Positive solutions of the Dirichlet problem for the prescribed mean curvature equation in Minkowski space *

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Abstract

We prove the existence of multiple positive solutions of the Dirichlet problem for the prescribed mean curvature equation in Minkowski space

$$\begin{cases} -\mathrm{div}\Big(\nabla u/\sqrt{1-|\nabla u|^2}\Big) = f(x,u,\nabla u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Here Ω is a bounded regular domain in $\mathbb{R}^N$ and the function $f = f(x,s,\xi)$ is either sublinear, or superlinear, or sub-superlinear near $s = 0$. The proof combines topological and variational methods.

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1 Introduction

Hypersurfaces of prescribed mean curvature in Minkowski space, with coordinates $(x_1,\ldots,x_N,t)$ and metric $\sum_{i=1}^N dx_i^2 - dt^2$, are of interest in differential geometry and

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in general relativity. In this paper we are concerned with the existence of such a kind of hypersurfaces which are graphs of solutions of the Dirichlet problem

$$\begin{cases} -\mathrm{div}\Big(\nabla u/\sqrt{1-|\nabla u|^2}\Big) = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1}$$

We assume throughout that Ω is a bounded domain in $\mathbb{R}^N$, with a boundary $\partial\Omega$ of class C^2, and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies the Carathéodory conditions. By a solution of (1) we mean a function $u \in W^{2,r}(\Omega)$, for some $r > N$, with $\|\nabla u\|_\infty < 1$, which satisfies the equation a.e. in Ω and vanishes on $\partial\Omega$. These are strong strictly spacelike solutions of (1) according to the terminology of, e.g., [5, 15, 2, 10].

In [2] and [10] some general solvability results for (1) were proved under the assumption that the function f is globally bounded. Yet, as all spacelike solutions are uniformly bounded by the quantity $\frac{1}{2}\mathrm{d}(\Omega)$, with $\mathrm{d}(\Omega)$ the diameter of Ω, one can always reduce to that situation by truncation. Nevertheless it should be observed that, if one already knows that problem (1) admits zero as a solution, the results in [2] and [10] provide no further information. Therefore it may be interesting to investigate in such cases the existence of nontrivial, in particular positive, solutions. We point out that, while this topic has been largely discussed in the literature for the Dirichlet problem associated with various classes of semilinear and quasilinear elliptic equations (including the prescribed mean curvature equation in Euclidean space), no result seems to be available for problem (1), at least when Ω is a general domain in $\mathbb{R}^N$.

Our aim here is indeed to extend to a genuine PDE setting what has been obtained in [6], for the one-dimensional problem, and in [3], [4], [7], for the radially symmetric problem in a ball. Namely, we will discuss the existence and the multiplicity of positive solutions of (1), assuming that the function $f = f(x,s)$ is sublinear, or superlinear, or sub-superlinear near $s = 0$.

In order to describe our results in a simple fashion, let us write the function f in the form

$$f(x,s) = \lambda\, a(x)(s^+)^p + \mu\, b(x)(s^+)^q, \tag{2}$$

where λ, μ are non-negative real parameters, $a, b : \bar{\Omega} \to \mathbb{R}$ are continuous functions, and p, q are given exponents satisfying $0 < p \le 1 < q$. The coefficients a, b are assumed to be simultaneously positive at some point of Ω, but they are allowed to vanish in parts of Ω or to change sign. The following conclusions are then obtained.

Take $\mu = 0$ in (2). If the exponent $p \in\,]0,1[$ is fixed, we prove that (1) has a positive solution for every $\lambda > 0$. If $p = 1$, we show that (1) has a positive solution for all large $\lambda > 0$, whereas non-existence of positive solutions is shown to occur for all sufficiently small $\lambda > 0$. It is immediately seen that in both cases the existence of positive solutions is guaranteed, with the same choices of λ, for any given $\mu > 0$.

Next, take $\lambda = 0$ in (2). If the exponent $q \in\,]1,+\infty[$ is fixed, we prove that (1) has at least two positive solutions for all large $\mu > 0$. Non-existence of positive solutions is also established for all sufficiently small $\mu > 0$.

Lastly, take $\lambda > 0$ and $\mu > 0$ in (2). Let the exponents $p \in]0, 1[$ and $q \in]1, +\infty[$ be given. Then (1) has at least three positive solutions for every large $\mu > 0$ and all sufficiently small $\lambda > 0$.

We point out that in all these statements no restriction is placed on the range of the exponent q.

Our results should be compared with similar ones obtained in [8] for a class of semilinear problems, in [9] and in [14] for a class of quasilinear problems driven by the p-Laplace operator and the mean curvature operator in Euclidean space, respectively. In these papers some kinds of local analogues to the classical conditions of "sublinearity" and of "superlinearity" have been introduced, extending in various directions some of the results proved in the celebrated work by Ambrosetti, Brezis and Cerami [1]. We observe however that the multiplicity and the non-existence results we obtain for (1) are peculiar of this problem, due to the specific structure of the differential operator, and have no analogue in all the above mentioned cases.

We remark that, unlike in [6] and [7], our approach here is topological. This allows us to introduce a dependence on the gradient of the solution into the right-hand side f of the equation, so that we can replace (1) with

$$\begin{cases} -\operatorname{div}\Big(\nabla u/\sqrt{1-|\nabla u|^2}\Big) = f(x,u,\nabla u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3}$$

where again $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ satisfies the Carathéodory conditions. Of course, this problem has not anymore a variational structure. However, our construction of the open sets, where we evaluate the degree of the solution operator associated with (3), relies on the knowledge of the radially symmetric solutions of suitable comparison problems, whose existence is proved by a minimization argument in [7].

We finally notice that the solvability of problem (3) has been explicitly raised as an open question in the recent work [13].

Notation. We list some additional notation that will be used throughout this paper. For $s \in \mathbb{R}$ we write $s^+ = \max\{s, 0\}$ and $s^- = -\min\{s, 0\}$. We denote by $B_R(x_0)$, or simply by B if no disambiguation is needed, the open ball in $\mathbb{R}^N$ centered at x_0 and having radius R. For functions $u, v : E \to \mathbb{R}$, with E a subset of $\mathbb{R}^N$ having positive measure, we write $u \le v$ (in E) if $u(x) \le v(x)$ a.e. in E, and $u < v$ (in E) if $u \le v$ and $u(x) < v(x)$ in a subset of E having positive measure. A function u such that $u > 0$ is called positive. Assume that $\mathcal{O}$ is an open bounded set with a boundary $\partial\mathcal{O}$ of class C^1; for functions $u, v \in C^1(\bar{\mathcal{O}})$, we write $u \ll v$ (in $\bar{\mathcal{O}}$) if $u(x) < v(x)$ for every $x \in \mathcal{O}$ and, if $u(x) = v(x)$ for some $x \in \partial\mathcal{O}$, then $\frac{\partial v}{\partial \nu}(x) < \frac{\partial u}{\partial \nu}(x)$, where $\nu = \nu(x)$ denotes the unit outer normal to $\mathcal{O}$ at $x \in \partial\mathcal{O}$. A function u such that $u \gg 0$ is called strictly positive. We also set $C^1_0(\bar{\mathcal{O}}) = \{u \in C^1(\bar{\mathcal{O}}) : u = 0 \text{ on } \partial\mathcal{O}\}$. Finally, we denote by $\mathcal{I}$ the identity operator.

2 Preliminaries

We collect in this section some results that will be repeatedly used in the proof of our main result. We start with a comparison principle, which is a direct consequence of [2, Lemma 1.2].

Lemma 2.1. *Assume that $\mathcal{O}$ is a bounded domain in $\mathbb{R}^N$, with a boundary $\partial\mathcal{O}$ of class C^1. Suppose that $v_1, v_2 \in L^\infty(\mathcal{O})$ satisfy $v_1 \le v_2$. Let, for $i = 1, 2$, $u_i \in W^{2,r}(\mathcal{O})$, for some $r > N$, be such that $\|\nabla u_i\|_\infty < 1$ and*

$$-\mathrm{div}\Big(\nabla u/\sqrt{1-|\nabla u|^2}\Big) = v_i \quad \textit{a.e. in } \mathcal{O}.$$

Then

$$u_1 \le u_2 - \min_{\partial\mathcal{O}}(u_2 - u_1). \tag{4}$$

Proof. Fix $v \in L^\infty(\mathcal{O})$ and suppose that $u \in W^{2,r}(\mathcal{O})$, for some $r > N$, is such that $\|\nabla u\|_\infty < 1$ and

$$-\mathrm{div}\Big(\nabla u/\sqrt{1-|\nabla u|^2}\Big) = v \quad \text{a.e. in } \mathcal{O}. \tag{5}$$

Let us set

$$\mathcal{C}_u = \{z \in C^{0,1}(\bar{\mathcal{O}}) : \|\nabla z\|_\infty \le 1 \text{ and } z = u \text{ on } \partial\mathcal{O}\}$$

and define the functional $\mathcal{J}_v : \mathcal{C}_u \to \mathbb{R}$ by

$$\mathcal{J}_v(w) = \int_{\mathcal{O}} \sqrt{1-|\nabla w|^2}\, dx + \int_{\mathcal{O}} vw\, dx,$$

for all $w \in \mathcal{C}_u$. We claim that u maximizes $\mathcal{J}_v$ in $\mathcal{C}_u$. Indeed, pick any $z \in \mathcal{C}_u$; multiplying (5) by $u - z$ and integrating by parts, we get

$$\int_{\mathcal{O}} \frac{\nabla u \cdot \nabla(u-z)}{\sqrt{1-|\nabla u|^2}}\, dx = \int_{\mathcal{O}} v(u-z)\, dx. \tag{6}$$

By the concavity of the function $y \mapsto \sqrt{1-|y|^2}$, we obtain

$$\int_{\mathcal{O}} \sqrt{1-|\nabla z|^2}\, dx - \int_{\mathcal{O}} \sqrt{1-|\nabla u|^2}\, dx \le \int_{\mathcal{O}} \frac{\nabla u \cdot \nabla(u-z)}{\sqrt{1-|\nabla u|^2}}\, dx. \tag{7}$$

Combining (6) and (7) yields

$$\mathcal{J}_v(z) \le \mathcal{J}_v(u).$$

Accordingly, we have that u_1 and u_2 are maximizers of $\mathcal{J}_{v_1}$ in $\mathcal{C}_{u_1}$ and of $\mathcal{J}_{v_2}$ in $\mathcal{C}_{u_2}$, respectively. Hence Lemma 1.2 in [2] applies, implying that (4) holds. $\square$

Next we prove a well-posedness result, which is based on the gradient estimates obtained in [2, Corollary 3.4, Theorem 3.5].

Lemma 2.2. *Assume that $\mathcal{O}$ is a bounded domain in $\mathbb{R}^N$, with a boundary $\partial\mathcal{O}$ of class C^2, and suppose that $v \in L^\infty(\mathcal{O})$. Then the problem*

$$\begin{cases} -\mathrm{div}\Big(\nabla u/\sqrt{1-|\nabla u|^2}\Big) = v & \text{in } \mathcal{O}, \\ u = 0 & \text{on } \partial\mathcal{O} \end{cases} \tag{8}$$

has a unique solution $u \in W^{2,r}(\mathcal{O})$ for all finite $r \geq 1$. Moreover, for any given $\Lambda > 0$ and $r > N$, there exist constants $\vartheta = \vartheta(\mathcal{O}, \Lambda) \in]0,1[$ and $c = c(\mathcal{O}, \Lambda, r) > 0$ such that, for every $v \in L^\infty(\mathcal{O})$ with $\|v\|_\infty \leq \Lambda$, the following estimates hold:

$$\|\nabla u\|_\infty < 1 - \vartheta \tag{9}$$

and

$$\|u\|_{W^{2,r}} \leq c\,\|v\|_\infty. \tag{10}$$

Proof. Uniqueness. The uniqueness of solutions of (8) immediately follows from Lemma 2.1.

Existence. Let $\Lambda > 0$ and $r > N$ be fixed. Take a function $v \in L^\infty(\mathcal{O})$, with $\|v\|_\infty \leq \Lambda$. We first assume that v further satisfies $v \in C^{0,1}(\bar{\mathcal{O}})$. By [2, Corollary 3.4, Theorem 3.5] there exists a constant $\vartheta = \vartheta(\mathcal{O}, \Lambda) \in]0,1[$ such that any solution $u \in C^2(\mathcal{O}) \cap C^1(\bar{\mathcal{O}})$ of (8) satisfies (9) and $\|u\|_\infty < \frac{1}{2}\mathrm{d}(\Omega)$. Accordingly, we can modify the differential operator on the left of the equation in (8) in such a way that [12, Theorem 1] applies, yielding the existence of constants $\alpha = \alpha(\mathcal{O}, \Lambda) \in]0,1]$ and $c_1 = c_1(\mathcal{O}, \Lambda) > 0$ such that $u \in C^{1,\alpha}(\bar{\mathcal{O}})$ and

$$\|u\|_{C^{1,\alpha}} < c_1.$$

We can suppose α has been taken so small that $W^{2,r}(\mathcal{O})$ is compactly imbedded into $C^{1,\alpha}(\bar{\mathcal{O}})$; as a consequence, α and c_1 now depend on $\mathcal{O}, \Lambda$ and r too. Let us define

$$\mathcal{C} = \{w \in C^{1,\alpha}(\bar{\mathcal{O}}) : \|\nabla w\|_\infty < 1 - \vartheta, \|w\|_{C^{1,\alpha}} < c_1\}.$$

$\mathcal{C}$ is an open bounded subset of $C^{1,\alpha}(\bar{\mathcal{O}})$ with $0 \in \mathcal{C}$. Pick any $w \in \bar{\mathcal{C}}$ and set, for $i, j = 1, \ldots, N$,

$$a_{ij} = \delta_{ij} a(|\nabla w|^2) + 2a'(|\nabla w|^2)\partial_{x_i} w\, \partial_{x_j} w, \tag{11}$$

where δ_{ij} is the Kronecker delta and $a(s) = (1-s)^{-\frac{1}{2}}$. Consider the Dirichlet problem

$$\begin{cases} -\displaystyle\sum_{i,j=1}^N a_{ij}\partial_{x_i x_j} u = v & \text{in } \mathcal{O}, \\ u = 0 & \text{on } \partial\mathcal{O}. \end{cases} \tag{12}$$

Note that the coefficients a_{ij} belong to $C^{0,\alpha}(\bar{\mathcal{O}})$ and they are uniformly bounded in $C^{0,\alpha}(\bar{\mathcal{O}})$, with bound independent of $w \in \bar{\mathcal{C}}$ and ultimately depending only on

$\mathcal{O}$, Λ and r; moreover, the ellipticity constant can be taken equal to 1. According to the L^r-regularity theory [11, Theorem 9.15, Theorem 9.13], problem (12) has a unique solution $u \in W^{2,r}(\mathcal{O})$ (depending on v and w) and there exists a constant $c_2 = c_2(\mathcal{O}, \Lambda, r) > 0$ such that

$$\|u\|_{W^{2,r}} \leq c_2(\|u\|_{L^r} + \|v\|_{L^r}).$$

By [11, Theorem 9.1] there is also a constant $c_3 = c_3(\mathcal{O}, \Lambda, r) > 0$ such that

$$\|u\|_\infty \leq c_3 \|v\|_{L^r}.$$

Combining these two estimates yields

$$\|u\|_{W^{2,r}} \leq c\,\|v\|_{L^r} \tag{13}$$

for some constant $c = c(\mathcal{O}, \Lambda, r) > 0$ (depending only on the indicated quantities). Moreover, as $u \in C^{1,\alpha}(\bar{\mathcal{O}})$, $v \in C^{0,1}(\bar{\mathcal{O}})$ and $a_{i,j} \in C^{0,\alpha}(\bar{\mathcal{O}})$, for $i, j = 1, \dots, N$, the Schauder regularity theory [11, Corollary 6.9] applies locally and allows us to conclude that $u \in C^{2,\alpha}(\mathcal{O})$; hence, in particular, $u \in W^{2,r}(\mathcal{O}) \cap C^2(\mathcal{O})$.

Let us denote by $\mathcal{L} : \bar{\mathcal{C}} \to C^{1,\alpha}(\bar{\mathcal{O}})$ the operator which sends each $w \in \bar{\mathcal{C}}$ onto the unique solution $u \in C^{1,\alpha}(\bar{\mathcal{O}})$ of (12). Let us verify that $\mathcal{L}$ is completely continuous. We first prove that $\mathcal{L}$ has a relatively compact range. Let $(w_n)_n$ be a sequence in $\bar{\mathcal{C}}$. By (13) the sequence $(\mathcal{L}(w_n))_n$ is bounded in $W^{2,r}(\mathcal{O})$. Hence there exists a subsequence $(\mathcal{L}(w_{n_k}))_k$ which converges weakly in $W^{2,r}(\mathcal{O})$ and strongly in $C^{1,\alpha}(\bar{\mathcal{O}})$ to some $u \in W^{2,r}(\mathcal{O})$. The continuity can be verified as follows. Let $(w_n)_n$ be a sequence in $\bar{\mathcal{C}}$ converging in $C^{1,\alpha}(\bar{\mathcal{O}})$ to some $w \in \bar{\mathcal{C}}$. We want to prove that $(\mathcal{L}(w_n))_n$ converges in $C^{1,\alpha}(\bar{\mathcal{O}})$ to $\mathcal{L}(w)$. Let us consider any subsequence $(\mathcal{L}(w_{n_k}))_k$ of $(\mathcal{L}(w_n))_n$ and verify that it has a subsequence converging to $\mathcal{L}(w)$. Arguing as above, there exists a subsequence $(\mathcal{L}(w_{n_{k_j}}))_j$ which converges weakly in $W^{2,r}(\mathcal{O})$ and strongly in $C^{1,\alpha}(\bar{\mathcal{O}})$ to some $u \in W^{2,r}(\mathcal{O})$. As each $u_{n_{k_j}} = \mathcal{L}(w_{n_{k_j}})$ satisfies problem (12), we can pass to the limit, concluding that u is a solution of (12) and hence, by uniqueness, $u = \mathcal{L}(w)$.

We further observe that u is a solution of (8) if and only if u is a fixed point of $\mathcal{L}$. In order to prove the existence of a fixed point of $\mathcal{L}$, we show that every solution $u \in \bar{\mathcal{C}}$ of

$$u = t\mathcal{L}(u), \tag{14}$$

for some $t \in [0, 1]$, belongs to $\mathcal{C}$. Note that (14) is equivalent to

$$\begin{cases} -\mathrm{div}\Big(\nabla u / \sqrt{1 - |\nabla u|^2}\Big) = tv & \text{in } \mathcal{O}, \\ u = 0 & \text{on } \partial\mathcal{O}. \end{cases} \tag{15}$$

As $\|tv\|_\infty \leq \Lambda$ and $v \in C^{0,1}(\bar{\mathcal{O}})$, we conclude by the previous argument that any solution u of (15) is such that $u \in W^{2,r}(\mathcal{O}) \cap C^2(\mathcal{O})$, $\|\nabla u\|_\infty < 1 - \vartheta$, $\|u\|_{C^{1,\alpha}} < c_1$, and hence $u \in \mathcal{C}$. Accordingly, the Leray-Schauder continuation theorem yields the existence of a fixed point $u \in \mathcal{C}$ of $\mathcal{L}$ and therefore of a solution of (8), which satisfies (9) and (13).

The general case of a function $v \in L^\infty(\mathcal{O})$, with $\|v\|_\infty \leq \Lambda$, can be easily dealt with by approximation. Fix $r > N$ and let $(v_n)_n$ be sequence in $C^{0,1}(\bar{\mathcal{O}})$ converging to v in $L^r(\mathcal{O})$ and satisfying $\|v_n\|_\infty \leq \Lambda$ for all n. The corresponding solutions $(u_n)_n$ of (8) satisfy (9) and (13). Arguing as above, we can extract a subsequence of $(u_n)_n$ which converges weakly in $W^{2,r}(\mathcal{O})$ to a solution u of (8). Clearly, estimate (9) is valid, possibly reducing ϑ. By the weak lower semicontinuity of the $W^{2,r}$-norm, (13) and hence (10) hold true as well. □

By Lemma 2.2 we can define an operator $\mathcal{K} : L^\infty(\mathcal{O}) \to C^1_0(\bar{\mathcal{O}})$ which sends any function $v \in L^\infty(\mathcal{O})$ onto the unique solution $u \in C^1_0(\bar{\mathcal{O}})$ of (8). Arguing as in the proof of Lemma 2.2 the following statement can be proved.

Lemma 2.3. *Assume that $\mathcal{O}$ is a bounded domain in $\mathbb{R}^N$, with a boundary $\partial\mathcal{O}$ of class C^2. Then $\mathcal{K} : L^\infty(\mathcal{O}) \to C^1_0(\bar{\mathcal{O}})$ is completely continuous.*

The following results follow from the maximum principle.

Lemma 2.4. *Assume that $\mathcal{O}$ is a bounded domain in $\mathbb{R}^N$, with a boundary $\partial\mathcal{O}$ of class C^2. Then, for any given $\Lambda > 0$, there exists a constant $d = d(\mathcal{O}, \Lambda) > 0$ such that, for every $v \in L^\infty(\mathcal{O})$ with $\|v\|_\infty \leq \Lambda$, the solution u of (8) satisfies*

$$\|u^+\|_\infty \leq d\,\|v^+\|_\infty.$$

Proof. As already observed in the proof of Lemma 2.2, u satisfies (12), where now the coefficients a_{ij}, for $i, j = 1, \dots, N$, are given by (11) with w replaced by u. Then [11, Theorem 9.1] immediately yields the conclusion. □

Lemma 2.5. *Assume that $\mathcal{O}$ and $\mathcal{O}_0$ are bounded domains in $\mathbb{R}^N$, with boundaries $\partial\mathcal{O}, \partial\mathcal{O}_0$ of class C^2, satisfying $\bar{\mathcal{O}}_0 \subset \mathcal{O}$. Let $v \in L^\infty(\mathcal{O})$ be such that $v > 0$ in $\mathcal{O}_0$ and suppose that the solution u of (8) satisfies $u \geq 0$ in $\mathcal{O}$. Then $\min_{\bar{\mathcal{O}}_0} u > 0$.*

Proof. As already observed in the proof of Lemma 2.2, u satisfies (12). Then the strong maximum principle (see, e.g., [16, Theorem 3.27]) implies that $u(x) > 0$ for every $x \in \mathcal{O}_0$. Suppose that $u(x_0) = 0$ at some $x_0 \in \partial\mathcal{O}_0 \subset \mathcal{O}$. By the Hopf boundary lemma (see, e.g., [16, Lemma 3.26]), we have $\frac{\partial u}{\partial \nu}(x_0) < 0$, thus contradicting the assumption $u \in C^1(\bar{\mathcal{O}})$ and $u \geq 0$ in $\mathcal{O}$. □

Lemma 2.6. *Assume that $\mathcal{O}$ is a bounded domain in $\mathbb{R}^N$, with a boundary $\partial\mathcal{O}$ of class C^2. Fix a constant $k \geq 0$. Let $v \in L^\infty(\mathcal{O})$ be such that $v > 0$ in $\mathcal{O}$ and let u be a solution of*

$$\begin{cases} -\mathrm{div}\Big(\nabla u/\sqrt{1-|\nabla u|^2}\Big) + ku = v & \text{in } \mathcal{O}, \\ u = 0 & \text{on } \partial\mathcal{O}. \end{cases}$$

Then $u \gg 0$.

Proof. The conclusions follow as in the proof of Lemma 2.5 from the strong maximum principle and the Hopf boundary lemma. □

We conclude with an existence result for the radially symmetric problem taken from [7].

Proposition 2.7. *Let us consider the Dirichlet problem*

$$\begin{cases} -\mathrm{div}\Big(\nabla u/\sqrt{1-|\nabla u|^2}\Big) = \nu\, u^p & \text{in } B, \\ u = 0 & \text{on } \partial B, \end{cases} \tag{16}$$

where B is an open ball in $\mathbb{R}^N$ and $\nu > 0$, $p > 0$ are given. The following conclusions hold:

(*i*) *if $p \in]0,1[$, then for every $\nu > 0$ problem (16) has at least one (radially symmetric) solution $u \in C^2(\bar{B})$ satisfying $u \gg 0$ in $\bar{B}$;*

(*ii*) *if $p \geq 1$, then there exists $\nu^* > 0$ such that, for every $\nu > \nu^*$, problem (16) has at least one (radially symmetric) solution $u \in C^2(\bar{B})$ satisfying $u \gg 0$ in $\bar{B}$.*

Proof. Looking for radially symmetric solutions of (16) we consider the one-dimensional problem

$$\begin{cases} -\Big(r^{N-1}u'/\sqrt{1-(u')^2}\Big)' = \nu\, r^{N-1}u^p & \text{in }]0,R[, \\ u'(0) = 0, \quad u(R) = 0, \end{cases} \tag{17}$$

where R is the radius of the ball B. An a-priori estimate devised in [6] and [7] allows to reduce (17) to an equivalent non-singular problem. Then positive solutions can be found as minimizers of the associated action functional. In particular, if $p \in]0,1[$, applying [7, Proposition 3.4] yields the existence, for every $\nu > 0$, of a positive solution of (17). If $p \geq 1$, applying [7, Proposition 3.3] yields the existence of $\nu^* > 0$ such that, for every $\nu > \nu^*$, there is a positive solution of (17). These solutions give rise to positive solutions of (16). It is observed in [7, Remark 3.4] that all such solutions belong to $C^2(\bar{B})$. Finally, Lemma 2.6 implies that they are strictly positive in $\bar{B}$. □

3 Existence and multiplicity results

Let us consider the Dirichlet problem (3) with $f(x,s,\xi) = \lambda a(x,s,\xi) + \mu b(x,s,\xi)$, that is

$$\begin{cases} -\mathrm{div}\Big(\nabla u/\sqrt{1-|\nabla u|^2}\Big) = \lambda a(x,u,\nabla u) + \mu b(x,u,\nabla u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{18}$$

We assume that $\lambda \geq 0$, $\mu \geq 0$,

(h_1) *Ω is a bounded domain in $\mathbb{R}^N$, with a boundary $\partial\Omega$ of class C^2,*

and

(h_2) *$a, b : \Omega \times [0, \frac{1}{2}\mathrm{d}(\Omega)] \times \bar{B}_1(0) \to \mathbb{R}$ satisfy the Carathéodory conditions and*

$$\operatorname*{ess\,sup}_{\Omega\times[0,\frac{1}{2}\mathrm{d}(\Omega)]\times\bar{B}_1(0)} |a| < +\infty \qquad \text{and} \qquad \operatorname*{ess\,sup}_{\Omega\times[0,\frac{1}{2}\mathrm{d}(\Omega)]\times\bar{B}_1(0)} |b| < +\infty.$$

We look here for positive (strong strictly spacelike) solutions u of (18). We recall that u is positive if it is non-trivial and non-negative, i.e., $u > 0$. In some cases we will be able to show that it is strictly positive, i.e., $u \gg 0$.

The following assumptions will be considered:

(a_1) *there exist an open ball $B \subseteq \Omega$, $a_1 > 0$ and $p_1 \in\,]0, 1[$ such that $a_1 s^{p_1} \le a(x, s, \xi)$ for a.e. $x \in B$, all $s \in [0, \frac{1}{2}\mathrm{d}(\Omega)]$ and all $\xi \in \bar{B}_1(0)$;*

(a_2) *$0 \le a(x, 0, \xi)$ for a.e. $x \in \Omega$ and all $\xi \in \bar{B}_1(0)$;*

(a_3) *there exist $a_2 > 0$ and $p_2 \in\,]0, 1[$ such that $a(x, s, \xi) \le a_2 s^{p_2}$ for a.e. $x \in \Omega$, all $s \in [0, \frac{1}{2}\mathrm{d}(\Omega)]$ and all $\xi \in \bar{B}_1(0)$;*

(b_1) *there exist an open ball $B \subseteq \Omega$, $b_1 > 0$ and $q_1 \in [1, +\infty[$ such that $b_1 s^{q_1} \le b(x, s, \xi)$ for a.e. $x \in B$, all $s \in [0, \frac{1}{2}\mathrm{d}(\Omega)]$ and all $\xi \in \bar{B}_1(0)$;*

(b_2) *$0 \le b(x, 0, \xi)$ for a.e. $x \in \Omega$ and all $\xi \in \bar{B}_1(0)$;*

(b_3) *there exist $b_2 > 0$ and $q_2 \in\,]1, +\infty[$ such that $b(x, s, \xi) \le b_2 s^{q_2}$ for a.e. $x \in \Omega$, all $s \in [0, \frac{1}{2}\mathrm{d}(\Omega)]$ and all $\xi \in \bar{B}_1(0)$.*

Theorem 3.1. *Assume (h_1) and (h_2). The following conclusions hold:*

(i) if $\mu = 0$ and (a_1) and (a_2) are satisfied, then for every $\lambda > 0$ problem (18) has at least one positive solution;

(ii) if $\lambda = 0$ and (b_1) and (b_2) are satisfied, then there exists $\mu^* > 0$ such that, for every $\mu > \mu^*$, problem (18) has at least one positive solution;

(iii) if $\lambda = 0$ and (b_1), (b_2) and (b_3) are satisfied, then there exists $\mu^* > 0$ such that, for every $\mu > \mu^*$, problem (18) has at least two positive solutions;

(iv) if (a_1), (a_2), (a_3), (b_1), (b_2) and (b_3) are satisfied, B denoting the same ball in (a_1) and (b_1), then there exist $\mu^* > 0$ and a function $\lambda(\cdot) :]\mu^*, +\infty[\to \mathbb{R}$ such that, for every $\mu > \mu^*$ and all $\lambda \in\,]0, \lambda(\mu)[$, problem (18) has at least three positive solutions.

Proof. Step 1. An equivalent problem. Fix $\lambda \geq 0$ and $\mu \geq 0$. Assume (h_1), (h_2), and (a_2) in case $\lambda > 0$, (b_2) in case $\mu > 0$. Define the functions $\bar{a}, \bar{b}, \bar{f} : \Omega \times [-\frac{1}{2}\mathrm{d}(\Omega), \frac{1}{2}\mathrm{d}(\Omega)] \times \bar{B}_1(0) \to \mathbb{R}$ by setting, for a.e. $x \in \Omega$ and all $\xi \in \bar{B}_1(0)$,

$$\bar{a}(x, s, \xi) = \begin{cases} a(x, 0, \xi) & \text{if } -\frac{1}{2}\mathrm{d}(\Omega) \leq s < 0, \\ a(x, s, \xi) & \text{if } 0 \leq s \leq \frac{1}{2}\mathrm{d}(\Omega), \end{cases}$$

$$\bar{b}(x, s, \xi) = \begin{cases} b(x, 0, \xi) & \text{if } -\frac{1}{2}\mathrm{d}(\Omega) \leq s < 0, \\ b(x, s, \xi) & \text{if } 0 \leq s \leq \frac{1}{2}\mathrm{d}(\Omega), \end{cases}$$

and

$$\bar{f} = \lambda \bar{a} + \mu \bar{b}.$$

The modified functions $\bar{a}$ and $\bar{b}$ share the assumed properties of a and b, respectively.

Notice that any non-trivial solution u of the Dirichlet problem

$$\begin{cases} -\mathrm{div}\Big(\nabla u / \sqrt{1 - |\nabla u|^2}\Big) = \lambda \bar{a}(x, u, \nabla u) + \mu \bar{b}(x, u, \nabla u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{19}$$

is positive. Indeed, as $\bar{f}(x, s, \xi) \geq 0$ for a.e. $x \in \Omega$, all $s \in [-\frac{1}{2}\mathrm{d}(\Omega), 0]$ and all $\xi \in \bar{B}_1(0)$, multiplying the equation in (19) by $u^- \in C^{0,1}(\bar{\Omega})$ and integrating by parts, we have

$$0 \leq \int_\Omega \bar{f}(x, u, \nabla u) u^- \, dx = \int_\Omega \frac{\nabla u \cdot \nabla(u^-)}{\sqrt{1 - |\nabla u|^2}} \, dx = -\int_\Omega \frac{|\nabla(u^-)|^2}{\sqrt{1 - |\nabla(u^-)|^2}} \, dx \leq 0$$

and hence $\nabla(u^-) = 0$ a.e. in Ω. As $u^- = 0$ on $\partial\Omega$, we conclude that $u^- = 0$ in Ω. Therefore a function u is a positive solution of (18) if and only if it is a non-trivial solution of (19).

We set

$$\mathcal{D} = \{u \in C^1_0(\bar{\Omega}) : \|\nabla u\|_\infty < 1\}$$

and let $\mathcal{N}_{\lambda,\mu} : \bar{\mathcal{D}} \to L^\infty(\Omega)$ be the superposition operator associated with $\bar{f}$, that is,

$$\mathcal{N}_{\lambda,\mu}(u) = \bar{f}(\cdot, u, \nabla u).$$

By (h_2) $\mathcal{N}_{\lambda,\mu}$ is continuous and has a bounded range. Hence, by Lemma 2.3, the operator $\mathcal{T}_{\lambda,\mu} : \bar{\mathcal{D}} \to C^1_0(\bar{\Omega})$, defined by

$$\mathcal{T}_{\lambda,\mu} = \mathcal{K} \circ \mathcal{N}_{\lambda,\mu},$$

is completely continuous. Clearly, a function u is a positive solution of (18) if and only if $u \in \mathcal{D}$ and is a non-trivial fixed point of $\mathcal{T}_{\lambda,\mu}$.

Step 2. Proof of (i). Take $\mu = 0$ and fix $\lambda > 0$. Assume (h_1), (h_2), (a_1) and (a_2). For sake of simplicity, the operators $\mathcal{N}_{\lambda,0}$ and $\mathcal{T}_{\lambda,0}$ will be denoted by $\mathcal{N}$ and $\mathcal{T}$, respectively. Set

$$\Lambda_a = \lambda \|\bar{a}\|_\infty$$

and let $\vartheta_a \in]0,1[$ be the constant $\vartheta_a = \vartheta$ introduced in Lemma 2.2, with $\Lambda = \Lambda_a$.

Let us consider the Dirichlet problem

$$\begin{cases} -\mathrm{div}\Big(\nabla u/\sqrt{1-|\nabla u|^2}\Big) = \lambda a_1 u^{p_1} & \text{in } B, \\ u = 0 & \text{on } \partial B, \end{cases} \tag{20}$$

with B, a_1 and p_1 defined in (a_1). Without restrictions, we can suppose that $\bar{B} \subset \Omega$. By Proposition 2.7 there exists a solution $\alpha \in C^2(\bar{B})$ of (20) satisfying $\|\alpha\|_\infty \leq \frac{1}{2}\mathrm{d}(\Omega)$ and $\alpha \gg 0$ in $\bar{B}$. Let us extend α to a function $\tilde{\alpha} \in C^1(\bar{\Omega})$ satisfying $\|\nabla\tilde{\alpha}\|_\infty \leq 1$ and

$$-\tfrac{1}{2}\mathrm{d}(\Omega) < \tilde{\alpha}(x) < 0$$

for all $x \in \bar{\Omega} \setminus \bar{B}$. We define the open bounded subset of $C^1_0(\bar{\Omega})$

$$\mathcal{U}_0 = \{u \in C^1_0(\bar{\Omega}) : \, u \gg \tilde{\alpha}, \, \|\nabla u\|_\infty < 1 - \vartheta_a\}$$

and $v_0 \in L^\infty(\Omega)$ by setting, for a.e. $x \in \Omega$,

$$v_0(x) = \lambda \bar{a}\big(x, \tilde{\alpha}(x), \nabla\tilde{\alpha}(x)\big).$$

Observe that $v_0 \geq 0$ in Ω, by definition of $\bar{a}$, $v_0 > 0$ in B, by (a_1), and $\|v_0\|_\infty \leq \Lambda_a$. Let z_0 be the solution of the Dirichlet problem

$$\begin{cases} -\mathrm{div}\Big(\nabla u/\sqrt{1-|\nabla u|^2}\Big) = v_0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Notice that, by Lemma 2.6, $z_0 \gg 0$ in $\bar{\Omega}$.

Claim. $\mathcal{T}$ *has no fixed points on* $\partial\mathcal{U}_0$ *and*

$$\deg(\mathcal{I} - \mathcal{T}, \mathcal{U}_0, 0) = 1.$$

We first prove that

$$\deg(\mathcal{I} - z_0, \mathcal{U}_0, 0) = 1.$$

It suffices to show that z_0 belongs to $\mathcal{U}_0$. The condition $\|\nabla z_0\|_\infty < 1 - \vartheta_a$ is satisfied by the definition of ϑ_a. It remains to prove that $z_0 \gg \tilde{\alpha}$ in $\bar{\Omega}$. Since $z_0 \geq 0$ in Ω and $\tilde{\alpha}(x) < 0$ for all $x \in \bar{\Omega} \setminus \bar{B}$, we only need to verify that $z_0(x) > \alpha(x)$ for all $x \in \bar{B}$. Since $z_0 \gg 0$ in $\bar{\Omega}$, we have $\min_{\bar{B}} z_0 > 0$. Moreover, as $\lambda a_1 \alpha^{p_1} \leq v_0$ in B by (a_1), we get, by Lemma 2.1,

$$\alpha(x) \leq z_0(x) - \min_{\partial B} z_0 < z_0(x),$$

for all $x \in \bar{B}$.

Next we consider the homotopy $\mathcal{H} : [0,1] \times \bar{\mathcal{D}} \to C^1_0(\bar{\Omega})$ defined by

$$\mathcal{H}(t,u) = \mathcal{K}\big(t\mathcal{N}(u) + (1-t)v_0\big).$$

By the properties of the operators $\mathcal{K}$ and $\mathcal{N}$, $\mathcal{H}$ is completely continuous. Observe that

$$\mathcal{H}(0,u) = z_0 \quad \text{and} \quad \mathcal{H}(1,u) = \mathcal{T}(u),$$

for all $u \in \bar{\mathcal{D}}$. Fix now $t \in [0,1]$ and suppose that $u \in \bar{\mathcal{U}}_0$ is a fixed point of $\mathcal{H}(t,\cdot)$. We will prove that $u \in \mathcal{U}_0$. Since u is a fixed point of $\mathcal{H}(t,\cdot)$, u is a solution of

$$\begin{cases} -\operatorname{div}\Big(\nabla u/\sqrt{1-|\nabla u|^2}\Big) = t\lambda \bar{a}(x,u,\nabla u) + (1-t)v_0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Observe that

$$t\lambda\bar{a}(x,s,\xi) + (1-t)v_0(x) \geq 0$$

for a.e. $x \in \Omega$, all $s \in [-\frac{1}{2}\mathrm{d}(\Omega), 0]$ and all $\xi \in \bar{B}_1(0)$. Arguing as in Step 1, we see that $u \geq 0$ in Ω. Moreover, as

$$t\lambda\bar{a}(\cdot,u,\nabla u) + (1-t)v_0 > 0$$

in B, by Lemma 2.5 we deduce that

$$\min_{\bar{B}} u > 0. \tag{21}$$

Let us prove that $u \gg \tilde{\alpha}$ in $\bar{\Omega}$. As above we observe that, since $u \geq 0$ in Ω and $\tilde{\alpha}(x) < 0$ for all $x \in \bar{\Omega} \setminus \bar{B}$, we only need to verify that $u(x) > \alpha(x)$ for all $x \in \bar{B}$. Note that, using (a_1) and $u \in \bar{\mathcal{U}}_0$, we have

$$t\lambda\bar{a}(\cdot,u,\nabla u) + (1-t)v_0 \geq t\lambda a_1 u^{p_1} + (1-t)\lambda\bar{a}\big(\cdot,\alpha,\nabla\alpha\big) \geq \lambda a_1 \alpha^{p_1}$$

in B. Applying Lemma 2.1 and recalling (21), we get

$$\alpha(x) \leq u(x) - \min_{\partial B} u < u(x)$$

for all $x \in \bar{B}$.

Furthermore, as

$$\|t\lambda\bar{a}(\cdot,u,\nabla u) + (1-t)v_0\|_\infty \leq \Lambda_a,$$

Lemma 2.2 yields

$$\|\nabla u\|_\infty < 1 - \vartheta_a.$$

In conclusion, $u \in \mathcal{U}_0$. The homotopy invariance of the degree implies that

$$\deg(\mathcal{I} - \mathcal{T}, \mathcal{U}_0, 0) = \deg(\mathcal{I} - z_0, \mathcal{U}_0, 0) = 1.$$

This concludes the proof of the claim.

Therefore, for every $\lambda > 0$, there exists a non-trivial fixed point u of the operator $\mathcal{T}$ in $\mathcal{U}_0$, i.e., there exists a positive solution u of (18) satisfying $u \gg \tilde{\alpha}$ in $\bar{\Omega}$.

Step 3. Proof of (ii). The proof is essentially the same as the proof of (i) in Step 2. Take $\lambda = 0$ and $\mu > 0$. Assume (h_1), (h_2), (b_1) and (b_2). For sake of simplicity, the operators $\mathcal{N}_{0,\mu}$ and $\mathcal{T}_{0,\mu}$ will be denoted simply by $\mathcal{N}$ and $\mathcal{T}$, respectively. Set

$$\Lambda_b = \mu \|\bar{b}\|_\infty$$

and let $\vartheta_b \in]0,1[$ be the constant $\vartheta_b = \vartheta$ introduced in Lemma 2.2, with $\Lambda = \Lambda_b$.

Let us consider the Dirichlet problem

$$\begin{cases} -\mathrm{div}\Big(\nabla u / \sqrt{1 - |\nabla u|^2}\Big) = \mu b_1 u^{q_1} & \text{in } B, \\ u = 0 & \text{on } \partial B, \end{cases} \tag{22}$$

with B, b_1 and q_1 defined in (b_1). Again, we can suppose that $\bar{B} \subset \Omega$. By Proposition 2.7 there exists a constant $\mu^* > 0$ such that, for any $\mu > \mu^*$, problem (22) has at least one solution $\alpha_1 \in C^2(\bar{B})$ satisfying $\alpha_1 \gg 0$ in $\bar{B}$ and $\|\alpha_1\|_\infty \le \frac{1}{2}\mathrm{d}(\Omega)$. As in Step 2, we extend α_1 to a function $\tilde{\alpha}_1 \in C^1(\bar{\Omega})$ satisfying $\|\nabla \tilde{\alpha}_1\|_\infty \le 1$ and

$$-\tfrac{1}{2}\mathrm{d}(\Omega) < \tilde{\alpha}_1(x) < 0$$

for all $x \in \bar{\Omega} \setminus \bar{B}$. We define the open bounded set

$$\mathcal{U}_1 = \{u \in C^1_0(\bar{\Omega}) : \, u \gg \tilde{\alpha}_1, \, \|\nabla u\|_\infty < 1 - \vartheta_b\}$$

and $v_1 \in L^\infty(\Omega)$ by setting, for a.e. $x \in \Omega$,

$$v_1(x) = \mu \bar{b}\big(x, \tilde{\alpha}_1(x), \nabla \tilde{\alpha}_1(x)\big).$$

The proof continues exactly as in Step 2, by showing that $\mathcal{T}$ has no fixed points on $\partial \mathcal{U}_1$ and

$$\deg(\mathcal{I} - \mathcal{T}, \mathcal{U}_1, 0) = 1.$$

Therefore we conclude that, for all $\mu > \mu^*$, there exists a non-trivial fixed point u of $\mathcal{T}$ in $\mathcal{U}_1$, i.e., there exists a positive solution u of (18) satisfying $u \gg \tilde{\alpha}_1$ in $\bar{\Omega}$.

Step 4. Proof of (iii). Take $\lambda = 0$ and $\mu > 0$. Assume (h_1), (h_2), (b_1), (b_2) and (b_3). Note that (b_1) and (b_3) together imply $q_1 > 1$. As in Step 3, the operators $\mathcal{N}_{0,\mu}$ and $\mathcal{T}_{0,\mu}$ will be denoted simply by $\mathcal{N}$ and $\mathcal{T}$, respectively. Let μ^* be the constant, whose existence was proved in Step 3, such that problem (18) has at least one positive solution for all $\mu > \mu^*$. Fix $\mu > \mu^*$ and let $u_1 \in \mathcal{U}_1$ be a corresponding solution. Let us prove the existence of a second positive solution.

For each $r > 0$ we set

$$\mathcal{U}_2^r = \{u \in C^1_0(\bar{\Omega}) : \, \|u\|_\infty < r, \, \|\nabla u\|_\infty < 1 - \vartheta_b\},$$

with ϑ_b defined in Step 3.

Claim. There exists $\hat{r} > 0$ such that, for each $r \in \,]0, \hat{r}]$, $\mathcal{T}$ has no fixed points on $\partial\mathcal{U}_2^r$ and

$$\deg\bigl(\mathcal{I} - \mathcal{T}, \mathcal{U}_2^r, 0\bigr) = 1.$$

Consider the homotopy $\mathcal{H} : [0,1] \times \bar{\mathcal{D}} \to C_0^1(\bar{\Omega})$ defined by

$$\mathcal{H}(t,u) = \mathcal{K}\bigl(t\mathcal{N}(u)\bigr).$$

By the properties of the operators $\mathcal{K}$ and $\mathcal{N}$, $\mathcal{H}$ is completely continuous. We have

$$\mathcal{H}(0,u) = 0 \quad \text{and} \quad \mathcal{H}(1,u) = \mathcal{T}(u),$$

for all $u \in \bar{\mathcal{D}}$. Fix $t \in [0,1]$ and suppose that $u \in \bar{\mathcal{U}_2^r}$ is a fixed point of $\mathcal{H}(t,\cdot)$. We will prove that $u \in \mathcal{U}_2^r$. Since u is a fixed point of $\mathcal{H}(t,\cdot)$, u is a solution of

$$\begin{cases} -\mathrm{div}\Bigl(\nabla u/\sqrt{1 - |\nabla u|^2}\Bigr) = t\mu\bar{b}(x, u, \nabla u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{23}$$

Multiplying the equation in (23) by u and integrating by parts, we obtain by (b_3)

$$\begin{aligned} \|\nabla u\|_{L^2}^2 &\le \int_\Omega \frac{\nabla u \cdot \nabla u}{\sqrt{1 - |\nabla u|^2}}\, dx = \int_\Omega t\mu\bar{b}(x, u, \nabla u) u \, dx \\ &\le \mu b_2 \int_\Omega |u|^{q_2+1}\, dx \le \mu b_2 r^{q_2 - 1} \int_\Omega u^2\, dx \le \mu b_2 c_P r^{q_2-1} \|\nabla u\|_{L^2}^2 , \end{aligned}$$

where $c_P > 0$ is the Poincaré constant. Hence there exists $\hat{r} > 0$ sufficiently small such that, for every $r \in \,]0, \hat{r}]$, we have $\|\nabla u\|_{L^2} = 0$ and therefore $u = 0$. The homotopy invariance of the degree implies that

$$\deg(\mathcal{I} - \mathcal{T}, \mathcal{U}_2^r, 0) = 1.$$

This concludes the proof of the claim.

We finally set

$$\mathcal{U}_3 = \{u \in C_0^1(\bar{\Omega}) : \|\nabla u\|_\infty < 1 - \vartheta_b\}.$$

Using the definition of ϑ_b and arguing as above, we easily see that

$$\deg(\mathcal{I} - \mathcal{T}, \mathcal{U}_3, 0) = 1.$$

Let us fix $r \in \,]0, \min\{\|\tilde{\alpha}_1\|_\infty, \hat{r}\}]$, with $\tilde{\alpha}_1$ defined in Step 3. Notice that the sets $\mathcal{U}_1$ and $\mathcal{U}_2^r$ previously defined are disjoint and both contained in $\mathcal{U}_3$. Let us define

$$\mathcal{W}^r = \mathcal{U}_3 \setminus (\overline{\mathcal{U}_1 \cup \mathcal{U}_2^r}).$$

As $\mathcal{T}$ has no fixed point in $\partial\mathcal{U}_1 \cup \partial\mathcal{U}_2^r \cup \partial\mathcal{U}_3$, by the excision and the additivity properties of the degree, we have

$$\begin{aligned}\deg(\mathcal{I}-\mathcal{T},\mathcal{U}_3,0) &= \deg(\mathcal{I}-\mathcal{T},\mathcal{U}_3 \setminus (\partial\mathcal{U}_1 \cup \partial\mathcal{U}_2^r),0)\\ &= \deg(\mathcal{I}-\mathcal{T},\mathcal{U}_1,0) + \deg(\mathcal{I}-\mathcal{T},\mathcal{U}_2^r,0) + \deg(\mathcal{I}-\mathcal{T},\mathcal{W}^r,0)\end{aligned}$$

and hence

$$\deg(\mathcal{I}-\mathcal{T},\mathcal{W}^r,0) = -1.$$

In particular, there exists a fixed point u_2 of $\mathcal{T}$ such that $\|u_2\|_\infty > r$ and for which the condition $u_2 \gg \tilde{\alpha}_1$ in $\bar{\Omega}$ does not hold. Therefore u_2 is a positive solution of (18) which differs from u_1. We conclude that, for all $\mu > \mu^*$, there exist at least two positive solutions of (18).

Step 5. Proof of (iv). Take $\lambda > 0$ and $\mu > 0$. Assume (h_1), (h_2), (a_1), (a_2), (a_3), (b_1), (b_2) and (b_3), B denoting the same ball in (a_1) and (b_1). Suppose also that $\bar{B} \subset \Omega$. As already noticed in Step 4, we have $q_1 > 1$. Let μ^* be the constant, introduced in Step 3, such that problem (18), with $\lambda = 0$, has at least one positive solution for all $\mu > \mu^*$. Fix $\mu > \mu^*$, set

$$\Lambda = \|\bar{a}\|_\infty + \mu\|\bar{b}\|_\infty$$

and let $\vartheta \in \,]0,1[$ be the constant introduced in Lemma 2.2. Let us take an open ball B_2, with $\bar{B}_2 \subset B$, and consider, for $\lambda \in \,]0,1]$, the Dirichlet problem

$$\begin{cases} -\mathrm{div}\Big(\nabla u/\sqrt{1-|\nabla u|^2}\Big) = \lambda a_1 u^{p_1} & \text{in } B_2,\\ u = 0 & \text{on } \partial B_2. \end{cases} \tag{24}$$

By Proposition 2.7 there exists a solution $\alpha_2^\lambda \in C^2(\bar{B}_2)$ of (24) satisfying $\alpha_2^\lambda \gg 0$ in $\bar{B}_2$ and $\|\alpha_2^\lambda\|_\infty \le \frac{1}{2}\mathrm{d}(\Omega)$.

Fix $r > N$ and denote by $c_0' > 0$ the constant, dependent on Λ, B_2 and r, whose existence follows from Lemma 2.2, with $\mathcal{O} = B_2$, such that

$$\|u\|_\infty \le c_0'\|v\|_\infty$$

holds for all $v \in L^\infty(B_2)$ satisfying $\|v\|_\infty \le \Lambda$. Similarly, denote by $c_0'' > 0$ the constant, dependent on Λ, Ω and r, whose existence follows from Lemma 2.2, with $\mathcal{O} = \Omega$, such that

$$\|u\|_\infty \le c_0''\|v\|_\infty$$

holds for any $v \in L^\infty(\Omega)$ satisfying $\|v\|_\infty \le \Lambda$. Set

$$c_1 = \Lambda \max\{c_0', c_0''\}$$

and

$$r_\lambda = \lambda(c_1 + 1).$$

Observe that, since by (a_1)

$$\|a_1(\alpha_2^\lambda)^{p_1}\|_\infty \leq \Lambda,$$

we have

$$\|\alpha_2^\lambda\|_\infty \leq c_0'\Lambda\lambda \leq c_1\lambda.$$

As in Step 3, let α_1 be a solution of the Dirichlet problem

$$\begin{cases} -\mathrm{div}\Big(\nabla u/\sqrt{1-|\nabla u|^2}\Big) = \mu b_1 u^{q_1} & \text{in } B, \\ u = 0 & \text{on } \partial B. \end{cases}$$

Since $\alpha_1 \gg 0$ in $\bar{B}$, we have $\min\limits_{\bar{B}_2} \alpha_1 > 0$. Therefore we can take $\bar{\lambda} \in]0,1]$ such that, for all $\lambda \in]0, \bar{\lambda}[$,

$$r_\lambda < \min_{\bar{B}_2} \alpha_1.$$

For all $\lambda \in]0, \bar{\lambda}[$ we extend α_1 to a function $\tilde{\alpha}_1^\lambda \in C^1(\bar{\Omega})$ and α_2^λ to a function $\tilde{\alpha}_2^\lambda \in C^1(\bar{\Omega})$ such that $\|\nabla\tilde{\alpha}_1^\lambda\|_\infty \leq 1$, $\|\nabla\tilde{\alpha}_2^\lambda\|_\infty \leq 1$,

$$\|\tilde{\alpha}_2^\lambda\|_{L^\infty(\Omega)} \leq r_\lambda,$$

$$-\tfrac{1}{2}\mathrm{d}(\Omega) < \tilde{\alpha}_2^\lambda(x) < \tilde{\alpha}_1^\lambda(x) < 0$$

for all $x \in \bar{\Omega} \setminus \bar{B}$ and

$$-\tfrac{1}{2}\mathrm{d}(\Omega) < \tilde{\alpha}_2^\lambda(x) < 0$$

for all $x \in \bar{B} \setminus \bar{B}_2$.

We define, for every $\lambda \in]0, \bar{\lambda}[$, the open bounded sets

$$\mathcal{V}_1^\lambda = \{u \in C_0^1(\bar{\Omega}) : u \gg \tilde{\alpha}_1^\lambda,\ \|\nabla u\|_\infty < 1 - \vartheta\}$$

and

$$\mathcal{V}_2^\lambda = \{u \in C_0^1(\bar{\Omega}) : u \gg \tilde{\alpha}_2^\lambda,\ \|u\|_\infty < r_\lambda,\ \|\nabla u\|_\infty < 1 - \vartheta\}.$$

We also set, for a.e. $x \in \Omega$,

$$v_1^\lambda(x) = \mu\bar{b}\big(x, \tilde{\alpha}_1^\lambda(x), \nabla\tilde{\alpha}_1^\lambda(x)\big)$$

and

$$v_2^\lambda(x) = \lambda\bar{a}\big(x, \tilde{\alpha}_2^\lambda(x), \nabla\tilde{\alpha}_2^\lambda(x)\big).$$

For every $\lambda \in]0, \bar{\lambda}[$ let z_1^λ be the solution of the Dirichlet problem

$$\begin{cases} -\mathrm{div}\Big(\nabla u/\sqrt{1-|\nabla u|^2}\Big) = v_1^\lambda & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Arguing as in the Claim of Step 2, we easily verify that $\mathcal{T}_{\lambda,\mu}$ has no fixed points on $\partial\mathcal{V}_1^\lambda$ and

$$\deg(\mathcal{I} - \mathcal{T}_{\lambda,\mu}, \mathcal{V}_1^\lambda, 0) = 1.$$

For every $\lambda \in \,]0, \bar{\lambda}[$ let z_2^λ be the solution of the Dirichlet problem

$$\begin{cases} -\mathrm{div}\Big(\nabla u/\sqrt{1-|\nabla u|^2}\Big) = v_2^\lambda & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Claim. There exists $\lambda(\mu) \in \,]0, \bar{\lambda}]$ such that, for all $\lambda \in \,]0, \lambda(\mu)[$, $\mathcal{T}_{\lambda,\mu}$ has no fixed points on $\partial\mathcal{V}_2^\lambda$ and

$$\deg(\mathcal{I} - \mathcal{T}_{\lambda,\mu}, \mathcal{V}_2^\lambda, 0) = 1.$$

We first prove that

$$\deg(\mathcal{I} - z_2^\lambda, \mathcal{V}_2^\lambda, 0) = 1.$$

It suffices to show that $z_2^\lambda \in \mathcal{V}_2^\lambda$. Arguing as in Step 2, we easily see that $\|\nabla z_2^\lambda\|_\infty < 1 - \vartheta$ and $z_2^\lambda \gg \alpha_2^\lambda$ in $\bar{\Omega}$. Furthermore we have, as remarked above,

$$\|z_2^\lambda\|_\infty \le c_0''\Lambda\lambda \le c_1\lambda < r_\lambda.$$

Next we consider the homotopy $\mathcal{H} : [0,1] \times \bar{\mathcal{D}} \to C_0^1(\bar{\Omega})$ defined by

$$\mathcal{H}(t,u) = \mathcal{K}\big(t\mathcal{N}_{\lambda,\mu}(u) + (1-t)v_2^\lambda\big).$$

By the properties of the operators $\mathcal{K}$ and $\mathcal{N}_{\lambda,\mu}$, $\mathcal{H}$ is completely continuous. Observe that

$$\mathcal{H}(0,u) = z_2^\lambda \quad \text{and} \quad \mathcal{H}(1,u) = \mathcal{T}_{\lambda,\mu}(u),$$

for all $u \in \bar{\mathcal{D}}$.

Fix now $t \in [0,1]$ and suppose that $u \in \bar{\mathcal{V}}_2^\lambda$ is a fixed point of $\mathcal{H}(t,\cdot)$. We will prove that $u \in \mathcal{V}_2^\lambda$. Arguing as in Step 2 we easily verify that $u \ge 0$ in Ω and

$$\min_{\bar{B}_2} u > 0. \tag{25}$$

Let us prove that $u \gg \tilde{\alpha}_2^\lambda$ in $\bar{\Omega}$. Since $u \ge 0$ in Ω and $\tilde{\alpha}_2^\lambda(x) < 0$ for all $x \in \bar{\Omega} \setminus \bar{B}_2$, we only need to verify that $u(x) > \alpha_2^\lambda(x)$ for all $x \in \bar{B}_2$. Note that

$$t\bar{f}(\cdot, u, \nabla u) + (1-t)v_2^\lambda \ge t\lambda a_1 u^{p_1} + (1-t)\lambda\bar{a}\big(\cdot, \alpha_2^\lambda, \nabla\alpha_2^\lambda\big) \ge \lambda a_1(\alpha_2^\lambda)^{p_1}$$

in B_2. Applying Lemma 2.1 and recalling that (25) holds, we get

$$\alpha_2^\lambda(x) \le u(x) - \min_{\partial B_2} u < u(x)$$

for all $x \in \bar{B}_2$.

Furthermore, as

$$\left\| t\bar{f}(\cdot, u, \nabla u) + (1-t)v_2^\lambda \right\|_\infty \le \Lambda,$$

Lemma 2.2 yields

$$\|\nabla u\|_\infty < 1 - \vartheta.$$

Finally, we verify that $\|u\|_\infty < r_\lambda$ if λ is sufficiently small. Since both

$$\|u\|_\infty \le r_\lambda$$

and

$$\|\tilde{\alpha}_2^\lambda\|_{L^\infty(\Omega)} \le r_\lambda$$

hold, we have

$$\begin{aligned} t\bar{f}(\cdot, u, \nabla u) + (1-t)v_2^\lambda &\le t\big(\lambda a_2 \|u\|_\infty^{p_2} + \mu b_2 \|u\|_\infty^{q_2}\big) + (1-t)\lambda a_2 \|\tilde{\alpha}_2^\lambda\|_\infty^{p_2} \\ &\le t\big(a_2(c_1+1)^{p_2}\lambda^{p_2+1} + \mu b_2 (c_1+1)^{q_2}\lambda^{q_2}\big) + (1-t)a_2(c_1+1)^{p_2}\lambda^{p_2+1} \\ &\le a_2(c_1+1)^{p_2}\lambda^{p_2+1} + \mu b_2(c_1+1)^{q_2}\lambda^{q_2} \le c_2\lambda^{1+\varepsilon} \end{aligned}$$

in Ω, where $c_2 > 0$ is a constant independent of λ and $\varepsilon = \min\{p_2, q_2 - 1\}$. Applying Lemma 2.4, we obtain

$$\|u\|_\infty \le c_3\lambda^{1+\varepsilon},$$

where $c_3 > 0$ is a constant independent of λ. Let $\lambda(\mu) \in \,]0, \bar{\lambda}[$ be such that $\lambda(\mu) \le \left(\frac{c_1+1}{c_3}\right)^{\frac{1}{\varepsilon}}$. Then, for each $\lambda \in \,]0, \lambda(\mu)[$, the inequality $\|u\|_\infty < r_\lambda$ holds and, hence, $u \in \mathcal{V}_2^\lambda$. The homotopy invariance of the degree implies then that

$$\deg(\mathcal{I} - \mathcal{T}_{\lambda,\mu}, \mathcal{V}_2^\lambda, 0) = 1.$$

This concludes the proof of the claim.

Observe that $\mathcal{V}_1^\lambda$ and $\mathcal{V}_2^\lambda$ are disjoint because of the choice of λ. Therefore problem (18) has at least two positive solutions u_1 and u_2, such that $u_1 \gg \tilde{\alpha}_1^\lambda$ and $\|u_2\|_\infty < r_\lambda$. To conclude the proof we define, for all $\lambda \in \,]0, \lambda(\mu)[$,

$$\mathcal{V}_3^\lambda = \{u \in C_0^1(\bar{\Omega}) : u \gg \tilde{\alpha}_2^\lambda,\ \|\nabla u\|_\infty < 1 - \vartheta\}.$$

We also set

$$\mathcal{W}^\lambda = \mathcal{V}_3^\lambda \setminus (\overline{\mathcal{V}_1^\lambda \cup \mathcal{V}_2^\lambda}).$$

Fix $\lambda \in \,]0, \lambda(\mu)[$. Arguing as in the first part of the previous claim, we easily verify that

$$\deg(\mathcal{I} - \mathcal{T}_{\lambda,\mu}, \mathcal{V}_3^\lambda, 0) = 1.$$

By the excision and the additivity properties of the degree, we obtain

$$\deg(\mathcal{I} - \mathcal{T}_{\lambda,\mu}, \mathcal{W}^\lambda, 0) = -1.$$

In particular, there exists a fixed point u_3 of $\mathcal{T}_{\lambda,\mu}$ such that $\|u_3\|_\infty > r_\lambda$ and for which the condition $u_3 \gg \tilde{\alpha}_1^\lambda$ in $\bar{\Omega}$ does not hold. Therefore u_3 is a positive solution of (18) which differs both from u_1 and from u_2. We conclude that, for every $\mu > \mu^*$ and all $\lambda \in]0, \lambda(\mu)[$, problem (18) has at least three positive solutions. □

Remark 3.1 Assume (h_1), (h_2) and

(b_4) *there exists $b_3 > 0$ such that $b(x,s,\xi) \geq -b_3 s$ for a.e. $x \in \Omega$, all $s \in [0, \frac{1}{2}\mathrm{d}(\Omega)]$ and all $\xi \in \bar{B}_1(0)$.*

Let u be a positive solution of the problem

$$\begin{cases} -\mathrm{div}\Big(\nabla u/\sqrt{1-|\nabla u|^2}\Big) = \mu b(x,u,\nabla u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{26}$$

for some $\mu > 0$. Then u is strictly positive. Indeed, rewrite the equation in (26) as

$$-\mathrm{div}\Big(\nabla u/\sqrt{1-|\nabla u|^2}\Big) + ku = \mu b(x,u,\nabla u) + ku$$

with $k = \mu b_3 + 1$. As the right-hand side of the equation is positive, Lemma 2.6 yields the conclusion.

The following non-existence result for problem (26) holds.

Proposition 3.2. *Assume (h_1), (h_2) and*

(b_5) *there exists $b_4 > 0$ such that $b(x,s,\xi) \leq b_4 s$ for a.e. $x \in \Omega$, all $s \in [0, \frac{1}{2}\mathrm{d}(\Omega)]$ and all $\xi \in \bar{B}_1(0)$.*

Then there exists $\mu_* > 0$ such that, for every $\mu \in]0, \mu_*[$, problem (26) has no positive solutions.

Proof. Let u be a positive solution of (26) for some $\mu > 0$. Multiplying the equation in (26) by u and integrating by parts, we easily obtain by (b_5)

$$\|\nabla u\|_{L^2}^2 \leq \mu b_4 \int_\Omega u^2\, dx \leq \mu b_4 c_P \|\nabla u\|_{L^2}^2,$$

where $c_P > 0$ is the Poincaré constant. This implies that $\mu \geq (b_4 c_P)^{-1}$. □

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