thus, the upper bound on the time estimates must be enlarged of the value 2δ.

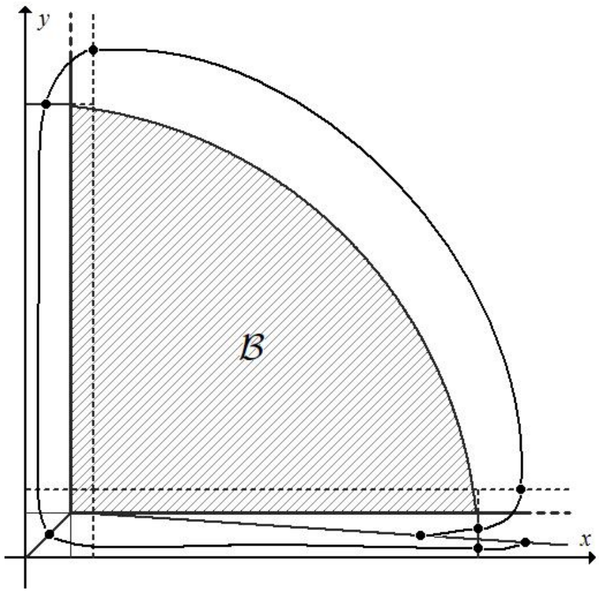

Figure 5: The computation of the time needed by a solution to perform a rotation in the general case.

We have proved that a solution $u(t)$ which does not enter the region $\mathcal{B}$ performs one rotation around the point $(1,1)$ in a time which belongs to the interval $(T/(N+1), T/N)$. Consequently, it cannot perform an integer number of rotations in the period T.

We are now ready to conclude our proof. Consider the closed curves Γ_{k_0}, Γ_{k_1} and Γ_{k_2} defined as in (7), where $k_1 = k_0 + N + 1$ and $k_2 = k_0 + 2N + 2$. The Poincaré-Bohl Theorem applies to the Poincaré map $\mathscr{P}$ associated to system (1) in the set $\overline{\Omega}_{k_1}$. First of all, this map is well-defined, because any solution starting from a point of $\overline{\Omega}_{k_1}$ must perform at least $N+1$ rotations around the origin in order to exit from the set Ω_{k_2}, so spending a time larger than T to do it. Moreover we can prove that

$$\mathscr{P}(u_0) - (1,1) \neq \mu(u_0 - (1,1)) \quad \text{for every } \mu \geq 1 \text{ and } u_0 \in \Gamma_{k_1}\,. \tag{20}$$

In order to do it, we consider two cases. If the solution to system (1) does not enter the region Ω_{k_0} the previous estimate tells us that it cannot perform an integer number of rotations around the point $(1,1)$, so $\mathscr{P}(u_0)-(1,1)\neq\mu(u_0-(1,1))$ for every positive constant μ. Otherwise, if the solution u enter the region Ω_{k_0} in a certain time $\bar{t}\in[0,T]$ then the solution needs to perform $N+1$ rotation to exit the region Ω_{k_1} spending a time larger than T to do it, so also in this case condition (20) is satisfied.

The proof of Theorem 1.3 is thus concluded.

References

[1] S. Bolotin and R.S. MacKay, *Isochronous potentials* in *Proceedings of the third conference: Localization and Energy Transfer in Nonlinear Systems*, World Scientific, London 2003, pp. 217–224.

[2] A. Fonda and M. Garrione, *A Landesman-Lazer type condition for asymptotically linear second order equations with a singulrity*, Proc. Roy. Soc. Edinburgh Sect. A **142** (2012), 1263–1277.

[3] A. Fonda and J. Mawhin, *Planar differential systems at resonance*, Adv. Differential Equations **11** (2006), 1111–1133.

[4] A. Fonda and A. Sfecci, *A general method for the existence of periodic solutions of differential equations in the plane*, J. Differential Equations **252** (2012), 1369–1391.

[5] A. Fonda and A. Sfecci, *Periodic solutions of a system of coupled oscillators with one-sided superlinear retraction forces*, Differential Integral Equations **25** (2012), 993–1010.

[6] A. Fonda and A. Sfecci, *Periodic bouncing solutions for nonlinear impact oscillators*, Adv. Nonlin. Stud. **13** (2013), 179–190.

[7] A. Fonda and R. Toader, *Radially symmetric systems with a singularity and asymptotically linear growth*, Nonlinear Anal. **74** (2011), 2485–2496.

[8] A. Fonda and R. Toader, *Periodic orbits of radially symmetric systems with a singularity: the repulsive case*, Adv. Nonlinear Stud. **11** (2011), 853–874.

[9] R. Ortega, *Periodic perturbations of an isochronous center*, Qual. Theory Dyn. Syst. **3** (2002), 83–91.

[10] E. Pinney, *The nonlinear differential equation* $y''+p(x)y+cy^{-3}=0$, Proc. Amer. Math. Soc **1** (1950), 681.

[11] R. Redheffer, *Steen's equation and its generalizations*, Aequationes Math. **58** (1999), 60–72.

[12] A. Sfecci, *From isochronous potentials to isochronous systems*, to appear.

Author's address:

Andrea Sfecci
Università degli Studi di Trieste
Dipartimento di Matematica e Geoscienze
Via Valerio 12/1, I-34127 Trieste, Italy
e-mail: asfecci@units.it