A nonresonance condition for radial solutions of a nonlinear Neumann elliptic problem

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Abstract

We prove an existence result for radial solutions of a Neumann elliptic problem whose nonlinearity asymptotically lies between the first two eigenvalues. To this aim, we introduce an alternative nonresonance condition with respect to the second eigenvalue which, in the scalar case, generalizes the classical one, in the spirit of [9]. Our approach also applies for nonlinearities which do not necessarily satisfy a subcritical growth assumption.

1 Introduction

In this paper we look for radial solutions of the Neumann problem

$$\begin{cases} -\Delta u = g(u) + e(|x|) & \text{in } B_1 \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial B_1\,, \end{cases} \tag{1}$$

where $B_1 = \{x \in \mathbb{R}^N : |x| < 1\}$, $g : \mathbb{R} \to \mathbb{R}$ and $e : [0,1] \to \mathbb{R}$ are continuous functions, and $|\cdot|$ indicates the euclidean norm (we have chosen the ball of radius 1, just for simplicity). The aim of this paper is to introduce a nonresonance condition with respect to the first *positive* eigenvalue, in order to guarantee the existence of a solution to (1).

Concerning the Dirichlet problem, denoting by λ_1 the first eigenvalue of $-\Delta$, and setting

$$G(u) = \int_0^u g(\xi)d\xi\,,$$

a classical result by Hammerstein [14] states that the assumption

$$\limsup_{|x|\to\infty} \frac{2G(s)}{s^2} < \lambda_1\,, \tag{2}$$

together with some growth restriction on g connected with the Sobolev embeddings, implies the existence of a solution. In [9], Fonda, Gossez and Zanolin replaced condition (2) by

$$\liminf_{x\to-\infty} \frac{2G(s)}{s^2} < \frac{\pi^2}{4}, \quad \text{and} \quad \liminf_{x\to+\infty} \frac{2G(s)}{s^2} < \frac{\pi^2}{4},$$

without needing further assumptions on the growth of g. Notice that, even if the limsup is here replaced by a liminf, one has that $\pi^2/4 < \lambda_1$, unless the dimension is equal to 1, in which case $B_1 = (-1, 1)$ and $\lambda_1 = \pi^2/4$. The case $N = 1$ has been first considered by Fernandes, Omari and Zanolin, in [6] (see also [17]). A similar condition for a parabolic problem has been considered by Grossinho and Omari in [12].

The situation for the Neumann problem is different, since the first eigenvalue is equal to zero, so that a similar result could be obtained easily by the use of constant upper and lower solutions. A more interesting situation arises when considering the first *positive* eigenvalue. For the scalar case, the situation is similar to the periodic boundary value problem. In this setting Fernandes and Zanolin [7] were the first to propose a "liminf" nonresonance condition related to the first Fučík curve (see also [3, 4, 5, 8, 13, 16, 18]). In higher dimension, nonresonance conditions for the Neumann problem have been considered by many authors, see e.g. [2, 11, 15]. However, it seems that a "liminf" existence result, in the spirit of above quoted papers, has not been carried out yet, not even in the case $N = 1$.

We will prove the following.

Theorem 1.1 *Let the following assumptions hold:*

$$\liminf_{u\to+\infty} \frac{2G(u)}{u^2} < \frac{\pi^2}{4}\,, \tag{3}$$

$$\limsup_{u\to-\infty} \frac{g(u)}{u} < \frac{\pi^2}{4}\,. \tag{4}$$

Moreover assume that there exists $d > 0$ such that

$$\big(g(u) + \bar{e}\big)\operatorname{sgn} u > 0 \qquad \text{when } |u| \geq d\,, \tag{5}$$

where $\bar{e} = N \int_0^1 s^{N-1} e(s)\, ds$. Then, problem (1) has at least one solution.

Let us make a brief comment on the assumptions in the above theorem. The sign condition (5) is needed in order to avoid resonance with respect to the zero eigenvalue; notice that $\bar{e}$ is a weighted mean and, in the case $N = 1$, $\bar{e}$ is exactly the mean value of $e(t)$. In (3) and (4) the value $\pi^2/4$ is the first positive eigenvalue in dimension 1, since in this case $B_1 = (-1, 1)$. However, if $N \geq 2$, the first positive eigenvalue of our differential operator is strictly larger than $\pi^2/4$ (see the Appendix). We emphasize the fact that, in (3), only a liminf condition is assumed on $G(u)$, and no further growth restrictions are imposed on $g(u)$ at $+\infty$.

We also propose the following variant of Theorem 1.1.

Theorem 1.2 *Assume that (3) and (5) hold. Instead of (4), let*

$$\lim_{u\to-\infty} \frac{2G(u)}{u^2} < \frac{\pi^2}{4}\,, \tag{6}$$

assuming that such a limit exists. Under these assumptions, problem (1) has at least one solution.

Clearly, we can switch the conditions at $+\infty$ and $-\infty$ in both theorems.

Our main result, presented in Section 2, makes use of a nonresonance condition with respect to the second eigenvalue which is related to the so-called time-map, and is stated for a more general nonlinearity $g(|x|, u)$. Theorems 1.1 and 1.2 will follow directly as corollaries, since the conditions (3), (4) and (6) give the correct estimates for the time-map. Variants of these condition can be considered, as well. Section 3 is dedicated to the proof of the main theorem. In Section 4 we provide a variant of our results by a lower and upper solutions approach. As a consequence, we obtain a necessary and sufficient condition for the existence of a solution to problem (1), in the spirit of [11, Theorem 1.1]. In the Appendix, we briefly recall the properties of the eigenvalues of our differential operator, related to the zeros of some Bessel functions.

2 Main results

Consider the following problem in the unitary ball:

$$\begin{cases} -\Delta u = g(|x|, u) + e(|x|) & \text{in } B_1 \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial B_1 \,, \end{cases} \tag{7}$$

where $g : [0,1] \times \mathbb{R} \to \mathbb{R}$ and $e : [0,1] \to \mathbb{R}$ are continuous functions. A radial solution $u(x) = v(|x|)$ to this problem satisfies

$$\begin{cases} -v'' - \dfrac{N-1}{t} v' = g(t, v) + e(t) \,, & t \in (0,1] \,, \\ v'(0) = 0 = v'(1) \,. \end{cases} \tag{8}$$

Define $\bar{e} = N \int_0^1 s^{N-1} e(s)\, ds$ and $\tilde{e}(t) = e(t) - \bar{e}$, so that $\int_0^1 s^{N-1} \tilde{e}(s)\, ds = 0$. Assume that

(H1) There exist a continuous function $\phi : \mathbb{R} \to \mathbb{R}$, and $d > 0$ such that

$$-\bar{e} < g(t, v) \le \phi(v) \qquad \text{for every } t \in [0,1] \text{ and every } v \ge d \,, \tag{9}$$

$$\phi(v) \le g(t, v) < -\bar{e} \qquad \text{for every } t \in [0,1] \text{ and every } v \le -d \,, \tag{10}$$

and moreover, for a suitable $\bar{\varepsilon} > 0$,

$$\phi(v) v \ge \bar{\varepsilon} v^2 \qquad \text{for every } |v| \ge d \,. \tag{11}$$

Set $\Phi(v) = \int_0^v \phi(\xi)d\xi$. By (11), we can assume d large enough to permit us to define, for every v such that $|v| \geq d$,

$$\tau(v) = \mathrm{sgn}(v)\frac{1}{\sqrt{2}}\int_0^v \frac{d\xi}{\sqrt{\Phi(v) - \Phi(\xi)}}\,.$$

The value $2\tau(v)$ is often defined as the *time-map* associated to the planar system

$$\begin{cases} x' = y \\ y' = -\phi(x)\,. \end{cases}$$

Roughly speaking, $\tau(v)$ is the time needed by a particle to reach $x = 0$ starting with null velocity from $x = v$. This tool has been used, e.g., in [5, 10, 16, 19, 20] in order to find periodic solutions to scalar problems.

Defining

$$\tau^{\pm} = \limsup_{v\to\pm\infty} \tau(v)\,, \qquad \tau_{\pm} = \liminf_{v\to\pm\infty} \tau(v)\,,$$

we are now ready to state our main result.

Theorem 2.1 *Let assumption (H1) hold and assume that either*

$$\tau^{+} > 1 \quad \textit{and} \quad \tau_{-} > 1\,, \tag{12}$$

or

$$\tau_{+} > 1 \quad \textit{and} \quad \tau^{-} > 1\,. \tag{13}$$

Then (7) has at least one radial solution.

We will present the proof of this theorem in Section 3. In order to see how Theorems 1.1 and 1.2 can be deduced, we recall the following estimates on the time-map.

Proposition 2.2 ([10, Corollary 1]) *Assume that for some positive constants ϱ_+, ϱ_- one has*

$$\limsup_{v\to\pm\infty} \frac{\phi(v)}{v} \leq \varrho_{\pm}\,.$$

Then, $\tau_{\pm} \geq \pi/2\sqrt{\varrho_{\pm}}$.

Proposition 2.3 ([10, Corollary 2]) *Assume that for some positive constants ϱ_+, ϱ_- one has*

$$\liminf_{v\to\pm\infty} \frac{2\Phi(v)}{v^2} \leq \varrho_{\pm}\,.$$

Then, $\tau^{\pm} \geq \pi/2\sqrt{\varrho_{\pm}}$.

Proposition 2.4 ([19, Corollary 8]) *Assume that for some positive constants ϱ_+, ϱ_- the following limits exist and*

$$\lim_{v\to\pm\infty} \frac{2\Phi(v)}{v^2} \leq \varrho_\pm \,.$$

Then, $\tau_\pm \geq \pi/2\sqrt{\varrho_\pm}$.

It is now easy to see that Theorems 1.1 and 1.2 follow directly from Theorem 2.1 and the above propositions. Indeed setting, for $\bar\varepsilon$ sufficiently small,

$$\phi(v) = \begin{cases} \max\{g(t,v) : t \in [0,1]\,,\, \bar\varepsilon v\} & \text{if } v \geq d \\ \min\{g(t,v) : t \in [0,1]\,,\, \bar\varepsilon v\} & \text{if } v \leq -d\,, \end{cases} \tag{14}$$

and extending it by continuity to the real line, assumption (H1) is directly verified and (3), (4) or (6) give the correct estimates for the time-maps.

Remark. Other conditions on ϕ can be given in order to find the required estimates on the values $\tau^\pm$ and $\tau_\pm$, but they are not presented in this paper for briefness. We refer to [10] for details.

3 Proof of Theorem 2.1

Before starting the proof of Theorem 2.1, let us define the following function in $(-\infty,-d]\cup[d,+\infty)$:

$$T(v) = \frac{1}{\sqrt{2}}\int_d^v \frac{d\xi}{\sqrt{\Phi(v)-\Phi(\xi)+\|e\|_\infty(v-\xi)}}\,, \qquad \text{for } v \geq d\,,$$

$$T(v) = \frac{1}{\sqrt{2}}\int_v^{-d} \frac{d\xi}{\sqrt{\Phi(v)-\Phi(\xi)-\|e\|_\infty(v-\xi)}}\,, \qquad \text{for } v \leq -d\,.$$

We have the following estimate.

Lemma 3.1 *For every $\epsilon > 0$ there exists $v_\epsilon > d$ such that, for every v with $|v| > v_\epsilon$, the following inequalities hold:*

$$T(v) \leq \tau(v) \leq (1+\epsilon)\,T(v) + \epsilon\,.$$

Proof. It is clear that $T(v) \leq \tau(v)$ for every v with $|v| > d$. We fix $\epsilon > 0$ and prove the lemma for positive values of v, the other case being specular. By (11), we can assume that there exists $d' > d$ such that

$$\Phi(d') > \Phi(s) \qquad \text{for every } s \in [0,d')\,, \tag{15}$$

and

$$\phi(s) > \frac{1}{\epsilon^2}\|e\|_\infty \qquad \text{for every } s \in [d',+\infty)\,. \tag{16}$$

If $v > d'$, we have

$$\tau(v) = \frac{1}{\sqrt{2}} \int_0^{d'} \frac{d\xi}{\sqrt{\Phi(v) - \Phi(\xi)}} + \frac{1}{\sqrt{2}} \int_{d'}^{v} \frac{d\xi}{\sqrt{\Phi(v) - \Phi(\xi)}} .$$

By (15), there exists $v_\epsilon > d'$ such that, for every $v > v_\epsilon$,

$$\frac{1}{\sqrt{2}} \int_0^{d'} \frac{d\xi}{\sqrt{\Phi(v) - \Phi(\xi)}} \leq \frac{1}{\sqrt{2}} \int_0^{d'} \frac{d\xi}{\sqrt{\Phi(v) - \Phi(d')}} \leq \epsilon .$$

Moreover, using (16),

$$\begin{aligned} \int_{d'}^{v} \frac{d\xi}{\sqrt{\Phi(v) - \Phi(\xi)}} &= \sqrt{1+\epsilon^2} \int_{d'}^{v} \frac{1}{\sqrt{(1+\epsilon^2) \int_\xi^v \phi(\sigma) d\sigma}} \, d\xi \\ &\leq \sqrt{1+\epsilon^2} \int_{d'}^{v} \frac{1}{\sqrt{\int_\xi^v (\phi(\sigma) + \|e\|_\infty) \, d\sigma}} \, d\xi \\ &= \sqrt{1+\epsilon^2} \int_{d'}^{v} \frac{1}{\sqrt{\Phi(v) - \Phi(\xi) + \|e\|_\infty (v - \xi)}} \, d\xi \\ &\leq \sqrt{1+\epsilon^2} \, \sqrt{2} \, T(v) \leq (1+\epsilon) \sqrt{2} \, T(v) . \end{aligned}$$

Summing the two integrals, we conclude the proof. ■

We are now ready to start the proof of Theorem 2.1. Let us prove it under assumption (12), the other case being specular. We can find a sufficiently small $\epsilon > 0$ such that $\tau^+ > 1 + 3\epsilon$ and $\tau_- > 1 + 3\epsilon$. Hence, there exists an increasing sequence of positive real values $(R_n)_n$ such that

$$\lim_n R_n = +\infty ,$$

with the following property:

$$\tau(R_n) > 1 + 2\epsilon \qquad \text{for every } n \in \mathbb{N} .$$

Besides, there exists $R_\natural > 0$ such that

$$\tau(v) > 1 + 2\epsilon \qquad \text{for every } v < -R_\natural .$$

Without loss of generality we can assume $R_\natural$ and R_0 to be greater than $d+1$ and v_ϵ, where the value v_ϵ is defined as in Lemma 3.1. In this way we have that

$$T(R_n) \geq \frac{\tau(R_n) - \epsilon}{1 + \epsilon} > 1 \qquad \text{for every } n \in \mathbb{N} , \tag{17}$$

$$T(v) \geq \frac{\tau(v) - \epsilon}{1 + \epsilon} > 1 \qquad \text{for every } v < -R_\natural . \tag{18}$$

We introduce the following problem, for every $\lambda \in [0,1]$,

$$\begin{cases} -v'' - \dfrac{N-1}{t}\, v' = \lambda\big(g(t,v)+e(t)\big) + (1-\lambda)\bar{\varepsilon} v\,, & t \in (0,1]\,, \\ v'(0) = 0 = v'(1)\,, \end{cases} \tag{19}$$

where $\bar{\epsilon}$ was introduced in (11). Define the sets

$$C^k_\sharp([0,1]) = \{v \in C^k([0,1]) \,:\, v'(0) = 0 = v'(1)\}\,, \quad k = 1,2\,.$$

It has been shown in [2] that (19) is equivalent to a fixed point problem of the type

$$v = \mathcal{G}_\lambda(v)\,,$$

where $\mathcal{G}_\lambda : C^1_\sharp([0,1]) \to C^1_\sharp([0,1])$ is a completely continuous operator. Indeed, any fixed point of $\mathcal{G}_\lambda$ belongs to $C^2_\sharp([0,1])$ (see e.g. [9]). Choosing $\bar{\varepsilon} > 0$ sufficiently small we have that, being $I - \mathcal{G}_0$ linear and invertible (see the Appendix for details), $d_{LS}(I - \mathcal{G}_0, \Omega, 0) = 1$ for every open bounded set $\Omega \subset C^1_\sharp([0,1])$ such that $0 \in \Omega$. Hence, by Leray-Schauder degree theory, in order to prove the existence of a solution to (8), it will be sufficient to find a suitable open and bounded set $\Omega \subset C^1_\sharp([0,1])$ such that there is no solution of (19) on $\partial\Omega$, for every $\lambda \in [0,1]$.

The set we are looking for will be of the type

$$\Omega = \{v \in C^1_\sharp([0,1]) : -c < v(t) < R \text{ and } \|v'\|_\infty < D\}\,. \tag{20}$$

The following lemma gives us the impossibility for a solution of *remaining large*:

Lemma 3.2 *Let v be a solution of (19). Then, there exists $\bar{t} \in [0,1]$ such that $|v(\bar{t})| < d$.*

Proof. Suppose $v(t) \geq d$ for every $t \in [0,1]$. Being v a solution to (19), we have

$$\frac{d}{dt}\big(t^{N-1} v'(t)\big) = -t^{N-1}\left[\lambda\big(g(t,v(t))+e(t)\big) + (1-\lambda)\bar{\varepsilon} v(t)\right]\,, \tag{21}$$

for every $t \in [0,1]$. Integrating this equation in the interval $[0,1]$ we obtain a contradiction using (9):

$$0 = -\int_0^1 t^{N-1}\left[\lambda\big(g(t,v(t))+\bar{e}\big) + (1-\lambda)\bar{\varepsilon} v(t)\right]\, dt < 0\,.$$

The case $v(t) \leq -d$ for every $t \in [0,1]$ is treated similarly using (10). ■

The remaining part of the proof, essentially, consists of three propositions: each one gives the existence of one of the three values R, c and D.

Proposition 3.3 *There exists an integer n_0 such that, for every $n \geq n_0$, every solution v to (19), with $\lambda \in [0,1]$, satisfies $\max_{[0,1]} v \neq R_n$.*

To prove this proposition, we argue by contradiction and assume that there exist a sequence $(\lambda_n)_n$, with $\lambda_n \in [0,1]$ for every n, and a subsequence, still denoted $(R_n)_n$ (in what follows we will denote every subsequence as the sequence itself), with the property that, for every n, there exists a solution v_n to (19) with $\lambda = \lambda_n$ such that $\max_{[0,1]} v_n = R_n$. We will prove that this situation is not possible. Define

$$t_M^n = \max\{t \in [0,1] : v_n(t) = R_n\}\,.$$

Notice that $v_n'(t_M^n) = 0$. Moreover, by Lemma 3.2 we can define

$$t_d^n = \max\{t \in [0,1] : v_n(t) = d\}\,.$$

We now continue the proof considering two cases.

Case 1: $t_M^n < t_d^n$. Define

$$\tilde{t}_n = \min\{t \in [t_M^n, 1] : v_n(t) = d\}\,.$$

For every $t \in [t_M^n, \tilde{t}_n]$ such that $v_n'(t) < 0$, it is possible to find a value $s(t) \in [t_M^n, t)$ such that $v_n'(s(t)) = 0$ and $v_n'(s) < 0$ for every $s \in (s(t), t]$. Consider the differential equation in (19) with $v = v_n$ and $\lambda = \lambda_n$. Using (9) and (11), we can write

$$-v_n''(s) \leq \phi(v_n(s)) + \|e\|_\infty \qquad \text{for every } s \in [s(t), t]\,.$$

Multiplying by $v_n'(s) \leq 0$ and integrating in the interval $[s(t), t]$, we obtain

$$-\frac{1}{2} v_n'(t)^2 \geq \Phi(v_n(t)) - \Phi(v_n(s(t))) + \|e\|_\infty \big(v_n(t) - v_n(s(t))\big)\,.$$

Hence, being Φ increasing in $[d, +\infty)$, we have, for every $t \in [t_M^n, \tilde{t}_n]$ such that $v_n'(t) < 0$,

$$1 \geq \frac{1}{\sqrt{2}} \frac{-v_n'(t)}{\sqrt{\Phi(R_n) - \Phi(v_n(t)) + \|e\|_\infty (R_n - v_n(t))}}\,.$$

Clearly, the previous inequality holds also when $v_n' \geq 0$, so it holds for every $t \in [t_M^n, \tilde{t}_n]$, thus giving us the following contradiction using (17):

$$\begin{aligned}
\tilde{t}_n - t_M^n &\geq \frac{1}{\sqrt{2}} \int_{t_M^n}^{\tilde{t}_n} \frac{-v_n'(t)}{\sqrt{\Phi(R_n) - \Phi(v_n(t)) + \|e\|_\infty (R_n - v_n(t))}}\, dt \\
&= \frac{1}{\sqrt{2}} \int_d^{R_n} \frac{d\xi}{\sqrt{\Phi(R_n) - \Phi(\xi) + \|e\|_\infty (R_n - \xi)}} = T(R_n) > 1\,.
\end{aligned}$$

Case 2: $t^n_M > t^n_d$. Call $m_n = \min_{[0,1]} v_n$ and

$$t^n_m = \max\{t \in [0,1] : v_n(t) = m_n\}\,.$$

Notice that $v_n'(t^n_m) = 0$. By Lemma 3.2 we know that $m_n < d$. We want to prove that

$$\lim_n m_n = -\infty\,. \tag{22}$$

By contradiction assume that there exists a constant $C > 0$ such that, up to a subsequence,

$$v_n(t) \geq -C \qquad \text{for every } t \in [0,1] \text{ and every } n \in \mathbb{N}\,.$$

Defining

$$\tilde{g}_n(t) = -t^{N-1}\left[\lambda_n\big(g(t, v_n(t)) + \bar{e}\big) + (1-\lambda_n)\bar{\varepsilon} v_n(t)\right]\,, \tag{23}$$

we can write equation (21), with $\lambda = \lambda_n$, as

$$\frac{d}{dt}\big(t^{N-1}v_n'(t)\big) = \tilde{g}_n(t) - \lambda_n t^{N-1}\tilde{e}(t)\,. \tag{24}$$

Integrating between 0 and 1, we obtain

$$\int_0^1 \tilde{g}_n(s)ds = 0\,.$$

Then, since $\tilde{g}_n$ is negative when $v_n > d$,

$$\begin{aligned}\int_0^1 |\tilde{g}_n(s)|ds &= \int\limits_{v_n>d} -\tilde{g}_n(s)ds + \int\limits_{-C\leq v_n\leq d} |\tilde{g}_n(s)|ds \\ &= \int\limits_{-C\leq v_n\leq d} \tilde{g}_n(s)ds + \int\limits_{-C\leq v_n\leq d} |\tilde{g}_n(s)|ds \\ &\leq \; 2 \int\limits_{-C\leq v_n\leq d} |\tilde{g}_n(s)|ds\,,\end{aligned}$$

which is bounded. Hence,

$$\int_0^1 \left|\frac{d}{ds}(s^{N-1}v_n'(s))\right| ds \leq \int_0^1 |\tilde{g}_n(s)|ds + \int_0^1 s^{N-1}|\tilde{e}(s)|ds \leq C'\,,$$

for a suitable constant C', independent of n.

We thus obtain

$$\left\|\frac{d}{dt}(t^{N-1}v_n')\right\|_{L^1(0,1)} \leq C'\,, \tag{25}$$

for every n. So, for every $t \in (t_m^n, 1]$,

$$t^{N-1}v_n'(t) = (t_m^n)^{N-1}v_n'(t_m^n) + \int_{t_m^n}^{t} \big(s^{N-1}v_n'(s)\big)'ds \le C' \,. \tag{26}$$

We want to show now that there exists a small $\delta_1 > 0$ such that, for every $t \in [t_m^n, t_m^n + \delta_1]$ and for every n, we have $v_n(t) \le d+1$. Calling

$$M = \max\left\{|g(t,v)| + \bar{\varepsilon}|v| + \|e\|_\infty \,:\, t \in [0,1]\,, v \in [-C, d+1]\right\}\,,$$

we observe that in a right neighborhood of t_m^n, as long as $v_n(t) \in [-C, d+1]$, we have, by (21),

$$\left|\frac{d}{dt}\big(t^{N-1}v_n'(t)\big)\right| \le Mt^{N-1}\,.$$

Hence,

$$\begin{aligned} t^{N-1}v_n'(t) &\le \int_{t_m^n}^{t} \left|\frac{d}{ds}\big(s^{N-1}v_n'(s)\big)\right| ds \\ &\le \int_{t_m^n}^{t} Ms^{N-1}\,ds = \frac{M}{N}\big(t^N - (t_m^n)^N\big) \\ &\le \frac{M}{N}(t - t_m^n)\cdot Nt^{N-1}\,, \end{aligned}$$

thus giving us $v_n'(t) \le M(t - t_m^n)$ in a right neighborhood of t_m^n. So,

$$v_n(t) \le m_n + \frac{M}{2}(t - t_m^n)^2\,,$$

as long as $v_n(t) \le d+1$. In particular, since $m_n \le d$, setting $\delta_1 = \sqrt{2/M}$, we have thus proved that $v_n(t) \le d+1$ for every $t \in [t_m^n, t_m^n + \delta_1]$, for every n.

Now, if $t \ge t_m^n + \delta_1 \ge \delta_1$, by (26),

$$v_n'(t) = \frac{t^{N-1}v_n'(t)}{t^{N-1}} \le \frac{C'}{t^{N-1}} \le \frac{C'}{\delta_1^{N-1}}\,.$$

Hence, being $t_m^n + \delta_1 < t_M^n$,

$$R_n = v_n(t_M^n) = v_n(t_m^n + \delta_1) + \int_{t_m^n+\delta_1}^{t_M^n} v_n'(s)\,ds \le d + 1 + \frac{C'}{\delta_1^{N-1}}\,,$$

which contradicts the assumption $R_n \to +\infty$. So, we have proved that (22) holds.

We can assume $m_n < -d$ for every n. Set

$$\hat{t}_n = \min\{t \in (t_m^n, t_d^n) : v_n(t) = -d\}\,.$$

Arguing as in Case 1, we can find, for every $t \in [t_m^n, \hat{t}_n]$ such that $v_n'(t) > 0$, a value $s(t) \in [t_m^n, t)$ such that $v_n'(s(t)) = 0$ and $v_n'(s) > 0$ for every $s \in (s(t), t]$. Considering the differential equation in (19) with $v = v_n$ and $\lambda = \lambda_n$, we can write, using (10) and (11),

$$-v_n''(s) \geq \phi(v_n(s)) - \|e\|_\infty \qquad \text{for every } s \in [s(t), t]\,.$$

Multiplying it by $v_n'(s) \geq 0$ and integrating in the interval $[s(t), t]$, using that Φ is decreasing in $(-\infty, -d]$, we obtain, arguing as above,

$$1 \geq \frac{1}{\sqrt{2}} \frac{v_n'(t)}{\sqrt{\Phi(m_n) - \Phi(v_n(t)) - \|e\|_\infty(m_n - v_n(t))}}\,.$$

Clearly, the previous inequality holds when $v_n' \leq 0$, so it holds for every $t \in [t_m^n, \hat{t}_n]$, thus giving us the following contradiction when n is large enough, using (18):

$$\begin{aligned}
\hat{t}_n - t_m^n &\geq \frac{1}{\sqrt{2}} \int_{t_m^n}^{\hat{t}_n} \frac{v_n'(t)}{\sqrt{\Phi(m_n) - \Phi(v_n(t)) - \|e\|_\infty(m_n - v_n(t))}}\, dt \\
&\geq \frac{1}{\sqrt{2}} \int_{m_n}^{-d} \frac{d\xi}{\sqrt{\Phi(m_n) - \Phi(\xi) - \|e\|_\infty(m_n - \xi)}} = T(m_n) > 1\,.
\end{aligned}$$

Proposition 3.3 is thus proved. ■

We have proved that there cannot exist solutions to (19) such that $\max_{[0,1]} v_n = R_n$ if n is large enough. Once fixed such a suitable value $R = R_n$ we state the following proposition.

Proposition 3.4 *There exists a real value $c > R_\natural$ such that, for every solution v to (19), with $\lambda \in [0,1]$, satisfying $\max_{[0,1]} v < R$, it has to be $\min_{[0,1]} v \neq -c$.*

The proof of this proposition is rather similar to the one of Proposition 3.3. We argue by contradiction that for every $c > R_\natural$ there exists a solution v to (19) with $\max_{[0,1]} v < R$, such that $\min_{[0,1]} v = -c$. Call

$$t_c = \max\{t \in [0,1] : v(t) = -c\}\,.$$

The situation is similar to the one when we have treated t_M^n. Using Lemma 3.2, it is possible to define

$$t_0 = \max\{t \in [0,1] : v(t) = -d\}\,.$$

If $t_0 > t_c$, set

$$t_d = \min\{t \in [t_c, 1] : v(t) = -d\} .$$

Arguing as above, we can find, for every $t \in [t_c, t_d]$ such that $v'(t) > 0$, a value $s(t) \in [t_c, t)$ such that $v'(s(t)) = 0$ and $v'(s) > 0$ for every $s \in (s(t), t]$. Considering the differential equation in (19), we can write, using (10) and (11),

$$-v''(s) \geq \phi(v(s)) - \|e\|_\infty \qquad \text{for every } s \in [s(t), t] .$$

Multiplying it by $v'(s) \geq 0$ and integrating in the interval $[s(t), t]$, using that Φ is decreasing in $(-\infty, -d]$, we obtain, arguing as above,

$$1 \geq \frac{1}{\sqrt{2}} \frac{v'(t)}{\sqrt{\Phi(-c) - \Phi(v(t)) - \|e\|_\infty(-c - v(t))}} .$$

Clearly, the previous inequality holds when $v' \leq 0$, so it holds for every $t \in [t_c, t_d]$, thus giving us the following contradiction, using (18):

$$\begin{aligned} t_d - t_c &\geq \frac{1}{\sqrt{2}} \int_{t_c}^{t_d} \frac{v'(t)}{\sqrt{\Phi(-c) - \Phi(v(t)) - \|e\|_\infty(-c - v(t))}}\, dt \\ &\geq \frac{1}{\sqrt{2}} \int_{-c}^{-d} \frac{d\xi}{\sqrt{\Phi(-c) - \Phi(\xi) - \|e\|_\infty(-c - \xi)}} = T(-c) > 1 . \end{aligned}$$

Otherwise, if $t_0 < t_c$, define $a = \max_{[0,1]} v < R$ and set $t_a = \max\{t \in [0,1] : v(t) = a\} < t_c$. Notice that $v'(t_a) = 0$. Setting

$$M' = \max\left\{|g(t, v)| + \bar{\varepsilon}|v| + \|e\|_\infty \ : t \in [0,1], v \in [-d-1, R]\right\} ,$$

and arguing as above, we have in a right neighborhood of t_a

$$t^{N-1} v'(t) \geq -\int_{t_a}^{t} \left| \frac{d}{ds}(s^{N-1} v'(s)) \right| ds \geq -M'(t - t_a) t^{N-1} .$$

A brief computation shows that $v(t) \geq -d - 1$ for every $t \in [t_a, t_a + \delta_2]$, where $\delta_2 = \sqrt{2/M'}$. Following the procedure which has given us the estimate in (25), we can find that

$$\left\| \frac{d}{dt}(t^{N-1} v') \right\|_{L^1(0,1)} \leq C'' , \tag{27}$$

for a suitable constant $C'' > 0$. So, for every $t \in (t_a, 1]$,

$$t^{N-1} v'(t) = t_a^{N-1} v'(t_a) + \int_{t_a}^{t} \left(s^{N-1} v'(s)\right)' ds \geq -C'' .$$

Summing up, we have

$$-c = v(t_c) = v(t_a + \delta_2) + \int_{t_a + \delta_2}^{t_c} v'(s) ds \geq -d - 1 - \frac{C''}{{\delta_2}^{N-1}} ,$$

giving us a contradiction when c is large enough. Proposition 3.4 is thus proved. ■

The following proposition gives us the needed control on the derivative, once the constants R and c have been fixed.

Proposition 3.5 *There exists a constant $D > 0$ such that, for every solution v to (19), with $\lambda \in [0,1]$, satisfying $-c < v(t) < R$ for every $t \in [0,1]$, it has to be $\|v'\|_\infty < D$.*

Proof. Setting

$$M'' = \max\{|g(t,v)| + \bar{\varepsilon}|v| + \|e\|_\infty \, : t \in [0,1]\,, v \in [-c,R]\}\ ,$$

arguing as above we have that

$$t^{N-1}|v'(t)| \le \int_0^t \left| \frac{d}{ds}(s^{N-1}v'(s)) \right| ds \le M'' t^N\ ,$$

thus giving us $|v'(t)| \le M''t \le M''$. So, we can choose $D = M'' + 1$ and the proof is completed. ■

So, after all, we have found the three constants R, c and D permitting us to define the set Ω on which we can apply the Leray-Schauder degree theory. The proof of Theorem 2.1 is thus completed.

4 A lower and upper solutions approach

In this section we prove a necessary and sufficient condition for the existence of a solution to problem (1), in the spirit of the result stated in [11]. We modify the assumption (H1) in order to obtain a different existence result for problem (7). The necessary and sufficient condition will follow as a direct consequence. Hence, assume that

(H2) There exist $A < B$ such that, for every $t \in [0,1]$,

$$g(t,A) + e(t) < 0 < g(t,B) + e(t)\,. \tag{28}$$

Moreover, there exist a continuous function $\phi : \mathbb{R} \to \mathbb{R}$, and two constants $K > 0$, $d > 0$ such that

$$-K < g(t,v) \le \phi(v) \qquad \text{for every } t \in [0,1] \text{ and every } v \ge d\,, \tag{29}$$

$$\phi(v) \le g(t,v) < K \qquad \text{for every } t \in [0,1] \text{ and every } v \le -d\,, \tag{30}$$

and moreover, for a suitable $\bar{\varepsilon} > 0$,

$$\phi(v)v \ge \bar{\varepsilon}v^2 \qquad \text{for every } |v| \ge d\,. \tag{31}$$

Condition (28) gives us the existence of a constant upper solution $u \equiv A$ and of a constant lower solution $u \equiv B$ to problem (7). Notice that they are ordered in the wrong way, so we cannot deduce the existence of a solution u to (7) laying between A and B. The existence is given by the following variant of Theorem 2.1.

Theorem 4.1 *Let assumption (H2) hold and assume that either*

$$\tau^+ > 1 \quad \textit{and} \quad \tau_- > 1\,, \tag{32}$$

or

$$\tau_+ > 1 \quad \textit{and} \quad \tau^- > 1\,. \tag{33}$$

Then (7) has at least one radial solution.

The proof of this theorem is rather similar to the one of Theorem 2.1. We will explain in detail only where they differ. We can assume $d > \max\{-A, B\}$. Suppose $A < 0 < B$, the other case will be treated later. Lemma 3.1 holds under assumption (H2), too. So, we can define as above the sequence $(R_n)_n$ and $R_\natural$, thus giving us (17) and (18). We can introduce problem (19) and the operator $\mathcal{G}_\lambda$, but now we look for a set Ω which is different from the one introduced in (20):

$$\Omega = \{v \in C^1_\sharp([0,1]) : -c < v(t) < R\,,\ \|v'\|_\infty < D \\ \text{and } \exists t_0 \in [0,1]\ :\ A < v(t_0) < B\}\,.$$

Notice that, by (28), there cannot exist a solution v to (19) such that $\max v = A$ or $\min v = B$. Hence, the proof of Theorem 4.1 follows directly from the following three propositions.

Proposition 4.2 *There exists an integer n_0 such that for every $n \geq n_0$, for every solution v to (19), with $\lambda \in [0,1]$, satisfying $A < v(t_0) < B$ for a certain $t_0 \in [0,1]$, it has to be* $\max_{[0,1]} v \neq R_n$.

Once fixed such a suitable value $R = R_n$, we can state the following proposition.

Proposition 4.3 *There exists a real value $c > R_\natural$ such that, for every solution v to (19), with $\lambda \in [0,1]$, satisfying* $\max_{[0,1]} v < R$ *and $A < v(t_0) < B$ for a certain $t_0 \in [0,1]$, it has to be* $\min_{[0,1]} v \neq -c$.

Once fixed the values R and c, we can state the following one.

Proposition 4.4 *There exists a constant $D > 0$ such that, for every solution v to (19), with $\lambda \in [0,1]$, satisfying $-c < v(t) < R$ for every $t \in [0,1]$, and $A < v(t_0) < B$ for a certain $t_0 \in [0,1]$, it has to be $\|v'\|_\infty < D$.*

Notice that in this case we do not need Lemma 3.2. The proof of these propositions is the same as those of Proposition 3.4, 3.3 and 3.5, except where we use the hypothesis $\mathrm{sgn}(v)(g(t,v)+\bar{e}) > 0$. In particular, this condition is used only to find the estimate in (25) and (27). So, we just need to rewrite this part.

Rename the function $\tilde{g}_n$ which appears in (23) as

$$\tilde{g}_n(t) = -t^{N-1}\left[\lambda_n\big(g(t,v_n(t))+K\big) + (1-\lambda_n)\bar{\varepsilon}v_n(t)\right], \tag{34}$$

We can write equation (21), with $\lambda=\lambda_n$, as

$$\frac{d}{dt}(t^{N-1}v_n'(t))' = \tilde{g}_n(t) + \lambda_n t^{N-1}(K-e(t))\,.$$

Integrating between 0 and 1, we obtain, assuming without loss of generality $K > \bar{e} = N\int_0^1 s^{N-1}e(s)\,ds$,

$$\int_0^1 \tilde{g}_n(s)ds \geq -\frac{K-\bar{e}}{N}\,.$$

Hence, since $\tilde{g}_n$ is negative when $v_n > d$,

$$\begin{aligned}\int_0^1 |\tilde{g}_n(s)|ds &= \int_{v_n>d} -\tilde{g}_n(s)ds + \int_{-C\leq v_n\leq d} |\tilde{g}_n(s)|ds \\ &\leq \frac{K-\bar{e}}{N} + \int_{-C\leq v_n\leq d} \tilde{g}_n(s)ds + \int_{-C\leq v_n\leq d} |\tilde{g}_n(s)|ds \\ &\leq \frac{K-\bar{e}}{N} + 2\int_{-C\leq v_n\leq d} |\tilde{g}_n(s)|ds\,,\end{aligned}$$

which is bounded. Then,

$$\int_0^1 \left|\frac{d}{ds}(s^{N-1}v_n'(s))\right| ds \leq \int_0^1 |\tilde{g}_n(s)|ds + \frac{K}{N} + \int_0^1 s^{N-1}|e(s)|\,ds \leq C'\,,$$

for a suitable constant C', independent of n. We thus obtain

$$\left\|\frac{d}{dt}(t^{N-1}v_n')\right\|_{L^1(0,1)} \leq C'\,, \tag{35}$$

for every n. Similarly, one can obtain (27).

Suppose now that $A < 0 < B$ is not satisfied. Choose $\eta \in (A,B)$ and define $h(t,v) = g(t,v+\eta)$. This function satisfies (H2) with $A_1 = A-\eta <$

$0 < B-\eta = B_1$ and $\phi_1 = \phi(\,\cdot\,+\eta)$, even slightly modifying the other values. By the above argument, we can find a solution z to the problem

$$\begin{cases} -\Delta z = h(|x|, z) + e(|x|) & \text{in } B_1 \\ \dfrac{\partial z}{\partial \nu} = 0 & \text{on } \partial B_1 \,. \end{cases}$$

The function $u = z + \eta$ is a solution to (7). The proof of Theorem 4.1 is thus completed.

We are now ready to state the following result, in the spirit of [11, Theorem 1.1], in the case when g does only depend on u.

Theorem 4.5 *Assume (3) and (4). Then (1) has a solution for every continuous function* $e(\cdot)$ *if and only if* $g(\mathbb{R}) = \mathbb{R}$.

Proof. The unboundness of g is clearly a necessary condition. Let us prove that this condition is sufficient, too. Since $g(\mathbb{R}) = \mathbb{R}$, for every continuous function e, we can find two real numbers α and β such that $g(\alpha) \geq \|e\|_\infty$ and $g(\beta) \leq -\|e\|_\infty$, which are respectively a lower and an upper solution to (7). If $\alpha \leq \beta$, the existence follows by the classical theory of upper and lower solutions. So, assume $\alpha > \beta$. If g is unbounded from below on $(\alpha, +\infty)$ then we can find a constant upper solution $\beta' > \alpha$, thus concluding. Similarly, if g is unbounded from above on $(-\infty, \beta)$. So, we just need to consider the case when there exists a constant $K > 0$ such that $g(v)\,\mathrm{sgn}(v) \geq -K$. Taking $A = \beta$ and $B = \alpha$, we easily deduce that (H2) holds, with ϕ defined as in (14). The conclusion follows by Theorem 4.1, in view of the time-map estimates given by Proposition 2.2 and 2.3. ■

An analogous statement holds if we replace condition (4) with (6) in view of Proposition 2.4.

Appendix. The eigenvalue problem

Here we study the following eigenvalue problem:

$$\mathcal{L}(v) = \lambda v \,, \qquad v \in C^2_\sharp([0,1]) \,, \tag{36}$$

where the operator $\mathcal{L} : C^2_\sharp([0,1]) \to C([0,1])$ is defined, for a fixed integer $N \geq 2$, as

$$\mathcal{L}(v)(t) = -v''(t) - \frac{N-1}{t} v'(t) \,, \qquad t \in (0,1] \,,$$

and $\mathcal{L}(v)(0) = -Nv''(0)$. The regularity at zero follows by the use of L'Hôpital's rule. We will show, in particular, that the first positive eigenvalue of this problem is greater than $\pi^2/4$.

Multiplying equation (36) by t^2 we obtain a Bessel-type equation

$$t^2 v'' + (N-1)tv' + \lambda t^2 v = 0\,,$$

which is equivalent to the equation

$$z^2 \frac{d^2 w}{dz^2} + z\frac{dw}{dz} + (z^2 - \nu^2)w = 0\,, \tag{37}$$

setting $\nu+1 = N/2$ and $\lambda = 1/\mu^2$, and using the following change of variable:

$$w(z) = z^\nu v(\mu z) \quad \text{and} \quad t = \mu z\,.$$

A solution of equation (37) is the Bessel function of the first kind:

$$J_\nu(z) = \sum_{m:0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m+\nu+1)} \left(\frac{z}{2}\right)^{2m+\nu}\,,$$

where Γ is the Euler function. In particular,

$$\lim_{z\to 0} \frac{J_\nu(z)}{z^\nu} = \frac{1}{\Gamma(\nu+1)2^\nu}\,, \tag{38}$$

and

$$\frac{d}{dz} J_\nu(z) = -J_{\nu+1}(z) + \frac{\nu}{z} J_\nu(z)\,. \tag{39}$$

We have, by (39),

$$\frac{d}{dz}\left(\frac{J_\nu(z)}{z^\nu}\right) = \frac{1}{z^\nu}\left(\frac{d}{dz} J_\nu(z) - \frac{\nu}{z} J_\nu(z)\right) = \frac{-J_{\nu+1}(z)}{z^\nu}\,.$$

Denote by z_ν the first positive zero of $J_{\nu+1}$. Defining

$$v_\nu(t) = \frac{J_\nu(z_\nu t)}{(z_\nu t)^\nu}\,,$$

we have that $v_\nu'(1) = 0$, and by (38),

$$\lim_{t\to 0^+} v_\nu'(t) = 0\,.$$

Consequently one has that $v_\nu \in C^2_\sharp([0,1])$. So, choosing $\mu = 1/z_\nu$, v_ν is an eigenfunction with eigenvalue $\lambda = z_\nu^2$ for the operator $\mathcal{L}$.
The zeros of Bessel functions satisfy

$$\nu < z_\nu < z_{\nu+1} \qquad \text{for every } \nu \geq 0\,.$$

Being

$$z_0 \sim 3.8317 \quad \text{and} \quad z_{1/2} \sim 4.4934\,,$$

we have that, for every $N \geq 2$, the first eigenvalue of the problem (36) is greater than $\pi^2/4$.

See [1] for more details and properties about Bessel functions.

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