# ON THE LOWER AND UPPER SOLUTION METHOD FOR THE PRESCRIBED MEAN CURVATURE EQUATION IN MINKOWSKI SPACE 

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#### Abstract

We develop a lower and upper solution method for the Dirichlet problem associated with the prescribed mean curvature equation in Minkowski space $$
\begin{cases}-\operatorname{div}\left(\nabla u / \sqrt{1-|\nabla u|^{2}}\right)=f(x, u) & \text { in } \Omega, \\ u=0 & \text { on } \partial \Omega .\end{cases}
$$

Here $\Omega$ is a bounded regular domain in $\mathbb{R}^{N}$ and the function $f$ satisfies the Carathéodory conditions. The obtained results display various peculiarities due to the special features of the involved differential operator.


1. Introduction. Let us consider the Dirichlet problem associated with the prescribed mean curvature equation in Minkowski space

$$
\begin{cases}-\operatorname{div}\left(\nabla u / \sqrt{1-|\nabla u|^{2}}\right)=f(x, u) & \text { in } \Omega  \tag{1}\\ u=0 & \text { on } \partial \Omega\end{cases}
$$

where $\Omega$ is a bounded domain in $\mathbb{R}^{N}$, with a regular boundary $\partial \Omega$, and the function $f$ satisfies suitable Carathéodory conditions. We are concerned here with strictly spacelike solutions of (1), i.e., weak solutions $u$ of (1) satisfying $\|\nabla u\|_{\infty}<1$.

The aim of this paper is to work out a lower and upper solution method for (1). It is worth recalling that [1, Theorem 3.6] and [8, Theorem 5.1], combined with a truncation argument, imply that problem (1) has at least one solution for any given $f$. In the light of this general existence result, the interest of using lower and upper solutions in this context relies on the localization, the multiplicity, and the stability information that they may provide. In this respect, due to the special features of the mean curvature operator in Minkowski space, some peculiarities are displayed. In particular, the knowledge of a single lower solution $\alpha$, or a single upper solution $\beta$, allows to localize solutions, and the existence of a pair of lower and upper solutions $\alpha$, $\beta$, with $\alpha \not \leq \beta$, yields multiplicity of solutions, without assuming any additional condition. We point out that these statements have no analogues for

[^0]other quasilinear elliptic problems driven, e.g., by the p-Laplace operator, or the mean curvature operator in Euclidean space.

Due to space limitations we cannot address in this paper any stability issue: this topic will be discussed elsewhere. For the same reason we produce here just a few sample applications of our statements. In particular, we show how to extend, or to recover with a simpler proof, some results recently obtained in $[2,5,3,6,7]$, concerning the existence of multiple positive solutions of the problem

$$
\begin{cases}-\operatorname{div}\left(\nabla u / \sqrt{1-|\nabla u|^{2}}\right)=\lambda u^{p} & \text { in } \Omega  \tag{2}\\ u=0 & \text { on } \partial \Omega\end{cases}
$$

where $p>0$ is a given exponent and $\lambda>0$ is a parameter.
Notation. We list some notation that are used throughout this paper. For $s \in \mathbb{R}$ we set $s^{+}=\max \{s, 0\}$ and $s^{-}=-\min \{s, 0\}$. For functions $u, v: \bar{\Omega} \rightarrow \mathbb{R}$, we write: $u \leq v$ if $u(x) \leq v(x)$ a.e. in $\Omega$, and $u<v$ if $u \leq v$ and $u(x)<v(x)$ in a subset of $\Omega$ having positive measure. We also write $u \ll v$ if there is $\varepsilon>0$ such that $u(x)+\varepsilon \operatorname{dist}(x, \partial \Omega) \leq v(x)$ for every $x \in \bar{\Omega}$. We set $C_{0}^{1}(\bar{\Omega})=\left\{u \in C^{1}(\bar{\Omega}): u=\right.$ 0 on $\partial \Omega\}$.
2. Statements. We assume that
$\left(h_{1}\right) \Omega$ is a bounded domain in $\mathbb{R}^{N}$, with a boundary $\partial \Omega$ of class $C^{2}$, and
$\left(h_{2}\right) f: \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ satisfies the $L^{\infty}$-Carathéodory conditions.
Notion of solution. By a solution of (1) we mean a function $u \in C^{0,1}(\bar{\Omega})$, with $\|\nabla u\|_{\infty}<1$ and $u=0$ on $\partial \Omega$, such that

$$
\begin{equation*}
\int_{\Omega} \frac{\nabla u \cdot \nabla w}{\sqrt{1-|\nabla u|^{2}}} d x=\int_{\Omega} f(x, u) w d x \tag{3}
\end{equation*}
$$

for all $w \in W_{0}^{1,1}(\Omega)$.
Remark 1. Clearly, any solution $u$ of (1) satisfies $\|u\|_{\infty}<\frac{1}{2} \operatorname{diam}(\Omega)$.
Remark 2. If $u$ is a solution of (1), according to the previous definition, then $u \in W^{2, r}(\Omega)$, for all finite $r \geq 1$. In particular, $u$ satisfies the equation a.e. in $\Omega$ and the boundary condition everywhere on $\partial \Omega$. Indeed, set $v=f(\cdot, u) \in L^{\infty}(\Omega)$. By [7, Lemma 2.2] the problem

$$
\begin{cases}-\operatorname{div}\left(\nabla z / \sqrt{1-|\nabla z|^{2}}\right)=v & \text { in } \Omega  \tag{4}\\ u=0 & \text { on } \partial \Omega\end{cases}
$$

has a unique solution $z \in W^{2, r}(\Omega)$, for all finite $r \geq 1$, satisfying the equation a.e. in $\Omega$ and the boundary condition everywhere on $\partial \Omega$. Of course, $z$ satisfies

$$
\int_{\Omega} \frac{\nabla z \cdot \nabla w}{\sqrt{1-|\nabla z|^{2}}} d x=\int_{\Omega} v w d x
$$

for all $w \in W_{0}^{1,1}(\Omega)$. Using the concavity of the function $y \mapsto \sqrt{1-|y|^{2}}$, we easily see that both $u$ and $z$ maximize the functional

$$
\begin{equation*}
\phi_{v}: w \mapsto \int_{\Omega} \sqrt{1-|\nabla w|^{2}} d x+\int_{\Omega} v w d x \tag{5}
\end{equation*}
$$

over the set

$$
\begin{equation*}
\mathcal{C}=\left\{w \in C^{0,1}(\bar{\Omega}):\|\nabla w\|_{\infty} \leq 1 \text { and } w=0 \text { on } \partial \Omega\right\} . \tag{6}
\end{equation*}
$$

This means that both $u$ and $z$ are variational solutions of (4) in the sense of [1]. Hence [1, Lemma 1.2] implies that $u=z$ and therefore $u \in W^{2, r}(\Omega)$, for all finite $r \geq 1$.
Lower and upper solutions. We say that a function $\alpha: \bar{\Omega} \rightarrow \mathbb{R}$ is a lower solution of (1) if there exist $\alpha_{1}, \ldots, \alpha_{m} \in C^{0,1}(\bar{\Omega})$, such that $\alpha=\max \left\{\alpha_{1}, \ldots, \alpha_{m}\right\}$ and, for each $i=1, \ldots, m$,

- $\left\|\nabla \alpha_{i}\right\|_{\infty}<1$,
- for every $w \in W_{0}^{1,1}(\Omega)$, with $w \geq 0$,

$$
\begin{equation*}
\int_{\Omega} \frac{\nabla \alpha_{i} \cdot \nabla w}{\sqrt{1-\left|\nabla \alpha_{i}\right|^{2}}} d x \leq \int_{\Omega} f\left(x, \alpha_{i}\right) w d x \tag{7}
\end{equation*}
$$

- $\alpha_{i} \leq 0$ on $\partial \Omega$.

We say that a lower solution $\alpha=\max \left\{\alpha_{1}, \ldots, \alpha_{m}\right\}$ of (1) is strict if every solution $u$ of (1), with $u \geq \alpha$, satisfies $u \gg \alpha_{i}$ for every $i=1, \ldots, m$.

Similarly, we say that a function $\beta: \bar{\Omega} \rightarrow \mathbb{R}$ is an upper solution of (1) if there exist $\beta_{1}, \ldots, \beta_{n} \in C^{0,1}(\bar{\Omega})$, such that $\beta=\min \left\{\beta_{1}, \ldots, \beta_{n}\right\}$ and, for each $j=1, \ldots, n$,

- $\left\|\nabla \beta_{j}\right\|_{\infty}<1$,
- for every $w \in W_{0}^{1,1}(\Omega)$, with $w \geq 0$,

$$
\int_{\Omega} \frac{\nabla \beta_{j} \cdot \nabla w}{\sqrt{1-\left|\nabla \beta_{j}\right|^{2}}} d x \geq \int_{\Omega} f\left(x, \beta_{j}\right) w d x
$$

- $\beta_{j} \geq 0$ on $\partial \Omega$.

We say that an upper solution $\beta=\min \left\{\beta_{1}, \ldots, \beta_{n}\right\}$ of (1) is strict if every solution $u$ of (1), with $u \leq \beta$, satisfies $u \ll \beta_{j}$ for every $j=1, \ldots, n$.
Remark 3. If $u$ is simultaneously a lower solution, with $m=1$, and an upper solution, with $n=1$, then $u$ is a solution. Of course, also the converse implication holds.

Theorem 2.1. Assume ( $h_{1}$ ) and $\left(h_{2}\right)$. The following conclusions hold.
(i) Suppose there exists a lower solution $\alpha$. Then problem (1) has at least one solution $u$, with

$$
u \geq \alpha .
$$

(ii) Suppose there exists an upper solution $\beta$. Then problem (1) has at least one solution $u$, with

$$
u \leq \beta .
$$

(iii) Suppose there exist a strict lower solution $\alpha$ and a strict upper solution $\beta$, with $\alpha \not \leq \beta$. Then problem (1) has at least three solutions $u_{1}, u_{2}, u_{3}$, with

$$
u_{1}<u_{2}<u_{3}, \quad u_{1} \ll \beta, \quad u_{2} \nsupseteq \alpha, \quad u_{2} \npreceq \beta, \quad u_{3} \gg \alpha .
$$

Remark 4. The following statements are related to the conclusions of Theorem 2.1; more details are given in Propositions 1 and 2 below.
(iv) Suppose there exist a lower solution $\alpha$ and an upper solution $\beta$, with $\alpha \leq \beta$. Then problem (1) has a solution $u$, with

$$
\alpha \leq u \leq \beta .
$$

(v) Suppose there exist a lower solution $\alpha$ and an upper solution $\beta$, with $\alpha \not \leq \beta$. Then problem (1) has at least two solutions $u_{1}, u_{2}$, with

$$
u_{1}<u_{2}, \quad u_{1} \leq \beta, \quad u_{2} \geq \alpha
$$

(vi) Suppose there exist lower solutions $\alpha, \bar{\alpha}$ and upper solutions $\beta, \bar{\beta}$, with $\alpha, \beta$ strict, $\bar{\alpha} \leq \min \{\alpha, \beta\} \leq \max \{\alpha, \beta\} \leq \bar{\beta}$, and $\alpha \not \leq \beta$. Then problem (1) has at least three solutions $u_{1}, u_{2}, u_{3}$, with

$$
\bar{\alpha} \leq u_{1}<u_{2}<u_{3} \leq \bar{\beta}, \quad u_{1} \ll \beta, \quad u_{2} \nsupseteq \alpha, \quad u_{2} \not \leq \beta, \quad u_{3} \gg \alpha
$$

Remark 5. In cases (i), (ii) and (iv) a variational characterization of the solutions is provided by Proposition 3 below. Define $\mathcal{C}$ as in (6) and, for $w \in \mathcal{C}$, set

$$
\begin{equation*}
\psi(w)=\int_{\Omega} \sqrt{1-|\nabla w|^{2}} d x+\int_{\Omega} F(x, w) d x \tag{8}
\end{equation*}
$$

where $F(x, s)=\int_{0}^{s} f(x, \xi) d \xi$. Then we have:

- in case (i), there is a solution $u$ that maximizes the functional $\psi$ over the set

$$
\{w \in \mathcal{C}: w \geq \alpha\}
$$

- in case (ii), there is a solution u that maximizes the functional $\psi$ over the set

$$
\{w \in \mathcal{C}: w \leq \beta\}
$$

- in case (iv), there is a solution $u$ that maximizes the functional $\psi$ over the set

$$
\{w \in \mathcal{C}: \alpha \leq w \leq \beta\}
$$

3. Proofs. Assume $\left(h_{1}\right)$ and $\left(h_{2}\right)$. By [7, Lemma 2.2] we can define an operator $\mathcal{T}: C^{1}(\bar{\Omega}) \rightarrow C_{0}^{1}(\bar{\Omega})$ which sends any function $v \in C^{1}(\bar{\Omega})$ onto the unique solution $u \in W^{2, r}(\Omega)$, for all finite $r \geq 1$, of the problem

$$
\begin{cases}-\operatorname{div}\left(\nabla u / \sqrt{1-|\nabla u|^{2}}\right)=f(x, v) & \text { in } \Omega  \tag{9}\\ u=0 & \text { on } \partial \Omega\end{cases}
$$

Clearly, $u$ is a solution of (1) if and only if $u$ is a fixed point of $\mathcal{T}$. Let us also denote by

$$
\mathcal{B}=\left\{u \in C_{0}^{1}(\bar{\Omega}):\|\nabla u\|_{\infty}<1\right\}
$$

the unit open ball in $C_{0}^{1}(\bar{\Omega})$.
Lemma 3.1. Assume $\left(h_{1}\right)$ and $\left(h_{2}\right)$. Then the operator $\mathcal{T}$ is completely continuous and $\operatorname{deg}(I-\mathcal{T}, \mathcal{B}, 0)=1$, where $I$ is the identity operator.

Proof. The result in [7, Lemma 2.3] shows that, by conditions $\left(h_{1}\right)$ and $\left(h_{2}\right)$, the operator $\mathcal{T}$ is completely continuous. Let $\Lambda>0$ be a constant such that $\|f(\cdot, v)\|_{\infty} \leq \Lambda$ for all $v \in \overline{\mathcal{B}}$. According to $[7$, Lemma 2.2] there is a constant $\eta \in] 0,1[$ such that the solution $u=\mathcal{T} v$ of (9) satisfies $\|\nabla u\|_{\infty} \leq \eta$. Hence $\mathcal{T}$ maps $\overline{\mathcal{B}}$ into $\mathcal{B}$. Standard results of degree theory yield $\operatorname{deg}(I-\mathcal{T}, \mathcal{B}, 0)=1$.

Remark 6. Lemma 3.1 implies that, under $\left(h_{1}\right)$ and $\left(h_{2}\right), \mathcal{T}$ has a fixed point in $\mathcal{B}$ and hence problem (1) has a solution. Since the same proof is still valid for the more general problem

$$
\begin{cases}-\operatorname{div}\left(\nabla u / \sqrt{1-|\nabla u|^{2}}\right)=f(x, u, \nabla u) & \text { in } \Omega  \tag{10}\\ u=0 & \text { on } \partial \Omega\end{cases}
$$

where $f: \Omega \times \mathbb{R} \times \mathbb{R}^{N} \rightarrow \mathbb{R}$ satisfies the $L^{\infty}$-Carathéodory conditions, an extension of the existence results in $[1,8]$ to the non-variational problem (10) follows. We notice that the solvability of (10) has been explicitly raised in [9] as an open question.
Proposition 1. Assume $\left(h_{1}\right)$ and $\left(h_{2}\right)$. Suppose there exist a lower solution $\alpha$ and an upper solution $\beta$, with $\alpha \leq \beta$. Then problem (1) has solutions $v, w$, with $\alpha \leq v \leq w \leq \beta$, such that every solution $u$ of (1), with $\alpha \leq u \leq \beta$, satisfies $v \leq u \leq w$. Further, if $\alpha$ and $\beta$ are strict, then

$$
\begin{equation*}
\operatorname{deg}(I-\mathcal{T}, \mathcal{U}, 0)=1 \tag{11}
\end{equation*}
$$

where

$$
\begin{equation*}
\mathcal{U}=\left\{z \in C_{0}^{1}(\bar{\Omega}): \alpha \ll z \ll \beta \text { and }\|\nabla z\|_{\infty}<1\right\} . \tag{12}
\end{equation*}
$$

Proof. The proof is divided into three parts.
Part 1. Existence of a solution $u$ of (1) with $\alpha \leq u \leq \beta$.
Step 1. Construction of a modified problem. We set, for a.e. $x \in \Omega$ and every $s \in \mathbb{R}$,

$$
\bar{f}(x, s)= \begin{cases}\max _{1 \leq i \leq m} f\left(x, \alpha_{i}(x)\right) & \text { if } s \leq \alpha(x)  \tag{13}\\ f(x, s) & \text { if } \alpha(x)<s<\beta(x) \\ \min _{1 \leq j \leq n} f\left(x, \beta_{j}(x)\right) & \text { if } s \geq \beta(x)\end{cases}
$$

Note that $\bar{f}$ satisfies the $L^{\infty}$-Carathéodory conditions. Then we consider the modified problem

$$
\begin{cases}-\operatorname{div}\left(\nabla u / \sqrt{1-|\nabla u|^{2}}\right)=\bar{f}(x, u) & \text { in } \Omega  \tag{14}\\ u=0 & \text { on } \partial \Omega\end{cases}
$$

Step 2. Every solution $u$ of (14) satisfies $\alpha \leq u \leq \beta$. In order to prove that $u \geq \alpha$, we fix any $i \in\{1, \ldots, m\}$ and we show that $u \geq \alpha_{i}$. Set $w=\left(u-\alpha_{i}\right)^{-} \in W_{0}^{1,1}(\Omega)$ and $A=\left\{x \in \Omega: u(x) \leq \alpha_{i}(x)\right\}$. Taking $w$ as test function in both (3), with $f$ replaced by $\bar{f}$, and (7), we get

$$
\begin{aligned}
\int_{A} \frac{\nabla u \cdot \nabla\left(u-\alpha_{i}\right)}{\sqrt{1-|\nabla u|^{2}}} d x & =-\int_{\Omega} \frac{\nabla u \cdot \nabla\left(u-\alpha_{i}\right)^{-}}{\sqrt{1-|\nabla u|^{2}}} d x \\
& =-\int_{\Omega} \bar{f}(x, u)\left(u-\alpha_{i}\right)^{-} d x=\int_{A} \bar{f}(x, u)\left(u-\alpha_{i}\right) d x
\end{aligned}
$$

and

$$
\begin{aligned}
-\int_{A} \frac{\nabla \alpha_{i} \cdot \nabla\left(u-\alpha_{i}\right)}{\sqrt{1-\left|\nabla \alpha_{i}\right|^{2}}} d x & =\int_{\Omega} \frac{\nabla \alpha_{i} \cdot \nabla\left(u-\alpha_{i}\right)^{-}}{\sqrt{1-\left|\nabla \alpha_{i}\right|^{2}}} d x \\
& \leq \int_{\Omega} f\left(x, \alpha_{i}\right)\left(u-\alpha_{i}\right)^{-} d x=-\int_{A} f\left(x, \alpha_{i}\right)\left(u-\alpha_{i}\right) d x
\end{aligned}
$$

Summing up we obtain

$$
\begin{aligned}
\int_{A}\left(\frac{\nabla u}{\sqrt{1-|\nabla u|^{2}}}-\frac{\nabla \alpha_{i}}{\sqrt{1-\left|\nabla \alpha_{i}\right|^{2}}}\right) & \cdot\left(\nabla u-\nabla \alpha_{i}\right) d x \\
& \leq \int_{A}\left(\bar{f}(x, u)-f\left(x, \alpha_{i}\right)\right)\left(u-\alpha_{i}\right) d x \leq 0
\end{aligned}
$$

The strict monotonicity of the function $y \mapsto y / \sqrt{1-|y|^{2}}$ yields $\nabla\left(u-\alpha_{i}\right)=0$ a.e. in $A$. This implies $\left(u-\alpha_{i}\right)^{-}=0$ and hence $u \geq \alpha_{i}$. In a completely similar way we prove that $u \leq \beta$.

Step 3. Problem (1) has at least one solution $u$, with $\alpha \leq u \leq \beta$. Let us consider the operator $\overline{\mathcal{T}}: C^{1}(\bar{\Omega}) \rightarrow C_{0}^{1}(\bar{\Omega})$ which sends any function $v \in C^{1}(\bar{\Omega})$ onto the unique solution $u$ of

$$
\begin{cases}-\operatorname{div}\left(\nabla u / \sqrt{1-|\nabla u|^{2}}\right)=\bar{f}(x, v) & \text { in } \Omega \\ u=0 & \text { on } \partial \Omega\end{cases}
$$

Lemma 3.1 implies that

$$
\begin{equation*}
\operatorname{deg}(I-\overline{\mathcal{T}}, \mathcal{B}, 0)=1 \tag{15}
\end{equation*}
$$

Therefore $\overline{\mathcal{T}}$ has a fixed point $u$, which is a solution of (14). By Step 2 we know that $u$ satisfies $\alpha \leq u \leq \beta$ and hence it is a solution of (1) as well.

Part 2. Existence of extremal solutions. We know that the solutions of (1) are precisely the fixed points of the operator $\mathcal{T}$. By Lemma 3.1 the set

$$
\mathcal{S}=\left\{u \in C_{0}^{1}(\bar{\Omega}): u=\mathcal{T} u \text { and } \alpha \leq u \leq \beta\right\}
$$

is compact. In Part 1 we showed that $\mathcal{S}$ is non-empty. Let us prove that there exists $\min \mathcal{S}$; a similar $\operatorname{argument}$ yields the existence of $\max \mathcal{S}$. For each $u \in \mathcal{S}$, define the closed subset of $\mathcal{S}$

$$
\mathcal{K}_{u}=\{z \in \mathcal{S}: z \leq u\}
$$

The family $\left(\mathcal{K}_{u}\right)_{u \in \mathcal{S}}$ has the finite intersection property. Indeed, if $u_{1}, u_{2} \in \mathcal{S}$, then $\min \left\{u_{1}, u_{2}\right\}$ is an upper solution of (1) with $\alpha \leq \min \left\{u_{1}, u_{2}\right\}$. By Part 1 there is a solution $u$ of (1) with $\alpha \leq u \leq \min \left\{u_{1}, u_{2}\right\} \leq \beta$, that is $\mathcal{K}_{u_{1}} \cap \mathcal{K}_{u_{2}} \neq \emptyset$. By the compactness of $\mathcal{S}$ there exists $v \in \bigcap_{u \in \mathcal{S}} \mathcal{K}_{u}$. Clearly, $v$ is the minimum solution of (1) lying between $\alpha$ and $\beta$.

Part 3. Degree computation. Let us assume that $\alpha$ and $\beta$ are, respectively, a strict lower and a strict upper solution. Since there exists a solution $u$ of (1), with $\alpha \leq u \leq \beta$, and every such a solution satisfies $\alpha \ll u \ll \beta$, it follows that $\alpha \ll \beta$. Hence the set $\mathcal{U}$ defined in (12) is a non-empty open bounded subset of $C_{0}^{1}(\bar{\Omega})$ such that there is no fixed point either of $\mathcal{T}$ or of $\overline{\mathcal{T}}$ on its boundary $\partial \mathcal{U}$. Moreover, as $\mathcal{T}$ and $\overline{\mathcal{T}}$ coincide in $\mathcal{U}$, we have

$$
\operatorname{deg}(I-\mathcal{T}, \mathcal{U}, 0)=\operatorname{deg}(I-\overline{\mathcal{T}}, \mathcal{U}, 0)
$$

Since $\overline{\mathcal{T}}$ is fixed point free in $\overline{\mathcal{B}} \backslash \mathcal{U}$, the excision property of the degree and (15) imply that

$$
\operatorname{deg}(I-\overline{\mathcal{T}}, \mathcal{U}, 0)=\operatorname{deg}(I-\overline{\mathcal{T}}, \mathcal{B}, 0)=1
$$

Thus we conclude that (11) holds.
Proposition 2. Assume $\left(h_{1}\right)$ and $\left(h_{2}\right)$. Suppose there exist a strict lower solution $\alpha$ and a strict upper solution $\beta$, with $\alpha \not \leq \beta$. Then problem (1) has at least three solutions $u_{1}, u_{2}, u_{3}$, with

$$
\begin{equation*}
u_{1}<u_{2}<u_{3}, \quad u_{1} \ll \beta, \quad u_{2} \nsupseteq \alpha, \quad u_{2} \not \leq \beta, \quad u_{3} \gg \alpha . \tag{16}
\end{equation*}
$$

Proof. The proof is divided into three steps.
Step 1. Construction of a modified problem. Set

$$
R=\max \left\{\|\alpha\|_{\infty},\|\beta\|_{\infty}, \frac{1}{2} \operatorname{diam}(\Omega)\right\}
$$

and define, for a.e. $x \in \Omega$ and every $s \in \mathbb{R}$,

$$
f_{R}(x, s)= \begin{cases}f(x, s) & \text { if }|s| \leq R \\ 0 & \text { if }|s| \geq R+1 \\ \text { linear } & \text { if } R<|s|<R+1\end{cases}
$$

Note that $f_{R}$ satisfies the $L^{\infty}$ _Carathéodory conditions. We consider the modified problem

$$
\begin{cases}-\operatorname{div}\left(\nabla u / \sqrt{1-|\nabla u|^{2}}\right)=f_{R}(x, u) & \text { in } \Omega  \tag{17}\\ u=0 & \text { on } \partial \Omega\end{cases}
$$

Due to our choice of $R$, Remark 1 implies that any solution of (17) is a solution of (1), $\alpha$ and $\beta$ are strict lower and upper solutions of (17) as well, and the constants $\bar{\alpha}=-R-1$ and $\bar{\beta}=R+1$ are strict lower and upper solutions of (17).
Step 2. Degree computation. Let us define the following open bounded subsets of $C_{0}^{1}(\bar{\Omega})$ :

$$
\begin{aligned}
& \mathcal{U}_{\bar{\alpha}}^{\beta}=\left\{u \in C_{0}^{1}(\bar{\Omega}): \bar{\alpha} \ll u \ll \beta \text { and }\|\nabla u\|_{\infty}<1\right\}, \\
& \mathcal{U}_{\alpha}^{\bar{\beta}}=\left\{u \in C_{0}^{1}(\bar{\Omega}): \alpha \ll u \ll \bar{\beta} \text { and }\|\nabla u\|_{\infty}<1\right\}, \\
& \mathcal{U}_{\bar{\alpha}}^{\bar{\beta}}=\left\{u \in C_{0}^{1}(\bar{\Omega}): \bar{\alpha} \ll u \ll \bar{\beta} \text { and }\|\nabla u\|_{\infty}<1\right\} .
\end{aligned}
$$

Notice that $\mathcal{U}_{\bar{\alpha}}^{\beta} \subset \mathcal{U}_{\bar{\alpha}}^{\bar{\beta}}, \mathcal{U}_{\alpha}^{\bar{\beta}} \subset \mathcal{U}_{\bar{\alpha}}^{\bar{\beta}}$, and, as $\alpha \not \leq \beta, \mathcal{U}_{\bar{\alpha}}^{\beta} \cap \mathcal{U}_{\alpha}^{\bar{\beta}}=\emptyset$. Moreover, since both $\alpha$ and $\bar{\alpha}$ are strict lower solutions of (17), and $\beta$ and $\bar{\beta}$ are strict upper solutions of (17), we have

$$
\begin{equation*}
0 \notin\left(I-\mathcal{T}_{R}\right)\left(\partial \mathcal{U}_{\alpha}^{\bar{\beta}} \cup \partial \mathcal{U}_{\bar{\alpha}}^{\beta} \cup \partial \mathcal{U}_{\bar{\alpha}}^{\bar{\beta}}\right) \tag{18}
\end{equation*}
$$

where $\mathcal{T}_{R}: C^{1}(\bar{\Omega}) \rightarrow C_{0}^{1}(\bar{\Omega})$ is the operator which sends any function $v \in C^{1}(\bar{\Omega})$ onto the unique solution $u \in C_{0}^{1}(\bar{\Omega})$ of

$$
\begin{cases}-\operatorname{div}\left(\nabla u / \sqrt{1-|\nabla u|^{2}}\right)=f_{R}(x, v) & \text { in } \Omega \\ u=0 & \text { on } \partial \Omega\end{cases}
$$

Define now the open bounded subset of $C_{0}^{1}(\bar{\Omega})$

$$
\mathcal{V}=\mathcal{U}_{\bar{\alpha}}^{\bar{\beta}} \backslash \overline{\left(\mathcal{U}_{\alpha}^{\bar{\beta}} \cup \mathcal{U}_{\bar{\alpha}}^{\beta}\right)}
$$

By (18), using the excision property of the degree, we get

$$
\operatorname{deg}\left(I-\mathcal{T}_{R}, \mathcal{U}_{\bar{\alpha}}^{\bar{\beta}}, 0\right)=\operatorname{deg}\left(I-\mathcal{T}_{R}, \mathcal{U}_{\bar{\alpha}}^{\bar{\beta}} \backslash\left(\partial \mathcal{U}_{\alpha}^{\bar{\beta}} \cup \partial \mathcal{U}_{\bar{\alpha}}^{\beta}\right), 0\right)
$$

and hence, using the additivity property of the degree,

$$
\operatorname{deg}\left(I-\mathcal{T}_{R}, \mathcal{U}_{\bar{\alpha}}^{\bar{\beta}}, 0\right)=\operatorname{deg}\left(I-\mathcal{T}_{R}, \mathcal{U}_{\bar{\alpha}}^{\beta}, 0\right)+\operatorname{deg}\left(I-\mathcal{T}_{R}, \mathcal{U}_{\alpha}^{\bar{\beta}}, 0\right)+\operatorname{deg}\left(I-\mathcal{T}_{R}, \mathcal{V}, 0\right)
$$

Since, by Proposition 1, we have

$$
\operatorname{deg}\left(I-\mathcal{T}_{R}, \mathcal{U}_{\bar{\alpha}}^{\bar{\beta}}, 0\right)=\operatorname{deg}\left(I-\mathcal{T}_{R}, \mathcal{U}_{\bar{\alpha}}^{\beta}, 0\right)=\operatorname{deg}\left(I-\mathcal{T}_{R}, \mathcal{U}_{\alpha}^{\bar{\beta}}, 0\right)=1
$$

we finally get

$$
\operatorname{deg}\left(I-\mathcal{T}_{R}, \mathcal{V}, 0\right)=-1
$$

Step 3. Existence of solutions. Since $\mathcal{U}_{\bar{\alpha}}^{\beta}, \mathcal{U}_{\alpha}^{\bar{\beta}}, \mathcal{V}$ are pairwise disjoint, the previous degree calculations imply that there are three distinct fixed points $u_{1}, u_{2}, u_{3}$ of the operator $\mathcal{I}_{R}$, with

$$
u_{1} \in \mathcal{U}_{\bar{\alpha}}^{\beta}, \quad u_{2} \in \mathcal{V}, \quad u_{3} \in \mathcal{U}_{\alpha}^{\bar{\beta}}
$$

This means that

$$
u_{1} \ll \beta, \quad u_{2} \nsupseteq \alpha, \quad u_{2} \not \leq \beta, \quad u_{3} \gg \alpha .
$$

Let $v$ and $w$ be, respectively, the minimum and the maximum solution of (17) lying between $\bar{\alpha}$ and $\bar{\beta}$. Then, possibly replacing $u_{1}$ with $v$ and $u_{3}$ with $w$, we immediately conclude that (17) and, hence, (1) have three distinct solutions for which (16) holds.

Proof of Theorem 2.1. In order to prove ( $i$ ), we consider the modified problem (17) constructed in Step 1 of the proof of Proposition 2, with the choice

$$
R=\max \left\{\|\alpha\|_{\infty}, \frac{1}{2} \operatorname{diam}(\Omega)\right\}
$$

Let us set $\bar{\beta}=R+1$. We have that $\alpha$ is a lower solution and $\bar{\beta}$ is an upper solution of (17) with $\alpha \leq \bar{\beta}$. By Lemma 1 there exists at least one solution $u$ of (17), with $\alpha \leq u \leq \frac{1}{2} \operatorname{diam}(\Omega)$, and hence of (1). A symmetric argument implies the validity of (ii). Finally, conclusion (iii) is precisely the conclusion of Proposition 2.

Proposition 3. Assume $\left(h_{1}\right)$ and $\left(h_{2}\right)$. Suppose there exist a lower solution $\alpha$ and an upper solution $\beta$, with $\alpha \leq \beta$. Then there is a solution $u$ that maximizes the functional $\psi$ over the set $\{w \in \mathcal{C}: \alpha \leq w \leq \beta\}$, where $\mathcal{C}$ and $\psi$ are defined by (6) and (8), respectively.
Proof. Our argument is partially inspired by [4]. Let $\bar{f}$ be defined as in (13) and set $\bar{F}(x, s)=\int_{0}^{s} \bar{f}(x, \xi) d \xi$. For all $w \in \mathcal{C}$, set

$$
\bar{\psi}(w)=\int_{\Omega} \sqrt{1-|\nabla w|^{2}} d x+\int_{\Omega} \bar{F}(x, w) d x
$$

By [1, Proposition 1.1] there is $u \in \mathcal{C}$ maximizing $\bar{\psi}$ over $\mathcal{C}$, i.e.,

$$
\begin{equation*}
\int_{\Omega} \sqrt{1-|\nabla u|^{2}} d x+\int_{\Omega} \bar{F}(x, u) d x \geq \int_{\Omega} \sqrt{1-|\nabla v|^{2}} d x+\int_{\Omega} \bar{F}(x, v) d x \tag{19}
\end{equation*}
$$

for all $v \in \mathcal{C}$. Now, take any $w \in \mathcal{C}$, fix $\lambda \in] 0,1[$ and choose $v=u+\lambda(w-u)$ in (19). By concavity, we have

$$
\begin{aligned}
& \int_{\Omega} \sqrt{1-|\nabla u|^{2}} d x+\int_{\Omega} \bar{F}(x, u) d x \\
& \quad \geq \int_{\Omega} \sqrt{1-|\nabla u+\lambda(\nabla w-\nabla u)|^{2}} d x+\int_{\Omega} \bar{F}(x, u+\lambda(w-u)) d x \\
& \quad \geq \lambda \int_{\Omega} \sqrt{1-|\nabla w|^{2}} d x+(1-\lambda) \int_{\Omega} \sqrt{1-|\nabla u|^{2}} d x+\int_{\Omega} \bar{F}(x, u+\lambda(w-u)) d x
\end{aligned}
$$

and hence, rearranging and dividing by $\lambda$,

$$
\int_{\Omega} \sqrt{1-|\nabla u|^{2}} d x-\int_{\Omega} \sqrt{1-|\nabla w|^{2}} d x \geq \int_{\Omega}\left(\int_{0}^{1} \bar{f}(x, u+t \lambda(w-u))(w-u) d t\right) d x
$$

Taking the limit for $\lambda \rightarrow 0$ and using the dominated convergence theorem yields

$$
\int_{\Omega} \sqrt{1-|\nabla u|^{2}} d x-\int_{\Omega} \sqrt{1-|\nabla w|^{2}} d x \geq \int_{\Omega} \bar{f}(x, u)(w-u) d x
$$

that is,

$$
\int_{\Omega} \sqrt{1-|\nabla u|^{2}} d x+\int_{\Omega} \bar{f}(x, u) u d x \geq \int_{\Omega} \sqrt{1-|\nabla w|^{2}} d x+\int_{\Omega} \bar{f}(x, u) w d x
$$

holds for all $w \in \mathcal{C}$.
Let us set now $v=\bar{f}(\cdot, u)$ and denote by $z \in W^{2, r}(\Omega)$, for all finite $r \geq 1$, the solution of problem (4). As in Remark 2, we have that both $u$ and $z$ maximize the functional $\phi_{v}$ defined in (5) over $\mathcal{C}$ and therefore are variational solutions of (4) in
the sense of [1]. Again [1, Lemma 1.2] applies implying that $u=z$. Accordingly, $u \in W^{2, r}(\Omega)$, for all finite $r \geq 1$, is solution of the modified problem (1). From Step 2 of Proposition 1 we conclude that $u$ satisfies $\alpha \leq u \leq \beta$ and therefore it maximizes the functional $\psi$ over the set $\{w \in \mathcal{C}: \alpha \leq w \leq \beta\}$.
4. Applications. We now produce a few sample applications of our results. We do not look for the maximum of generality: our main purpose being here to illustrate through simple examples how strict lower and upper solutions can be constructed in the non-standard framework of the prescribed mean curvature equation in Minkowski space.
Example 1. Assume $p \in] 0,1[$. Then, for each $\lambda>0$, problem (2) has at least one solution $u \gg 0$. For convenience, we replace $u^{p}$ with $\left(u^{+}\right)^{p}$ at the right-hand side of (2). In the light of the previous results, it is clear that it is enough to construct a strict lower solution $\alpha>0$ of (2). Fix an open ball $B$ with $\bar{B} \subset \Omega$. It follows from, e.g., [6] that, for any given $\lambda>0$, the Dirichlet problem

$$
\begin{cases}-\operatorname{div}\left(\nabla u / \sqrt{1-|\nabla u|^{2}}\right)=\lambda u^{p} & \text { in } B \\ u=0 & \text { on } \partial B\end{cases}
$$

has a (radially symmetric) solution $z \in C^{2}(\bar{B})$ satisfying $z \gg 0$ in $\bar{B}$. Let us define a function $\alpha \in C^{0,1}(\bar{\Omega})$ by

$$
\alpha(x)= \begin{cases}z(x) & \text { if } x \in \bar{B} \\ 0 & \text { if } x \in \bar{\Omega} \backslash \bar{B}\end{cases}
$$

As the outer normal derivative of $z$ on $\partial B$ is negative and the test functions $w$ in (7) are non-negative, one can easily verify, integrating by parts, that $\alpha$ is a lower solution of (2). Let us show that it is strict. Suppose that $u$ is a solution of (2) with $u \geq \alpha$. As $u>0$ the strong maximum principle and the Hopf boundary lemma imply that $u \gg 0$ in $\bar{\Omega}$ (see [7, Lemma 2.6]). In particular, we have that $\min _{\bar{B}} u>0$. As moreover $\lambda u^{p} \geq \lambda \alpha^{p}$ in $B$, we can apply [1, Lemma 2.1] and conclude that $\alpha(x) \leq u(x)-\min _{\partial B} u<u(x)$ for every $x \in \bar{B}$. Thus we have proved that $u \gg \alpha$.
Example 2. Assume $p=1$. Then there exists $\lambda^{*}>0$ such that, for each $\lambda>\lambda^{*}$, problem (2) has at least one solution $u \gg 0$. The proof proceeds as in Example 1 still using [6].
Example 3. Assume $p \in] 1,+\infty\left[\right.$. Then there exists $\lambda^{*}>0$ such that, for each $\lambda>\lambda^{*}$, problem (2) has at least two solutions $u_{1}>u_{2} \gg 0$. As 0 is a (lower) solution, in order to get the conclusion, we have just to construct a strict lower solution $\alpha>0$ and a strict upper solution $\beta>0$ such that $\alpha \not \leq \beta$. The existence of a strict lower solution $\alpha>0$, for all sufficiently large $\lambda>0$, follows from the same argument as in Example 1, using again [6]. Let us prove the existence of a strict upper solution $\beta \gg 0$ with $\beta \nsupseteq \alpha$. Let ]a,b[ be the projection of $\Omega$ over the $x_{1}$-axis and set $g(s)=\lambda\left(s^{+}\right)^{p}$. It follows from the proof of [5, Theorem 2.5] that the equation

$$
-\left(v^{\prime} / \sqrt{1-v^{\prime 2}}\right)^{\prime}=g(v)
$$

has a sequence $\left(v_{k}\right)_{k}$ of solutions of class $C^{2}$ in $[a, b]$, such that $\min _{[a, b]} v_{k}>0$ for every $k$ and $\lim _{k \rightarrow+\infty}\left\|v_{k}\right\|_{\infty}=0$. For each $k$, define $\beta_{k}(x)=v_{k}\left(x_{1}\right)$ for all $x \in \bar{\Omega}$. Clearly, each $\beta_{k}$ is an upper solution of (2), with $\min _{\bar{\Omega}} \beta_{k}>0$ and, provided that
$k$ is large enough, $\beta_{k} \nsupseteq \alpha$. Using [1, Lemma 2.1] as in Example 1, we easily verify that $\beta_{k}$ is strict.
Example 4. Assume $f: \mathbb{R} \rightarrow \mathbb{R}$ is continuous and satisfies

$$
\liminf _{s \rightarrow 0^{+}} \frac{F(s)}{s^{2}}=0 \text { and } \limsup _{s \rightarrow 0^{+}} \frac{F(s)}{s^{2}}=+\infty
$$

where $F(s)=\int_{0}^{s} f(t) d t$. Then problem (1) has a sequence $\left(u_{k}\right)_{k}$ of solutions such that $u_{k}>0$ for every $k$ and $\lim _{k \rightarrow+\infty}\left\|u_{k}\right\|_{\infty}=0$. Let us first notice that $f(0)=0$ and hence $\alpha=0$ is a (lower) solution. As in Example 3 we construct a sequence $\left(\beta_{k}\right)_{k}$ of upper solution of (2), with $\min _{\bar{\Omega}} \beta_{k}>0$ for every $k$ and $\lim _{k \rightarrow+\infty}\left\|\beta_{k}\right\|_{\infty}=$ 0 . Then we use Proposition 3 to get, for each $k$, the existence of a solution $u_{k}$, with $0 \leq u_{k} \leq \beta_{k}$, maximizing the functional $\psi$ over the set $\left\{w \in \mathcal{C}: 0 \leq w \leq \beta_{k}\right\}$. Finally we construct, as in [10], a sequence of functions $\left(\zeta_{k}\right)_{k}$, with $\zeta_{k} \in \bar{C}_{0}^{1}(\bar{\Omega})$, $\left\|\nabla \zeta_{k}\right\|_{\infty}<1,0<\zeta_{k} \leq \beta_{k}$ and $\psi\left(\zeta_{k}\right)>\operatorname{meas}(\Omega)$ for every $k$. This allows us to conclude that $u_{k}>0$ for each $k$ and $\lim _{k \rightarrow+\infty}\left\|u_{k}\right\|_{\infty}=0$.

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