# Existence, regularity and stability properties of periodic solutions of a capillarity equation in the presence of lower and upper solutions 

Franco Obersnel, Pierpaolo Omari and Sabrina Rivetti *<br>Dipartimento di Matematica e Geoscienze<br>Università degli Studi di Trieste<br>Via A. Valerio 12/1, 34127 Trieste, Italy<br>E-mail: obersnel@units.it, omari@units.it, sabrina.rivetti@phd.units.it

January 17, 2012


#### Abstract

We develop a lower and upper solutions method for the periodic problem associated with the capillarity equation $$
-\left(u^{\prime} / \sqrt{1+u^{\prime 2}}\right)^{\prime}=f(t, u)
$$ in the space of bounded variation functions. We get the existence of periodic solutions both in the case where the lower solution $\alpha$ and the upper solution $\beta$ satisfy $\alpha \leq \beta$, and in the case where $\alpha \not \leq \beta$. In the former case we also prove regularity and order stability of solutions.

2010 Mathematics Subject Classification: 34C25, 34B15, 76D45, 47H07, 34D20, 49Q20. Keywords and Phrases: quasilinear ordinary differential equation, prescribed curvature equation, capillarity equation, periodic problem, bounded variation function, existence, lower and upper solutions, regularity, order stability, Lyapunov instability.


## 1 Introduction

Let us consider the quasilinear ordinary differential equation

$$
\begin{equation*}
-\left(u^{\prime} / \sqrt{1+u^{\prime 2}}\right)^{\prime}=f(t, u) \tag{1}
\end{equation*}
$$

This equation, together with its $N$-dimensional counterpart

$$
\begin{equation*}
-\operatorname{div}\left(\nabla u / \sqrt{1+|\nabla u|^{2}}\right)=f(x, u) \tag{2}
\end{equation*}
$$

[^0]plays an important role in various physical and geometrical questions: capillarity-type problems in fluid mechanics, flux limited diffusion phenomena, prescribed mean curvature problems (see, e.g., [18, 21, 19]).

The solvability of the $T$-periodic problem associated with (1), as well as of more general problems of the type

$$
\begin{equation*}
\left.-\left(\phi\left(u^{\prime}\right)\right)^{\prime}=f(t, u) \text { in }\right] 0, T\left[, \quad u(0)=u(T), u^{\prime}(0)=u^{\prime}(T),\right. \tag{3}
\end{equation*}
$$

with $\phi: \mathbb{R} \rightarrow \mathbb{R}$ a continuous bounded increasing function and $f:[0, T] \times \mathbb{R} \rightarrow \mathbb{R}$, say, continuous, has received considerable attention in some recent papers: the existence of classical solutions has been studied in [4, 7, [5, 8, ,9, 6, 25] using topological methods, whereas the existence of bounded variation solutions has been discussed in [28, 29] using nonsmooth critical point theory.

The aim of this paper is to develop a lower and upper solutions method for the $T$-periodic problem associated with (1) in a variational setting. Formally (1) is the Euler-Lagrange equation of the functional

$$
\mathcal{H}(v)=\int_{0}^{T} \sqrt{1+{v^{\prime}}^{2}} d t-\int_{0}^{T} F(t, v) d t
$$

where $\frac{\partial}{\partial s} F(t, s)=f(t, s)$. The functional $\mathcal{H}$ is well-defined in the space $W_{T}^{1,1}(0, T)$ of all absolutely continuous functions $v$ satisfying the periodicity conditions $v(0)=v(T)$. Yet this space, which could be a natural candidate where to settle the problem, is not a favourable framework to deal with critical point theory. Therefore, as we did in [28, 29, we replace the space $W_{T}^{1,1}(0, T)$ with the space $B V(0, T)$ of bounded variation functions and the functional $\mathcal{H}$ with an appropriate relaxation, which keeps record of the periodic boundary conditions. Since the relaxed functional is not differentiable in $B V(0, T)$, but is just the sum of a convex term and of a differentiable one, we suitably generalize the notion of critical point, interpreting it as the solution of a certain variational inequality. Thus the solutions of the $T$-periodic problem associated with (1) we find are weaker and less regular than the ones obtained, e.g., in [4, 7, 5, 8, ,9, 6, 25); hence an additional effort for studying the regularity of the obtained solutions is required.

The paper is organized as follows. Section 2 is devoted to introduce some definitions and to recall some preliminary results that will be extensively used in the sequel. In Section 3 we introduce two notions, of increasing generality, of lower and upper solutions for the $T$ periodic problem associated with (1). A discussion of some explicit conditions which yield the existence of lower and upper solutions is also performed. In Section 4 we present our existence results for the $T$-periodic problem associated with (1), in the presence of a lower solution $\alpha$ and of an upper solution $\beta$. If $\alpha \leq \beta$, we prove the existence of a minimum solution $v$ and of a maximum solution $w$ lying between $\alpha$ and $\beta$. Here we follow rather closely the approach we introduced in [27] to deal with the mixed problem for (22). In particular the existence of a solution which is a local minimizer of the associated functional is obtained. It is worth noticing that a similar existence result for classical solutions of (3) was obtained in [8, 9 using topological degree, but there a one-sided boundedness condition on the right-hand $f$ of the equation had to be assumed, with bound depending on the size of the range of $\phi$ and on the period $T$. We see that such an assumption can be completely removed using our method. If $\alpha \not \leq \beta$, we are still able to prove the existence of a solution, but now we must put a control on $f$ with respect to the first branch of the Dancer-Fučík spectrum of the $T$-periodic problem for the 1-Laplace operator, as defined in [29]. Our approach here is perturbative:
solutions of the $T$-periodic problem associated with (1) are obtained as limits in $B V(0, T)$ of solutions of an approximating sequence of regularized problems. In this context a stronger notion of lower and upper solutions is needed and no localization information is obtained. It remains an open question to prove this result by a more direct method, which could probably allow to overcome such limitations. Nevertheless, these existence results yield rather general and flexible tools to investigate the solvability of the $T$-periodic problem associated with (1), as we illustrate by few simple examples. A non-existence result, witnessing the sharpness of some of the considered assumptions, completes Section 4. In Section 5 we turn to prove the regularity of the bounded variation solutions, when a regularity and monotonicity condition is assumed on $f$. Here a regularity result for solutions of the Dirichlet problem associated with (1), we recently proved in [30, plays a central role. In Section 6 we show how certain stability properties of the solutions of the $T$-periodic problem associated with (1) can be detected by the use of lower and upper solutions. In particular we prove the order stability, as defined in [20], of the minimum and of the maximum solution lying between a pair of lower and upper solutions $\alpha$ and $\beta$ satisfying $\alpha \leq \beta$. The use of this concept of stability is motivated by the fact that, even in the case where all solutions between $\alpha$ and $\beta$ are regular, we cannot expect the existence of Lyapunov stable $T$-periodic solutions in between (see [17, 33): this fact is explicitly proven at the end of Section 6. It is also worth noting that our stability conclusions are obtained without assuming any additional regularity condition, like, e.g., Lipschitz continuity, on $f$, as it is usually done in the semilinear case in order to associate with the considered problem an order preserving operator (see, e.g., [1, 20]).

Notation. We list a few notations that will be used throughout this paper. We set $\mathbb{R}_{0}^{+}=$ $] 0,+\infty[$. For functions $u, v: E(\subseteq \mathbb{R}) \rightarrow \mathbb{R}$, where $E$ has positive 1-dimensional Lebesgue measure, we write $u \leq v$ if $u(t) \leq v(t)$ a.e. in $E$, and $u<v$ if $u \leq v$ and $u(t)<v(t)$ in a subset of $E$ having positive measure. We define the functions $u \vee v$ and $u \wedge v$ by setting $(u \vee v)(t)=\max \{u(t), v(t)\}$ and $(u \wedge v)(t)=\min \{u(t), v(t)\}$ a.e. in $E$. We also write $u^{+}=u \vee 0$ and $u^{-}=-(u \wedge 0)$. We set $W_{T}^{1,1}(0, T)=\left\{u \in W^{1,1}(0, T): u(0)=u(T)\right\}$, and $C_{T}^{1}([0, T])=\left\{u \in C^{1}([0, T]): u(0)=u(T), u^{\prime}(0)=u^{\prime}(T)\right\}$. Of course, we can identify $W_{T}^{1,1}(0, T)$ with $W_{l o c, T}^{1,1}(\mathbb{R})=\left\{u \in W_{l o c}^{1,1}(\mathbb{R}): u\right.$ is $T$-periodic $\}$ and $C_{T}^{1}([0, T])$ with $C_{T}^{1}(\mathbb{R})=$ $\left\{u \in C^{1}(\mathbb{R}): u\right.$ is $T$-periodic $\}$. For every $u \in B V(a, b)$, where $a<b$, we denote by $u\left(t_{0}^{+}\right)$the right trace of $u$ at $t_{0} \in\left[a, b\left[\right.\right.$ and by $u\left(t_{0}^{-}\right)$the left trace of $u$ at $\left.\left.t_{0} \in\right] a, b\right]$. The conventions $\pm \infty+r= \pm \infty$, for each $r \in \mathbb{R}$, are adopted.

## 2 Preliminaries

Let $a, b \in \mathbb{R}$, with $a<b$. We recall that $v \in B V(a, b)$ if $v \in L^{1}(a, b)$ and $\int_{a}^{b}|D v|<\infty$, where

$$
\int_{a}^{b}|D v|=\sup \left\{\int_{a}^{b} v w^{\prime} d t: w \in C_{0}^{1}(] a, b[) \text { and }\|w\|_{L^{\infty}} \leq 1\right\}
$$

For any $v \in B V(a, b)$ we set

$$
\begin{gathered}
\int_{a}^{b} \sqrt{1+|D v|^{2}}=\sup \left\{\int_{a}^{b}\left(v w_{1}^{\prime}+w_{2}\right) d t: w_{1}, w_{2} \in C_{0}^{1}(] a, b[)\right. \\
\text { and } \left.\left\|w_{1}^{2}+w_{2}^{2}\right\|_{L^{\infty}} \leq 1\right\} .
\end{gathered}
$$

Note that

$$
\begin{equation*}
\int_{a}^{b} \sqrt{1+|D v|^{2}} \geq \int_{a}^{b}|D v| \tag{4}
\end{equation*}
$$

For any fixed $r \in[1,+\infty[\cup\{\infty\}, B V(a, b)$ is a Banach space with respect to the norm

$$
\|v\|_{B V}=\int_{a}^{b}|D v|+\|v\|_{L^{r}}
$$

When $a=0$ and $b=T$ we set for convenience

$$
\mathcal{J}(v)=\int_{0}^{T} \sqrt{1+|D v|^{2}}+\left|v\left(T^{-}\right)-v\left(0^{+}\right)\right|
$$

The functional $\mathcal{J}: B V(0, T) \rightarrow \mathbb{R}$ is convex and, also by the continuity of the trace map [19, Theorem 2.11], is Lipschitz continuous [13, p. 362].

We now collect, for reader's convenience, some technical results which will be used in the sequel. Most proofs can be found in [29].
Proposition 2.1 (Approximation property). For any given $v \in B V(0, T)$ there exists a sequence $\left(v_{n}\right)_{n}$ in $W_{T}^{1,1}(0, T)$ such that

$$
\begin{gathered}
\lim _{n \rightarrow+\infty} v_{n}=v \text { in } L^{1}(0, T) \text { and a.e. in }[0, T] \\
\lim _{n \rightarrow+\infty} \int_{0}^{T}\left|v_{n}^{\prime}\right| d t=\int_{0}^{T}|D v|+\left|v\left(T^{-}\right)-v\left(0^{+}\right)\right|
\end{gathered}
$$

and

$$
\lim _{n \rightarrow+\infty} \int_{0}^{T} \sqrt{1+\left|v_{n}^{\prime}\right|^{2}} d t=\int_{0}^{T} \sqrt{1+|D v|^{2}}+\left|v\left(T^{-}\right)-v\left(0^{+}\right)\right|
$$

Proof. See [29, Proposition 2.1].
Proposition 2.2 (One-sided approximation property). For any given $v \in B V(0, T)$ there exist sequences $\left(v_{n}\right)_{n}$ and $\left(w_{n}\right)_{n}$ in $W^{1,1}(0, T)$ such that, for all $n, v_{n}(t) \geq v(t)$ and $w_{n}(t) \leq$ $v(t)$ a.e. in $[0, T]$,

$$
\begin{gather*}
\lim _{n \rightarrow+\infty} v_{n}=\lim _{n \rightarrow+\infty} w_{n}=v \text { in } L^{1}(0, T) \text { and a.e. in }[0, T],  \tag{5}\\
\lim _{n \rightarrow+\infty} \int_{0}^{T}\left|v_{n}^{\prime}\right| d t=\lim _{n \rightarrow+\infty} \int_{0}^{T}\left|w_{n}^{\prime}\right| d t=\int_{0}^{T}|D v|,  \tag{6}\\
\lim _{n \rightarrow+\infty} \int_{0}^{T} \sqrt{1+\left|v_{n}^{\prime}\right|^{2}} d t=\lim _{n \rightarrow+\infty} \int_{0}^{T} \sqrt{1+\left|w_{n}^{\prime}\right|^{2}} d t=\int_{0}^{T} \sqrt{1+|D v|^{2}} \tag{7}
\end{gather*}
$$

and

$$
\begin{equation*}
\lim _{n \rightarrow+\infty} \mathcal{J}\left(v_{n}\right)=\lim _{n \rightarrow+\infty} \mathcal{J}\left(w_{n}\right)=\mathcal{J}(v) \tag{8}
\end{equation*}
$$

Proof. We only prove the existence of the sequence $\left(v_{n}\right)_{n}$ as the result about the sequence $\left(w_{n}\right)_{n}$ can be similarly verified. By [13, Theorem 3.3] there exists a sequence $\left(v_{n}\right)_{n}$ in $W^{1,1}(0, T)$ satisfying (5), 7) and, for every $n, v_{n}(t) \geq v(t)$ a.e. in $[0, T]$. By [3, Fact 3.1], (6) also holds. As both (5) and (6) hold, by [19, Theorem 2.11], we also have

$$
\lim _{n \rightarrow+\infty} v_{n}\left(0^{+}\right)=v\left(0^{+}\right) \text {and } \lim _{n \rightarrow+\infty} v_{n}\left(T^{-}\right)=v\left(T^{-}\right) .
$$

Hence, (8) is also satisfied.
Proposition 2.3 (Semicontinuity of $\mathcal{J})$. If $\left(v_{n}\right)_{n}$ is a sequence in $B V(0, T)$ converging in $L^{1}(0, T)$ to $v \in B V(0, T)$, then

$$
\mathcal{J}(v) \leq \liminf _{n \rightarrow+\infty} \mathcal{J}\left(v_{n}\right)
$$

Proof. See [29, Proposition 2.4].
Proposition 2.4 (Lattice property). For every $u, v \in B V(0, T)$

$$
\mathcal{J}(u \vee v)+\mathcal{J}(u \wedge v) \leq \mathcal{J}(u)+\mathcal{J}(v)
$$

Proof. See [29, Proposition 2.5].
Proposition 2.5 (Asymmetric Wirtinger inequality). Let $\mu, \nu \in \mathbb{R}_{0}^{+}$satisfy

$$
\frac{1}{\sqrt{\mu}}+\frac{1}{\sqrt{\nu}}=\sqrt{T}
$$

For every $v \in B V(0, T)$ such that

$$
\mu \int_{0}^{T} v^{+} d t-\nu \int_{0}^{T} v^{-} d t=0
$$

we have

$$
\mu \int_{0}^{T} v^{+} d t+\nu \int_{0}^{T} v^{-} d t \leq \int_{0}^{T}|D v|+\left|v\left(T^{-}\right)-v\left(0^{+}\right)\right|
$$

Proof. See [29, Proposition 2.6].
Corollary 2.6 (Symmetric Wirtinger inequality). For every $v \in B V(0, T)$ such that $\int_{0}^{T} v d t$ $=0$ we have

$$
\|v\|_{L^{1}} \leq \frac{T}{4}\left(\int_{0}^{T}|D v|+\left|v\left(T^{-}\right)-v\left(0^{+}\right)\right|\right)
$$

Proof. See [29, Corollary 2.7].
Proposition 2.7 (An oscillation estimate). For every $v \in B V(0, T)$ we have

$$
2(\underset{[0, T]}{\operatorname{esssup}} v-\underset{[0, T]}{\operatorname{essinf}} v) \leq \int_{0}^{T}|D v|+\left|v\left(T^{-}\right)-v\left(0^{+}\right)\right| .
$$

Proof. See [29, Proposition 2.9].

Proposition 2.8 (A continuous projector). Fix $\mu, \nu \in \mathbb{R}_{0}^{+}$. For each $v \in L^{1}(0, T)$ there exists a unique $\mathcal{P}(v) \in \mathbb{R}$ such that

$$
\mu \int_{0}^{T}(v-\mathcal{P}(v))^{+} d t-\nu \int_{0}^{T}(v-\mathcal{P}(v))^{-} d t=0
$$

The map $\mathcal{P}: L^{1}(0, T) \rightarrow \mathbb{R}$ such that $v \mapsto \mathcal{P}(v)$ is a continuous projector.
Proof. See [29, Proposition 2.11].

## 3 Lower and upper solutions

Throughout we assume
$\left(h_{1}\right) f:[0, T] \times \mathbb{R} \rightarrow \mathbb{R}$, with $T>0$, satisfies the Carathéodory conditions, i.e., $f(t, \cdot): \mathbb{R} \rightarrow \mathbb{R}$ is continuous for a.e. $t \in[0, T]$ and $f(\cdot, s):[0, T] \rightarrow \mathbb{R}$ is measurable for every $s \in \mathbb{R}$; the same symbol $f$ will also be used for denoting the T-periodic extension, with respect to the first variable, of $f$ a.e. onto $\mathbb{R}$.

Notion of solution. We say that a function $u \in B V(0, T)$ is a solution of the $T$-periodic problem associated with (1) if $f(\cdot, u) \in L^{1}(0, T)$ and

$$
\begin{equation*}
\mathcal{J}(v)-\mathcal{J}(u) \geq \int_{0}^{T} f(t, u)(v-u) d t \tag{9}
\end{equation*}
$$

for all $v \in B V(0, T)$. Note that this means that $u$ is a minimizer in $B V(0, T)$ of the functional $v \mapsto \mathcal{J}(v)-\int_{0}^{T} f(t, u) v d t$.

Remark 3.1 It has ben proved in [28] that if $u \in W_{T}^{1,1}(0, T)$ is a solution of the $T$-periodic problem associated with (1), then it is a weak solution of the same problem, in the sense that

$$
\int_{0}^{T} \frac{u^{\prime} v^{\prime}}{\sqrt{1+{u^{\prime 2}}^{2}}} d t=\int_{0}^{T} f(t, u) v d t
$$

for every $v \in W_{T}^{1,1}(0, T)$, and hence one has

$$
\begin{gathered}
u^{\prime} / \sqrt{1+u^{\prime 2}} \in W_{T}^{1,1}(0, T) \\
\left.-\left(u^{\prime} / \sqrt{1+u^{\prime 2}}\right)^{\prime}=f(t, u) \quad \text { a.e. in }\right] 0, T[ \\
u(0)=u(T), \quad\left(u^{\prime} / \sqrt{1+u^{\prime 2}}\right)(0)=\left(u^{\prime} / \sqrt{1+u^{\prime 2}}\right)(T) .
\end{gathered}
$$

It is worth noting that $u$ may present a derivative blow up, but $u^{\prime} \in C^{0}([0, T],[-\infty,+\infty])$. Therefore $u^{\prime}$ satisfies the periodicity condition in an extended sense, i.e., possibly $u^{\prime}(0)=$ $u^{\prime}(T)=+\infty$, or $u^{\prime}(0)=u^{\prime}(T)=-\infty$. This kind of non-classical solutions of the prescribed curvature equation, even possibly discontinuous (hence belonging to $S B V(0, T)$, the space of special functions of bounded variation [2]), has already been considered in [11, 10, 26, 31, 12].

Conversely, it is easily seen, using the convexity of the function $s \mapsto \sqrt{1+s^{2}}$, that a weak solution of the $T$-periodic problem associated with (1) satisfies (9) for all $v \in B V(0, T)$.

Remark 3.2 Suppose $u$ is a solution of the $T$-periodic problem associated with (1) and extend $u$ a.e. onto $\mathbb{R}$ by $T$-periodicity. Fix any $t_{0} \in \mathbb{R}$ and denote by $\tilde{u}$ its restriction onto $\left[t_{0}, t_{0}+T\right]$. Then $\tilde{u}$ is a solution of the $T$-periodic problem associated with 11 on the interval $\left[t_{0}, t_{0}+T\right]$, that is

$$
\begin{aligned}
\int_{t_{0}}^{t_{0}+T} & \sqrt{1+|D v|^{2}}+\left|v\left(\left(t_{0}+T\right)^{-}\right)-v\left(t_{0}^{+}\right)\right| \\
& \quad-\int_{t_{0}}^{t_{0}+T} \sqrt{1+|D \tilde{u}|^{2}}-\left|\tilde{u}\left(\left(t_{0}+T\right)^{-}\right)-\tilde{u}\left(t_{0}^{+}\right)\right| \\
& \geq \int_{t_{0}}^{t_{0}+T} f(t, u)(v-\tilde{u}) d t
\end{aligned}
$$

for every $v \in B V\left(t_{0}, t_{0}+T\right)$. Indeed, for any $v \in B V\left(t_{0}, t_{0}+T\right)$ we can define $\tilde{v} \in B V(0, T)$ by setting $\tilde{v}(t)=v\left(t+t_{0}\right)$ a.e. in $[0, T]$. Then we have

$$
\begin{aligned}
\int_{t_{0}}^{t_{0}+T} & \sqrt{1+|D v|^{2}}+\left|v\left(\left(t_{0}+T\right)^{-}\right)-v\left(t_{0}^{+}\right)\right| \\
& \quad-\int_{t_{0}}^{t_{0}+T} \sqrt{1+|D \tilde{u}|^{2}}-\left|\tilde{u}\left(\left(t_{0}+T\right)^{-}\right)-\tilde{u}\left(t_{0}^{+}\right)\right| \\
= & \int_{0}^{T} \sqrt{1+|D \tilde{v}|^{2}}+\left|\tilde{v}\left(T^{-}\right)-\tilde{v}\left(0^{+}\right)\right|-\int_{0}^{T} \sqrt{1+|D u|^{2}}-\left|u\left(T^{-}\right)-u\left(0^{+}\right)\right| \\
\geq & \int_{0}^{T} f(t, u)(\tilde{v}-u) d t=\int_{t_{0}}^{t_{0}+T} f(t, \tilde{u})(v-\tilde{u}) d t .
\end{aligned}
$$

Lower and upper solutions. The following notions of lower and upper solutions are adopted.

- Fix $p \in\left[1,+\infty\left[\cup\{\infty\}\right.\right.$. We say that a function $\alpha \in W^{1, \infty}(0, T)$ is a $W^{2, p}$-lower solution of the $T$-periodic problem associated with (1) if there exist functions $\alpha_{1}, \ldots, \alpha_{m} \in$ $W^{2, p}(0, T)$ such that $\alpha=\alpha_{1} \vee \cdots \vee \alpha_{m}$ and, for each $i=1, \ldots, m$,

$$
\begin{equation*}
-\left(\alpha_{i}^{\prime} / \sqrt{1+\alpha_{i}^{\prime 2}}\right)^{\prime} \leq f\left(t, \alpha_{i}\right) \text { a.e. in }[0, T], \quad \alpha_{i}(0)=\alpha_{i}(T), \quad \alpha_{i}^{\prime}(0) \geq \alpha_{i}^{\prime}(T) \tag{10}
\end{equation*}
$$

- We say that a function $\alpha \in B V(0, T)$ is a $B V$-lower solution of the $T$-periodic problem associated with (11) if there exist functions $\alpha_{1}, \ldots, \alpha_{m} \in B V(0, T)$ such that $\alpha=\alpha_{1} \vee$ $\cdots \vee \alpha_{m}$ and, for each $i=1, \ldots, m, f\left(\cdot, \alpha_{i}\right) \in L^{1}(0, T)$ and

$$
\begin{equation*}
\mathcal{J}\left(\alpha_{i}+z\right)-\mathcal{J}\left(\alpha_{i}\right) \geq \int_{0}^{T} f\left(t, \alpha_{i}\right) z d t \tag{11}
\end{equation*}
$$

for all $z \in B V(0, T)$ with $z \leq 0$.

- Fix $p \in\left[1,+\infty\left[\cup\{\infty\}\right.\right.$. We say that a function $\beta \in W^{1, \infty}(0, T)$ is a $W^{2, p}$-upper solution of the $T$-periodic problem associated with (1) if there exist functions $\beta_{1}, \ldots, \beta_{n} \in$ $W^{2, p}(0, T)$ such that $\beta=\beta_{1} \wedge \cdots \wedge \beta_{n}$ and, for each $i=1, \ldots, n$,

$$
-\left(\beta_{i}^{\prime} / \sqrt{1+{\beta_{i}^{\prime}}^{2}}\right)^{\prime} \geq f\left(t, \beta_{i}\right) \text { a.e. in }[0, T], \quad \beta_{i}(0)=\beta_{i}(T), \beta_{i}^{\prime}(0) \leq \beta_{i}^{\prime}(T)
$$

- We say that a function $\beta \in B V(0, T)$ is a $B V$-upper solution of the $T$-periodic problem associated with (11) if there exist functions $\beta_{1}, \ldots, \beta_{n} \in B V(0, T)$ such that $\beta=\beta_{1} \wedge$ $\cdots \wedge \beta_{n}$ and, for each $j=1, \ldots, n, f\left(\cdot, \beta_{j}\right) \in L^{1}(0, T)$ and

$$
\begin{equation*}
\mathcal{J}\left(\beta_{j}+z\right)-\mathcal{J}\left(\beta_{j}\right) \geq \int_{0}^{T} f\left(t, \beta_{j}\right) z d t \tag{12}
\end{equation*}
$$

for all $z \in B V(0, T)$ with $z \geq 0$.

- We say that a lower solution $\alpha$ (respectively an upper solution $\beta$ ) of the $T$-periodic problem associated with (1) if proper if it is not a solution.

Remark 3.3 A function $\alpha \in B V(0, T)$, with $f(\cdot, \alpha) \in L^{1}(0, T)$, is a $B V$-lower solution of the $T$-periodic problem associated with (1), with $m=1$, if and only if $\alpha$ minimizes the functional $v \mapsto \mathcal{J}(v)-\int_{0}^{T} f(t, \alpha) v d t$ on the cone $\{v \in B V(0, T): v \leq \alpha\}$. Similarly, $\beta \in B V(0, T)$, with $f(\cdot, \beta) \in L^{1}(0, T)$, is a $B V$-upper solution of the $T$-periodic problem associated with (1), with $n=1$, if and only if $\beta$ minimizes the functional $v \mapsto \mathcal{J}(v)-\int_{0}^{T} f(t, \beta) v d t$ on the cone $\{v \in B V(0, T): v \geq \beta\}$. This notion of lower and upper solutions has already been used in [19. Section 12] for dealing with classical solutions of the minimal surface equation, as well as in [24, 27] for studying the Dirichlet, the Neumann and the mixed problems for the prescribed mean curvature equation in the setting of bounded variation functions.

Remark 3.4 A function $u \in B V(0, T)$ is a solution of the $T$-periodic problem associated with (1) if and only if it is simultaneously a $B V$-lower solution of the $T$-periodic problem associated with (11), with $m=1$, and a $B V$-upper solution of the $T$-periodic problem associated with (1), with $n=1$. The proof of this fact is similar to the one of [27, Remark 2.3].

Proposition 3.1. Suppose that $\alpha=\alpha_{1} \vee \cdots \vee \alpha_{m}$ is a $W^{2,1}$-lower solution of the $T$-periodic problem associated with (1) such that, for each $i=1, \ldots, m, f\left(\cdot, \alpha_{i}\right) \in L^{1}(0, T)$. Then $\alpha$ is a $B V$-lower solution of the T-periodic problem associated with (1).

Proof. We may suppose $m=1$. Let $z \in W^{1,1}(0, T)$ be such that $z \leq 0$. Multiplying the first inequality in 10 by $z$ and integrating by parts we obtain, as $\alpha(0)=\alpha(T)$,
$\int_{0}^{T} f(t, \alpha) z d t \leq \int_{0}^{T} \alpha^{\prime} z^{\prime} / \sqrt{1+{\alpha^{\prime}}^{2}} d t+\alpha^{\prime}(0) z(0) / \sqrt{1+\alpha^{\prime}(0)^{2}}-\alpha^{\prime}(T) z(T) / \sqrt{1+\alpha^{\prime}(T)^{2}}$.
Using the convexity of the function $s \mapsto \sqrt{1+s^{2}}$ and the assumption $\alpha^{\prime}(0) \geq \alpha^{\prime}(T)$ we get

$$
\begin{aligned}
\int_{0}^{T} f(t, \alpha) z d t & \leq \int_{0}^{T} \sqrt{1+(\alpha+z)^{\prime 2}} d t-\int_{0}^{T} \sqrt{1+z^{\prime 2}} d t+|z(T)-z(0)| \\
& =\mathcal{J}(\alpha+z)-\mathcal{J}(\alpha)
\end{aligned}
$$

Now, let $z \in B V(0, T)$ be such that $z \leq 0$. Set $v=\alpha+z$. By Proposition 2.2 there exists a sequence $\left(w_{n}\right)_{n}$ in $W^{1,1}(0, T)$ such that $w_{n} \leq \alpha+z$ for every $n, \lim _{n \rightarrow+\infty} w_{n}=\alpha+z$ in $L^{1}(0, T)$ and a.e. in $[0, T]$, and $\lim _{n \rightarrow+\infty} \mathcal{J}\left(w_{n}\right)=\mathcal{J}(\alpha+z)$. Note that $\left(w_{n}\right)_{n}$ is bounded in $B V(0, T)$ and, hence, in $L^{\infty}(0, T)$. Set, for each $n, z_{n}=w_{n}-\alpha$; we have $z_{n} \in W^{1,1}(0, T)$
and $z_{n} \leq z \leq 0$. Moreover, $\left(z_{n}\right)_{n}$ is bounded in $L^{\infty}(0, T)$ and $\lim _{n \rightarrow+\infty} z_{n}=z$ in $L^{1}(0, T)$ and a.e. in $[0, T]$. Hence we get, using the Lebesgue dominated convergence theorem,

$$
\begin{aligned}
\mathcal{J}(\alpha+z) & =\lim _{n \rightarrow+\infty} \mathcal{J}\left(\alpha+z_{n}\right) \\
& \geq \lim _{n \rightarrow+\infty} \int_{0}^{T} f(t, \alpha) z_{n} d t+\mathcal{J}(\alpha)=\int_{0}^{T} f(t, \alpha) z d t+\mathcal{J}(\alpha)
\end{aligned}
$$

i.e., $\alpha$ is a $B V$-lower solution of the $T$-periodic problem associated with (1).

A similar result can be proved for upper solutions.
Proposition 3.2. Suppose that $\beta=\beta_{1} \wedge \cdots \wedge \beta_{n}$ is a $W^{2,1}$-upper solution of the $T$-periodic problem associated with (1) such that, for each $i=1, \ldots, n, f\left(\cdot, \beta_{i}\right) \in L^{1}(0, T)$. Then $\beta$ is a $B V$-upper solution of the T-periodic problem associated with (1).

Remark 3.5 Let $\alpha$ be a $B V$-lower solution of the $T$-periodic problem associated with (1). Assume that $m=1$ and $\alpha \in W_{T}^{1,1}(0, T) \cap W^{2,1}(0, T)$. Then $\alpha$ is a $W^{2,1}$-lower solution. Indeed, fix $z \in W_{T}^{1,1}(0, T)$, with $z \leq 0$. From 11 we have, for every $s>0$,

$$
\frac{\mathcal{J}(\alpha+s z)-\mathcal{J}(\alpha)}{s} \geq \int_{0}^{T} f(t, \alpha) z d t
$$

Letting $s \rightarrow 0^{+}$we get, as $\mathcal{J}$ restricted to $W_{T}^{1,1}(0, T)$ is Gateaux differentiable,

$$
\int_{0}^{T} \alpha^{\prime} z^{\prime} / \sqrt{1+\alpha^{\prime 2}} d t \geq \int_{0}^{T} f(t, \alpha) z d t
$$

Integrating by parts we get

$$
-\int_{0}^{T}\left(\alpha^{\prime} / \sqrt{1+{\alpha^{\prime}}^{2}}\right)^{\prime} z d t+\left(\alpha^{\prime}(T) / \sqrt{1+\alpha^{\prime}(T)^{2}}-\alpha^{\prime}(0) / \sqrt{1+\alpha^{\prime}(0)^{2}}\right) z(0) \geq \int_{0}^{T} f(t, \alpha) z d t
$$

By choosing $z$ with compact support we obtain

$$
-\left(\alpha^{\prime} / \sqrt{1+\alpha^{\prime 2}}\right)^{\prime} \leq f(t, \alpha)
$$

a.e. in $[0, T]$. Since we can choose $z$ such that $z(0)=-1$ and the quantity

$$
\int_{0}^{T}\left(\left(\alpha^{\prime} / \sqrt{1+\alpha^{\prime 2}}\right)^{\prime}+f(t, \alpha)\right) z d t
$$

is as small as we wish, we easily conclude that

$$
\alpha^{\prime}(T) / \sqrt{1+\alpha^{\prime}(T)^{2}}-\alpha^{\prime}(0) / \sqrt{1+\alpha^{\prime}(0)^{2}} \leq 0
$$

i.e.,

$$
\alpha^{\prime}(T) \leq \alpha^{\prime}(0)
$$

A similar conclusion holds for upper solutions.

Construction of lower and upper solutions. Now we produce some explicit conditions on the function $f$ which guarantee the existence of a lower solution, or an upper solution. The first statement concerns the simplest case of constant lower and upper solutions.

Proposition 3.3. If $\alpha \in \mathbb{R}$ is such that $f(t, \alpha) \geq 0$ a.e. in $[0, T]$, then $\alpha$ is a $W^{2, \infty}$ lower solution of the $T$-periodic problem associated with (1). Similarly, if $\beta \in \mathbb{R}$ is such that $f(t, \beta) \geq 0$ a.e. in $[0, T]$, then $\beta$ is a $W^{2, \infty}$-upper solution of the $T$-periodic problem associated with (1).

Alternatively, the existence of a lower solution, or an upper solution, follows assuming suitable conditions of Landesman-Lazer type (see [22, 23]). It is convenient in this setting to split $f$ as

$$
\begin{equation*}
f(t, s)=g(t, s)-e(t) \tag{13}
\end{equation*}
$$

where $e \in L^{1}(0, T)$. In the sequel we set $\bar{e}=\frac{1}{T} \int_{0}^{T} e d t$ and $\tilde{e}=e-\bar{e}$.
Proposition 3.4. Assume ( $h_{1}$ ) and
( $h_{2}$ ) for each $r>0$ there exists $\gamma \in L^{1}(0, T)$ such that $|f(t, s)| \leq \gamma(t)$ for a.e. $t \in[0, T]$ and every $s \in[-r, r]$.
Take $e \in L^{\infty}(0, T)$ and define $g$ by (13). Assume further
$\left(h_{3}\right)$ there exist $c \in \mathbb{R} \cup\{-\infty\}$ and $d \in \mathbb{R} \cup\{+\infty\}$, with $c<d$, such that

$$
g(t, s) \geq \bar{e}
$$

for a.e. $t \in[0, T]$ and every $s \in] c, d[$,
and
$\left(h_{4}\right)$ there exist $\mu, \nu \in \mathbb{R}_{0}^{+}$, with

$$
\frac{1}{\sqrt{\mu}}+\frac{1}{\sqrt{\nu}}=\sqrt{T}
$$

and $\vartheta \in] 0,1[$ such that

$$
\underset{[0, T]}{\operatorname{essinf}} \tilde{e} \geq-\vartheta \mu \quad \text { and } \quad \underset{[0, T]}{\operatorname{esssup}} \tilde{e} \leq \vartheta \nu
$$

Finally, suppose that

$$
\begin{equation*}
\frac{T}{1-\vartheta} \leq d-c \tag{14}
\end{equation*}
$$

Then there exists a $B V$-lower solution $\alpha$ of the $T$-periodic problem associated with (1) such that $c \leq \alpha(t) \leq d$ a.e. in $[0, T]$.

Proof. We first show that, for any $h \in L^{\infty}(0, T)$ with $\int_{0}^{T} h d t=0, \underset{[0, T]}{\operatorname{ess} \sup } h \leq \vartheta \mu$ and $\underset{[0, T]}{\operatorname{ess} \inf } h \geq-\vartheta \nu$, the $T$-periodic problem associated with

$$
\begin{equation*}
-\left(u^{\prime} / \sqrt{1+u^{\prime 2}}\right)^{\prime}=h(t) \tag{15}
\end{equation*}
$$

has at least one solution $w \in B V(0, T)$ satisfying

$$
\mu \int_{0}^{T} w^{+} d t-\nu \int_{0}^{T} w^{-} d t=0
$$

and

$$
\begin{equation*}
\|w\|_{L^{\infty}} \leq \frac{1}{2} \frac{T}{1-\vartheta} \tag{16}
\end{equation*}
$$

To this end let us set

$$
\mathcal{S}=\left\{v \in B V(0, T): \mu \int_{0}^{T} v^{+} d t=\nu \int_{0}^{T} v^{-} d t\right\}
$$

We endow $\mathcal{S}$ with the norm

$$
\int_{0}^{T}|D v|+\left|v\left(T^{-}\right)-v\left(0^{+}\right)\right|
$$

which, by Proposition 2.5 is equivalent to

$$
\|v\|_{B V}=\int_{0}^{T}|D v|+\|v\|_{L^{1}}
$$

Observe that $\mathcal{S}$ is a Banach space. Define a functional $\mathcal{H}: B V(0, T) \rightarrow \mathbb{R}$ by setting

$$
\begin{equation*}
\mathcal{H}(v)=\mathcal{J}(v)-\int_{0}^{T} h v d t . \tag{17}
\end{equation*}
$$

Estimate (4) and Proposition 2.5 again imply that, for every $v \in \mathcal{S}$,

$$
\begin{aligned}
\mathcal{H}(v) & \geq \int_{0}^{T}|D v|+\left|v\left(T^{-}\right)-v\left(0^{+}\right)\right|-\vartheta\left(\mu \int_{0}^{T} v^{+} d t+\nu \int_{0}^{T} v^{-} d t\right) \\
& \geq(1-\vartheta)\left(\int_{0}^{T}|D v|+\left|v\left(T^{-}\right)-v\left(0^{+}\right)\right|\right)
\end{aligned}
$$

Hence $\mathcal{H}$ is bounded from below and coercive in $\mathcal{S}$. Let $\left(w_{n}\right)_{n}$ be a minimizing sequence. Since $\left(w_{n}\right)_{n}$ is bounded in $\mathcal{S}$, there exists a subsequence of $\left(w_{n}\right)_{n}$, which we still denote by $\left(w_{n}\right)_{n}$, and a function $w \in \mathcal{S}$ such that $\lim _{n \rightarrow+\infty} w_{n}=w$ in $L^{1}(0, T)$. As $\mathcal{H}$ is lower semicontinuous with respect to the $L^{1}$-convergence in $\mathcal{S}, \mathcal{H}$ has a global minimum at $w$, and $w$ satisfies

$$
\begin{equation*}
\mathcal{J}(v)-\mathcal{J}(w) \geq \int_{0}^{T} h(v-w) d t \tag{18}
\end{equation*}
$$

for every $v \in \mathcal{S}$ and, actually, for every $v \in B V(0, T)$, as $\mathcal{H}(v+k)=\mathcal{H}(v)$ for all $k \in \mathbb{R}$. Hence $w$ is a solution of 15. Now, taking $v=0$ in 18), we have

$$
\begin{aligned}
\int_{0}^{T}|D w| & +\left|w\left(T^{-}\right)-w\left(0^{+}\right)\right| \leq \mathcal{J}(w) \leq \mathcal{J}(0)+\int_{0}^{T} h w d t \\
& \leq T+\vartheta\left(\mu \int_{0}^{T} w^{+} d t+\nu \int_{0}^{T} w^{-} d t\right) \leq T+\vartheta\left(\int_{0}^{T}|D w|+\left|w\left(T^{-}\right)-w\left(0^{+}\right)\right|\right)
\end{aligned}
$$

and hence, using Proposition 2.7, estimate (16) follows.
Next we show how to construct a lower solution $\alpha$, with $c \leq \alpha \leq d$. Let $w$ be a solution of the $T$-periodic problem associated with (15), with $h=-\tilde{e}$. In case $c=-\infty$ or $d=+\infty$, we
can find a constant $b$ such that, setting $\alpha=w+b$, we have $c \leq \alpha \leq d$. Otherwise, we define $\alpha=\frac{1}{2}(c+d)+w$. We get, by 14, $c \leq \alpha \leq d$ and, by $\left(h_{3}\right)$,

$$
\mathcal{J}(\alpha+z)-\mathcal{J}(\alpha)=\mathcal{J}(w+z)-\mathcal{J}(w) \geq-\int_{0}^{T} \tilde{e} z d t \geq \int_{0}^{T} g(t, \alpha) z d t-\int_{0}^{T} e z d t
$$

for every $z \in B V(0, T)$ with $z \leq 0$. Hence $\alpha$ is a $B V$-lower solution of the $T$-periodic problem associated with (1).

Remark 3.6 We note that, in case $c=-\infty$ or $d=+\infty$, relation (14) is trivially satisfied.
Proposition 3.5. Assume $\left(h_{1}\right)$ and $\left(h_{2}\right)$. Take $e \in L^{\infty}(0, T)$ and define $g$ by 13). Assume further
( $h_{5}$ ) there exist $c \in \mathbb{R} \cup\{-\infty\}$ and $d \in \mathbb{R} \cup\{+\infty\}$, with $c<d$, such that

$$
g(t, s) \leq \bar{e}
$$

for a.e. $t \in[0, T]$ and every $s \in] c, d[$,
$\left(h_{4}\right)$ and suppose that 14 is satisfied. Then there exists a $B V$-upper solution $\beta$ of the $T$ periodic problem associated with (1) such that $c \leq \beta(t) \leq d$ a.e. in $[0, T]$.

We show now that the two-sided bound on $\tilde{e}$ required by $\left(h_{4}\right)$ can be replaced by a onesided bound, as expressed by $\left(h_{6}\right)$.

Proposition 3.6. Assume $\left(h_{1}\right)$ and $\left(h_{2}\right)$. Take $e \in L^{\infty}(0, T)$ and define $g$ by 13). Assume further ( $h_{3}$ ),
$\left(h_{6}\right)$ there exists $\left.\vartheta \in\right] 0,1\left[\right.$ such that either $\left\|\tilde{e}^{+}\right\|_{L^{1}} \leq 2 \vartheta$, or $\left\|\tilde{e}^{-}\right\|_{L^{1}} \leq 2 \vartheta$,
and suppose that (14) is satisfied. Then there exists a $B V$-lower solution $\alpha$ of the $T$-periodic problem associated with (1) such that $c \leq \alpha(t) \leq d$ a.e. in $[0, T]$.

Proof. We shall assume $\left\|\tilde{e}^{+}\right\|_{L^{1}} \leq 2 \vartheta$ in $\left(h_{6}\right)$. The proof in case $\left\|\tilde{e}^{-}\right\|_{L^{1}} \leq 2 \vartheta$ is similar. Arguing as in the proof of Proposition 3.4 we can show that, for any $h \in L^{1}(0, T)$ such that $\int_{0}^{T} h d t=0$ and $\left\|h^{-}\right\|_{L^{1}} \leq 2 \vartheta$, the $T$-periodic problem associated with 15 has at least one solution $w \in B V(0, T)$ satisfying $\underset{[0, T]}{\operatorname{ess} \sup } w=0$ and $\sqrt{16}$. To this end we set now

$$
\mathcal{S}=\{v \in B V(0, T): \underset{[0, T]}{\operatorname{ess} \sup } v=0\}
$$

and endow $\mathcal{S}$ with the norm

$$
\int_{0}^{T}|D v|+\left|v\left(T^{-}\right)-v\left(0^{+}\right)\right| .
$$

We define the functional $\mathcal{H}: B V(0, T) \rightarrow \mathbb{R}$ as in 17) and, using Proposition 2.7 and $\left(h_{6}\right)$, we show that, for every $v \in \mathcal{S}$,

$$
\begin{aligned}
\mathcal{H}(v) & \geq \int_{0}^{T}|D v|+\left|v\left(T^{-}\right)-v\left(0^{+}\right)\right|+\underset{[0, T]}{\operatorname{ess} \inf } v \int_{0}^{T} h^{-} d t \\
& \geq(1-\vartheta)\left(\int_{0}^{T}|D v|+\left|v\left(T^{-}\right)-v\left(0^{+}\right)\right|\right) .
\end{aligned}
$$

Hence $\mathcal{H}$ is bounded from below and coercive in $\mathcal{S}$. Arguing as in the proof of Proposition 3.4, we see that $\mathcal{H}$ has a global minimum at some $w \in \mathcal{S}$, which satisfies (18) for every $v \in \mathcal{S}$ and, actually, for every $v \in B V(0, T)$, as $\mathcal{H}(v+k)=\mathcal{H}(v)$ for all $k \in \mathbb{R}$. Hence $w$ is a solution of (15). Now, taking $v=0$ in (18), we obtain

$$
\begin{aligned}
\int_{0}^{T}|D w| & +\left|w\left(T^{-}\right)-w\left(0^{+}\right)\right| \leq \mathcal{J}(w) \leq \mathcal{J}(0)+\int_{0}^{T} h w d t \\
& \leq T-\int_{0}^{T} h^{-} w d t \leq T+\vartheta\left(\int_{0}^{T}|D w|+\left|w\left(T^{-}\right)-w\left(0^{+}\right)\right|\right)
\end{aligned}
$$

and hence, using Proposition 2.7, (16) is satisfied. The lower solution $\alpha$ is eventually constructed as in the proof of Proposition 3.4 .

Proposition 3.7. Assume ( $h_{1}$ ) and $\left(h_{2}\right)$. Take $e \in L^{\infty}(0, T)$ and define $g$ by 13). Suppose further that $\left(h_{5}\right)$ and $\left(h_{6}\right)$ hold and that (14) is satisfied. Then there exists a $B V$-upper solution $\beta$ of the $T$-periodic problem associated with (1) such that $c \leq \beta(t) \leq d$ a.e. in $[0, T]$.

## 4 Existence results

We develop in this section a lower and upper solutions method for the $T$-periodic problem associated with (1).

Well-ordered lower and upper solutions. The first result deals with the case where the lower solution is smaller than the upper solution. The approach is variational and provides the existence of a solution bracketed by the given lower and upper solutions. The proof is a simplified version of the argument in [27, Theorem 2.4]. Here we consider the functional $\mathcal{I}: B V(0, T) \rightarrow \mathbb{R}$ defined by

$$
\mathcal{I}(v)=\int_{0}^{T} \sqrt{1+{v^{\prime}}^{2}} d t-\int_{0}^{T} F(t, v) d t
$$

where $F(t, s)=\int_{0}^{s} f(t, \xi) d \xi$.
Theorem 4.1. Assume ( $h_{1}$ ),
$\left(h_{7}\right)$ there exists $p>1$ such that, for each $r>0$, there is $\gamma \in L^{p}(0, T)$ such that $|f(t, s)| \leq \gamma(t)$ for a.e. $t \in[0, T]$ and every $s \in[-r, r]$,
$\left(h_{8}\right)$ there exist a $B V$-lower solution $\alpha$ and a $B V$-upper solution $\beta$ of the $T$-periodic problem associated with (1) satisfying $\alpha \leq \beta$.

Then the T-periodic problem associated with (1) has at least one solution $u \in B V(0, T)$ such that

$$
\alpha \leq u \leq \beta \quad \text { and } \quad \mathcal{I}(u)=\min _{\substack{v \in B V(0, T) \\ \alpha \leq v \leq \beta}} \mathcal{I}(v)
$$

Moreover, there exist solutions $v, w$ of the T-periodic problem associated with (1), with $\alpha \leq$ $v \leq w \leq \beta$, such that every solution $u$ of the T-periodic problem associated with (1), with $\alpha \leq u \leq \beta$, satisfies $v \leq u \leq w$.

Proof. Without restriction we can assume $p \in] 1,2]$. Set $q=\frac{p}{p-1} \in[2,+\infty[$. Let $\alpha=$ $\alpha_{1} \vee \cdots \vee \alpha_{m}$ and $\beta=\beta_{1} \wedge \cdots \wedge \beta_{n}$ where, for each $i=1, \ldots, m, \alpha_{i}$ satisfies (11) for all $z \in B V(0, T)$ with $z \leq 0$ and, for all $j=1, \ldots, n, \beta_{j}$ satisfies 12$)$ for all $z \in B V(0, T)$ with $z \geq 0$.
Step 1. A modified problem. Let us set, for a.e. $t \in[0, T]$ and every $s \in \mathbb{R}$,

$$
\begin{gathered}
Q(s)=|s|^{q}, \\
h_{i}(t, s)= \begin{cases}f\left(t, \alpha_{i}(t)\right)+Q^{\prime}\left(\alpha_{i}(t)\right) & \text { if } s<\alpha_{i}(t), \\
f(t, s)+Q^{\prime}(s) & \text { if } s \geq \alpha_{i}(t)\end{cases} \\
k_{j}(t, s)= \begin{cases}f\left(t, \beta_{j}(t)\right)+Q^{\prime}\left(\beta_{j}(t)\right) & \text { if } s>\beta_{j}(t), \\
f(t, s)+Q^{\prime}(s) & \text { if } s \leq \beta_{j}(t),\end{cases}
\end{gathered}
$$

for $i=1, \ldots, m, j=1, \ldots, n$, and

$$
\ell(t, s)= \begin{cases}\max _{i=1, \ldots, m} h_{i}(t, s) & \text { if } s<\alpha(t) \\ f(t, s)+Q^{\prime}(s) & \text { if } \alpha(t) \leq s \leq \beta(t) \\ \min _{j=1, \ldots, n} k_{j}(t, s) & \text { if } s>\beta(t)\end{cases}
$$

Clearly, $Q$ is of class $C^{1}$ and strictly convex and $\ell$ satisfies the Carathéodory conditions ( $h_{1}$ ) and $\left(h_{7}\right)$. Moreover, there exists a function $\lambda \in L^{p}(0, T)$ such that, for a.e. $t \in[0, T]$ and every $s \in \mathbb{R}$,

$$
|\ell(t, s)| \leq \lambda(t)
$$

and hence, setting $L(t, s)=\int_{0}^{s} \ell(t, \xi) d \xi$,

$$
\begin{equation*}
|L(t, s)| \leq \lambda(t)|s| \tag{19}
\end{equation*}
$$

Let us consider the modified equation

$$
\begin{equation*}
-\left(u^{\prime} / \sqrt{1+u^{\prime 2}}\right)^{\prime}=\ell(t, u)-Q^{\prime}(u) \tag{20}
\end{equation*}
$$

Of course, a solution of the $T$-periodic problem associated with 20 is a function $u \in B V(0, T)$ such that

$$
\begin{equation*}
\mathcal{J}(v)-\mathcal{J}(u) \geq \int_{0}^{T}\left(\ell(t, u)-Q^{\prime}(u)\right)(v-u) d t \tag{21}
\end{equation*}
$$

for every $v \in B V(0, T)$.
Step 2. Existence of solutions of the modified problem. We define a functional $\mathcal{K}: B V(0, T) \rightarrow \mathbb{R}$ by setting

$$
\mathcal{K}(v)=\mathcal{J}(v)+\int_{0}^{T} Q(v) d t-\int_{0}^{T} L(t, v) d t
$$

We aim to show that there exists $\min _{v \in B V(0, T)} \mathcal{K}(v)$. We first observe that $\mathcal{K}$ is coercive and bounded from below in $B V(0, T)$. Indeed, using (4), (19) and standard inequalities, we can
find constants $d_{1}, d_{2}>0$ such that

$$
\begin{aligned}
\mathcal{K}(v) & \geq \int_{0}^{T}|D v|+\left|v\left(T^{-}\right)-v\left(0^{+}\right)\right|+\|v\|_{L^{q}}^{q}-\|\lambda\|_{L^{p}}\|v\|_{L^{q}} \\
& \geq d_{1}\|v\|_{B V}-d_{2},
\end{aligned}
$$

for every $v \in B V(0, T)$. Let $\left(u_{n}\right)_{n}$ be a minimizing sequence. Since $\left(u_{n}\right)_{n}$ is bounded in $B V(0, T)$, there exists a subsequence of $\left(u_{n}\right)_{n}$, which we still denote by $\left(u_{n}\right)_{n}$, and a function $u \in B V(0, T)$ such that $\lim _{n \rightarrow+\infty} u_{n}=u$ in $L^{q}(0, T)$. By Proposition 2.3 we have

$$
\liminf _{n \rightarrow+\infty} \mathcal{J}\left(u_{n}\right) \geq \mathcal{J}(u)
$$

and, as the functionals $v \mapsto \int_{0}^{T} Q(v) d t$ and $v \mapsto \int_{0}^{T} L(t, v) d t$ are continuous in $L^{q}(0, T)$,

$$
\lim _{n \rightarrow+\infty} \int_{0}^{T}\left(Q\left(u_{n}\right)-L\left(t, u_{n}\right)\right) d t=\int_{0}^{T}(Q(u)-L(t, u)) d t
$$

Hence, we conclude that

$$
\inf _{v \in B V(0, T)} \mathcal{K}(v)=\lim _{n \rightarrow+\infty} \mathcal{K}\left(u_{n}\right) \geq \mathcal{K}(u),
$$

that is, $\mathcal{K}(u)=\min _{v \in B V(0, T)} \mathcal{K}(v)$.
Observe that any minimizer $u$ of $\mathcal{K}$ satisfies 21 for every $v \in B V(0, T)$. Indeed, let $v \in$ $B V(0, T)$ and take $s \in] 0,1[$. By the convexity of $\mathcal{J}$ we obtain

$$
\begin{aligned}
(1-s) \mathcal{J}(u) & +s \mathcal{J}(v)-\mathcal{J}(u) \geq \mathcal{J}((1-s) u+s v)-\mathcal{J}(v) \\
& \geq-\int_{0}^{T}((Q(u+s(v-u))-L(t, u+s(v-u))-(Q(u)-L(t, u))) d t
\end{aligned}
$$

and, dividing by $s$,

$$
\mathcal{J}(v)-\mathcal{J}(u) \geq-\int_{0}^{T} \frac{1}{s}((Q(u+s(v-u))-L(t, u+s(v-u)))-(Q(u)-L(t, u))) d t
$$

Hence, letting $s \rightarrow 0^{+}$, we easily get

$$
\mathcal{J}(v)-\mathcal{J}(u) \geq-\int_{0}^{T}\left(Q^{\prime}(u)-\ell(t, u)\right)(v-u) d t
$$

i.e., (21) holds. We conclude that the $T$-periodic problem associated with (20) has at least one solution.
Step 3. Any solution $u$ of the $T$-periodic problem associated with 20 satisfies $\alpha \leq u \leq \beta$. Let us show that $u \leq \beta$; by a similar argument we can prove that $u \geq \alpha$. We fix $j \in\{1, \ldots, n\}$ and verify that $u \leq \beta_{j}$. Taking $v=u \wedge \beta_{j}=u-\left(u-\beta_{j}\right)^{+}$as a test function in (21) we obtain

$$
\begin{align*}
\mathcal{J}\left(u \wedge \beta_{j}\right) & -\mathcal{J}(u) \geq-\int_{0}^{T}\left(\ell(t, u)-Q^{\prime}(u)\right)\left(u-\beta_{j}\right)^{+} d t \\
& \geq-\int_{0}^{T} f\left(t, \beta_{j}\right)\left(u-\beta_{j}\right)^{+} d t+\int_{0}^{T}\left(Q^{\prime}(u)-Q^{\prime}\left(\beta_{j}\right)\right)\left(u-\beta_{j}\right)^{+} d t . \tag{22}
\end{align*}
$$

Taking $z=\left(u-\beta_{j}\right)^{+}$as a test function in we have, as $u \vee \beta_{j}=\beta_{j}+\left(u-\beta_{j}\right)^{+}$,

$$
\begin{equation*}
\mathcal{J}\left(u \vee \beta_{j}\right)-\mathcal{J}\left(\beta_{j}\right) \geq \int_{0}^{T} f\left(t, \beta_{j}\right)\left(u-\beta_{j}\right)^{+} d t \tag{23}
\end{equation*}
$$

Summing 22 and 23 and using Proposition 2.4 and the convexity of $Q$, we find

$$
0 \geq \mathcal{J}\left(u \wedge \beta_{j}\right)+\mathcal{J}\left(u \vee \beta_{j}\right)-\mathcal{J}\left(\beta_{j}\right)-\mathcal{J}(u) \geq \int_{0}^{T}\left(Q^{\prime}(u)-Q^{\prime}\left(\beta_{j}\right)\right)\left(u-\beta_{j}\right)^{+} d t \geq 0
$$

As $Q^{\prime}$ is strictly increasing, we conclude that $\left(u-\beta_{j}\right)^{+}(t)=0$ a.e. in $[0, T]$ and therefore $u \leq \beta_{j}$.
Step 4. There is a solution $u$ of the $T$-periodic problem associated with (1) such that

$$
\alpha \leq u \leq \beta \quad \text { and } \quad \mathcal{I}(u)=\min _{\substack{v \in B V(0, T) \\ \alpha \leq v \leq \beta}} \mathcal{I}(v)
$$

Let $u$ be a solution of the $T$-periodic problem associated with 20). As $u$ is such that $\alpha \leq$ $u \leq \beta$, we have $\ell(\cdot, u)-Q^{\prime}(u)=f(\cdot, u)$ and hence $u$ is a solution of the $T$-periodic problem associated with (11). Furthermore, for a.e. $t \in[0, T]$ and every $s$, with $\alpha(t) \leq s \leq \beta(t)$, we have

$$
L(t, s)=L(t, \alpha)+F(t, s)-F(t, \alpha)+Q(s)-Q(\alpha)
$$

Hence we obtain, for every $v \in B V(0, T)$, with $\alpha \leq v \leq \beta$,

$$
\mathcal{K}(v)=\mathcal{I}(v)+\int_{0}^{T}(Q(\alpha)+F(t, \alpha)-L(t, \alpha)) d t
$$

Since $u$ minimizes $\mathcal{K}$, we conclude that $u$ minimizes $\mathcal{I}$ on the set of all $v \in B V(0, T)$, with $\alpha \leq v \leq \beta$.
Step 5. Existence of extremum solutions. Let us set

$$
\begin{aligned}
\mathcal{U}=\{u \in B V(0, T): u & \text { is a solution of the } T \text {-periodic problem } \\
& \text { associated with (1) such that } \alpha \leq u \leq \beta\} .
\end{aligned}
$$

We notice that $\mathcal{U}$ is bounded in $B V(0, T)$. Indeed, if $u \in \mathcal{U}$, taking $v=0$ as a test function in (9), we obtain

$$
\mathcal{J}(u) \leq \mathcal{J}(0)+\int_{0}^{T} f(t, u) u d t \leq T+\|\gamma\|_{L^{1}}\|u\|_{L^{\infty}} \leq T+\|\gamma\|_{L^{1}} \max \left\{\|\alpha\|_{L^{\infty}},\|\beta\|_{L^{\infty}}\right\}
$$

To prove the compactness of $\mathcal{U}$ in $L^{q}(0, T)$, take any sequence $\left(u_{n}\right)_{n}$ in $\mathcal{U}$. Since $\left(u_{n}\right)_{n}$ is bounded in $B V(0, T)$, applying the same argument we used in Step 2, we easily deduce that there exists a subsequence of $\left(u_{n}\right)_{n}$ which converges in $L^{q}(0, T)$ to some $u \in \mathcal{U}$.

Let us prove that there exists $\min \mathcal{U}$; a similar argument shows the existence of $\max \mathcal{U}$. For each $u \in \mathcal{U}$ we define the closed subset of $L^{q}(0, T)$

$$
\mathcal{C}_{u}=\{v \in \mathcal{U}: v \leq u\}
$$

The family $\left(\mathcal{C}_{u}\right)_{u \in \mathcal{U}}$ has the finite intersection property. Indeed, if $u_{1}, u_{2} \in \mathcal{U}$, then $u_{1} \wedge u_{2}$ is an upper solution of the $T$-periodic problem associated with (1), with $\alpha \leq u_{1} \wedge u_{2}$. Hence, there is a solution $u$ of the $T$-periodic problem associated with 11, with $\alpha \leq u \leq u_{1} \wedge u_{2} \leq \beta$; i.e., $u \in \mathcal{C}_{u_{1}} \cap \mathcal{C}_{u_{2}}$. The compactness of $\mathcal{U}$ implies that there exists $v \in \mathcal{U}$ such that $v \in \bigcap_{u \in \mathcal{U}} \mathcal{C}_{u}$; that is $v \leq u$ for every $u \in \mathcal{U}$.

Remark 4.1 It is clear from the proof above that we can replace in Theorem 4.1 assumption $\left(h_{1}\right)$ with
$\left(h_{9}\right) f:[0, T] \times[-\rho, \rho] \rightarrow \mathbb{R}$ satisfies the Carathéodory conditions, i.e., $f(t, \cdot):[-\rho, \rho] \rightarrow \mathbb{R}$ is continuous for a.e. $t \in[0, T]$ and $f(\cdot, s):[0, T] \rightarrow \mathbb{R}$ is measurable for every $s \in[-\rho, \rho]$, where

$$
\rho>\max _{i=1, \ldots, m ; j=1, \ldots . n}\left\{\left\|\alpha_{i}\right\|_{L^{\infty}},\left\|\beta_{j}\right\|_{L^{\infty}}\right\} .
$$

and assumption $\left(h_{7}\right)$ with
( $h_{10}$ ) there exist $p>1$ and $\gamma \in L^{p}(0, T)$ such that $|f(t, s)| \leq \gamma(t)$ for a.e. $t \in[0, T]$ and every $s \in[-\rho, \rho]$.

Remark 4.2 Let the assumptions of Theorem 4.1 be satisfied. If a solution $u$ of the $T$ periodic problem associated with (11), such that $\alpha \leq u \leq \beta$ and

$$
\mathcal{I}(u)=\min _{\substack{v \in B V(0, T) \\ \alpha \leq v \leq \beta}} \mathcal{I}(v)
$$

satisfies

$$
\underset{[0, T]}{\operatorname{ess} \inf }(u-\alpha)>0>\underset{[0, T]}{\operatorname{ess} \sup }(u-\beta),
$$

then $u$ is a local minimum point of $\mathcal{I}$ in $B V(0, T)$. Indeed, there exists a number $\delta>0$ such that, if $v \in B V(0, T)$ satisfies $\|u-v\|_{B V}<\delta$, then $\alpha \leq v \leq \beta$ and hence

$$
\mathcal{I}(u)=\min _{\substack{v \in B V(0, T) \\\|u-v\|_{B V}<\delta}} \mathcal{I}(v)
$$

Non-well-ordered lower and upper solutions. Our next result deals with the case where $\alpha$ and $\beta$ may fail to satisfy the ordering condition $\alpha \leq \beta$, assumed in Theorem 4.1. We show that this restriction can be removed at the expense of assuming a stronger notion of lower and upper solutions, as well as of placing an additional control on $f$, with respect to the second eigenvalue $4 / T$ of the 1-Laplace operator with $T$-periodic boundary conditions, or more generally with respect to the first branch $\Sigma$ of the Dancer-Fučík spectrum of the same operator, defined by

$$
\Sigma=\left\{(\mu, \nu) \in \mathbb{R}_{0}^{+} \times \mathbb{R}_{0}^{+}: \frac{1}{\sqrt{\mu}}+\frac{1}{\sqrt{\nu}}=\sqrt{T}\right\}
$$

We refer to [29] for the introduction of $\Sigma$ and for a discussion of its variational properties.
Theorem 4.2. Assume ( $h_{1}$ ),
$\left(h_{11}\right)$ there exist $a W^{2, \infty}(0, T)$-lower solution $\alpha=\alpha_{1} \vee \cdots \vee \alpha_{m}$ and a $W^{2, \infty}(0, T)$-upper solution $\beta=\beta_{1} \wedge \cdots \wedge \beta_{n}$ of the $T$-periodic problem associated with (1) such that, for each $i=1, \ldots, m$,

$$
\underset{[0, T]}{\operatorname{essinf}}\left(\left(\alpha_{i}^{\prime} / \sqrt{1+\alpha_{i}^{\prime 2}}\right)^{\prime}+f\left(t, \alpha_{i}\right)\right)>0
$$

and, for each $j=1, \ldots, n$,

$$
\underset{[0, T]}{\operatorname{ess} \sup }\left(\left(\beta_{j}^{\prime} / \sqrt{1+\beta_{j}^{\prime 2}}\right)^{\prime}+f\left(t, \beta_{j}\right)\right)<0
$$

and
$\left(h_{12}\right)$ there exist $\mu, \nu \in \mathbb{R}_{0}^{+}$, with $\frac{1}{\sqrt{\mu}}+\frac{1}{\sqrt{\nu}}=\sqrt{T}$, such that

$$
\underset{[0, T] \times \mathbb{R}}{\operatorname{ess} \sup } f<\mu \quad \text { and } \quad \underset{[0, T] \times \mathbb{R}}{\operatorname{ess} \inf } f>-\nu .
$$

Then the T-periodic problem associated with (1) has at least one solution $u \in B V(0, T)$.
Proof. In case $\alpha \leq \beta$ Theorem 4.1 guarantees the existence of a solution $u \in B V(0, T)$ of the $T$-periodic problem associated with (1). Therefore, in the sequel, we may assume that there exists $t_{0} \in[0, T]$ such that

$$
\alpha\left(t_{0}\right)>\beta\left(t_{0}\right) .
$$

Step 1. A perturbed problem. Let us take a sequence $\left(\varepsilon_{n}\right)_{n}$, with $\varepsilon_{n}>0$ for all $n$, such that $\lim _{n \rightarrow+\infty} \varepsilon_{n}=0$. For each $n$ we consider the $T$-periodic problem associated with

$$
\begin{equation*}
-\varepsilon_{n} u^{\prime \prime}-\left(u^{\prime} / \sqrt{1+u^{\prime 2}}\right)^{\prime}=f(t, u) \tag{24}
\end{equation*}
$$

This equation can also be written as

$$
\begin{equation*}
-u^{\prime \prime}=\frac{\left(1+u^{\prime 2}\right)^{3 / 2}}{1+\varepsilon_{n}\left(1+u^{\prime 2}\right)^{3 / 2}} f(t, u) \tag{25}
\end{equation*}
$$

For sake of simplicity we set, for each $n$,

$$
\begin{equation*}
g_{n}(t, s, \xi)=\frac{\left(1+\xi^{2}\right)^{3 / 2}}{1+\varepsilon_{n}\left(1+\xi^{2}\right)^{3 / 2}} f(t, s) \tag{26}
\end{equation*}
$$

for a.e. $t \in[0, T]$, every $s \in \mathbb{R}$ and every $\xi \in \mathbb{R}$. Note that, thanks to $\left(h_{12}\right)$, we have for each $n$

$$
\left|g_{n}(t, s, \xi)\right| \leq \frac{1}{\varepsilon_{n}} \max \{\mu, \nu\}
$$

Let us verify that, for large $n, \alpha$ is a lower solution of the $T$-periodic problem associated with 25. Indeed, for each $i=1 \ldots m$, as $\alpha_{i} \in W^{2, \infty}(0, T)$ and $\lim _{n \rightarrow 0} \varepsilon_{n}=0$, we have, for a.e. $t \in[0, T]$,

$$
-\left(\alpha_{i}^{\prime} / \sqrt{1+{\alpha_{i}^{\prime}}^{2}}\right)^{\prime}-f\left(t, \alpha_{i}\right) \leq \varepsilon_{n} \alpha_{i}^{\prime \prime}
$$

and hence

$$
-\alpha_{i}^{\prime \prime} \leq g_{n}\left(t, \alpha_{i}, \alpha_{i}^{\prime}\right)
$$

provided that $n$ is taken sufficiently large. Similarly, we can prove that $\beta$ is an upper solution of the $T$-periodic problem associated to (25). By [14, pp. 173-174] we conclude that, possibly relabelling the sequence $\left(\varepsilon_{n}\right)_{n}$, for each $n$ there exists a solution $u_{n} \in W^{2, \infty}(0, T)$ of the $T$-periodic problem associated with 25 such that

$$
\begin{equation*}
u_{n}\left(t_{n}^{\prime}\right) \leq \alpha\left(t_{n}^{\prime}\right) \quad \text { and } \quad u_{n}\left(t_{n}^{\prime \prime}\right) \geq \beta\left(t_{n}^{\prime \prime}\right) \tag{27}
\end{equation*}
$$

for some $t_{n}^{\prime}, t_{n}^{\prime \prime} \in[0, T]$.

Step 2. Estimates. For each $n, u_{n}$ satisfies the following weak formulation of the $T$-periodic problem associated with (24)

$$
\begin{equation*}
\varepsilon_{n} \int_{0}^{T} u_{n}^{\prime} v^{\prime} d t+\int_{0}^{T} u_{n}^{\prime} v^{\prime} / \sqrt{1+u_{n}^{\prime 2}} d t=\int_{0}^{T} f\left(t, u_{n}\right) v d t \tag{28}
\end{equation*}
$$

for all $v \in W_{T}^{1,1}(0, T)$. Let $\mathcal{P}$ be the projector operator defined in Proposition 2.8. Taking $v=\left(u_{n}-\mathcal{P}\left(u_{n}\right)\right)^{+}$as a test function in 28 and setting $E_{n}^{+}=\left\{t \in[0, T]: u_{n}(t)-\mathcal{P}\left(u_{n}\right)>0\right\}$, we get

$$
\begin{align*}
\int_{0}^{T} f\left(t, u_{n}\right)\left(u_{n}-\mathcal{P}\left(u_{n}\right)\right)^{+} d t & =\varepsilon_{n} \int_{E_{n}^{+}} u_{n}^{\prime 2} d t+\int_{E_{n}^{+}} u_{n}^{\prime 2} / \sqrt{1+u_{n}^{\prime 2}} d t \\
& \geq \int_{E_{n}^{+}}\left|u_{n}^{\prime}\right| d t-c T \tag{29}
\end{align*}
$$

where

$$
\begin{equation*}
c=\max _{s \in \mathbb{R}}\left(|s|-s^{2} / \sqrt{1+s^{2}}\right)>0 . \tag{30}
\end{equation*}
$$

Similarly, taking $v=\left(u_{n}-\mathcal{P}\left(u_{n}\right)\right)^{-}$and setting $E_{n}^{-}=\left\{t \in[0, T]: u_{n}(t)-\mathcal{P}\left(u_{n}\right)<0\right\}$, we get

$$
\begin{align*}
-\int_{0}^{T} f\left(t, u_{n}\right)\left(u_{n}-\mathcal{P}\left(u_{n}\right)\right)^{-} d t & =\varepsilon_{n} \int_{E_{n}^{-}} u_{n}^{\prime 2} d t+\int_{E_{n}^{-}} u_{n}^{\prime 2} / \sqrt{1+u_{n}^{\prime 2}} d t \\
& \geq \int_{E_{n}^{-}}\left|u_{n}^{\prime}\right| d t-c T \tag{31}
\end{align*}
$$

By (29) and (31), also using [32, Theorem 1.56], we easily obtain

$$
\int_{0}^{T}\left|u_{n}^{\prime}\right| d t-2 c T \leq \int_{0}^{T} f\left(t, u_{n}\right)\left(u_{n}-\mathcal{P}\left(u_{n}\right)\right)^{+} d t-\int_{0}^{T} f\left(t, u_{n}\right)\left(u_{n}-\mathcal{P}\left(u_{n}\right)\right)^{-} d t
$$

Assumption $\left(h_{12}\right)$ yields the existence of a constant $\left.\vartheta \in\right] 0,1[$, independent of $n$, such that

$$
\int_{0}^{T}\left|u_{n}^{\prime}\right| d t-2 c T \leq \vartheta\left(\int_{0}^{T} \mu\left(u_{n}-\mathcal{P}\left(u_{n}\right)\right)^{+} d t+\int_{0}^{T} \nu\left(u_{n}-\mathcal{P}\left(u_{n}\right)\right)^{-} d t\right)
$$

By Proposition 2.5 we get

$$
\begin{equation*}
\left\|u_{n}^{\prime}\right\|_{L^{1}} \leq \frac{2 c T}{1-\vartheta} \tag{32}
\end{equation*}
$$

Using (27) and 32 we obtain, for all $t \in[0, T]$,

$$
u_{n}(t)=\int_{t_{n}^{\prime}}^{t} u_{n}^{\prime}(s) d s+u_{n}\left(t_{n}^{\prime}\right) \leq\left\|u_{n}^{\prime}\right\|_{L^{1}}+\alpha\left(t_{n}^{\prime}\right) \leq \frac{2 c T}{1-\vartheta}+\|\alpha\|_{L^{\infty}}
$$

and

$$
u_{n}(t)=\int_{t_{n}^{\prime \prime}}^{t} u_{n}^{\prime}(s) d s+u_{n}\left(t_{n}^{\prime \prime}\right) \geq-\left\|u_{n}^{\prime}\right\|_{L^{1}}+\beta\left(t_{n}^{\prime}\right) \geq-\frac{2 c T}{1-\vartheta}-\|\beta\|_{L^{\infty}}
$$

which lead to

$$
\begin{equation*}
\left\|u_{n}\right\|_{L^{\infty}} \leq \frac{2 c T}{1-\vartheta}+\max \left\{\|\alpha\|_{L^{\infty}},\|\beta\|_{L^{\infty}}\right\} \tag{33}
\end{equation*}
$$

This last estimate, combined with 32 , yields

$$
\begin{equation*}
\left\|u_{n}\right\|_{W^{1,1}} \leq K \tag{34}
\end{equation*}
$$

for some constant $K$.
Step 3. Existence of a solution. Fix any $w \in W_{T}^{1,1}(0, T)$. Taking $v=w-u_{n}$ as a test function in 28 , also using the convexity properties of the function $s \mapsto \sqrt{1+s^{2}}+\frac{\varepsilon_{n}}{2} s^{2}$, we get

$$
\begin{aligned}
& \int_{0}^{T} f\left(t, u_{n}\right)\left(w-u_{n}\right) d t=\int_{0}^{T} u_{n}^{\prime}\left(w-u_{n}\right)^{\prime} / \sqrt{1+u_{n}^{\prime 2}} d t+\varepsilon_{n} \int_{0}^{T} u_{n}^{\prime}\left(w-u_{n}\right)^{\prime} d t \\
& \quad \leq \int_{0}^{T} \sqrt{1+w^{\prime 2}} d t+\frac{\varepsilon_{n}}{2} \int_{0}^{T} w^{\prime 2} d t-\int_{0}^{T} \sqrt{1+u_{n}^{\prime 2}} d t-\frac{\varepsilon_{n}}{2} \int_{0}^{T}{u_{n}^{\prime 2}}^{2} d t
\end{aligned}
$$

Hence we have

$$
\mathcal{J}(w)-\mathcal{J}\left(u_{n}\right)+\frac{\varepsilon_{n}}{2} \int_{0}^{T} w^{\prime 2} d t \geq \int_{0}^{T} f\left(t, u_{n}\right)\left(w-u_{n}\right) d t
$$

for all $w \in W_{T}^{1,1}(0, T)$. As, by (34) the sequence $\left(u_{n}\right)_{n}$ is bounded in $W^{1,1}(0, T)$, we can extract a subsequence, we still denote by $\left(u_{n}\right)_{n}$, converging with respect to the $L^{1}$-topology and a.e. to a function $u \in B V(0, T)$. By $\left(h_{2}\right)$, which follows from $\left.\left(h_{12}\right), 33\right)$ and the Lebesgue dominated convergence theorem we get

$$
\lim _{n \rightarrow+\infty} \int_{0}^{T} f\left(t, u_{n}\right)\left(w-u_{n}\right) d t=\int_{0}^{T} f(t, u)(w-u) d t
$$

Hence, by Proposition 2.3 we conclude

$$
\begin{aligned}
\mathcal{J}(u) \leq \liminf _{n \rightarrow+\infty} \mathcal{J}\left(u_{n}\right) & \leq \mathcal{J}(w)-\lim _{n \rightarrow+\infty} \int_{0}^{T} f\left(t, u_{n}\right)\left(w-u_{n}\right) d t \\
& =\mathcal{J}(w)-\int_{0}^{T} f(t, u)(w-u) d t
\end{aligned}
$$

that is

$$
\mathcal{J}(w)-\mathcal{J}(u) \geq \int_{0}^{T} f(t, u)(w-u) d t
$$

Fix any $v \in B V(0, T)$. By Proposition 2.1 there exists a sequence $\left(w_{k}\right)_{k} \in W_{T}^{1,1}(0, T)$, bounded in $W_{T}^{1,1}(0, T)$, such that $\lim _{k \rightarrow+\infty} w_{k}=v$ in $L^{1}(0, T)$ and a.e. in $[0, T]$ and $\lim _{k \rightarrow+\infty} \mathcal{J}\left(w_{k}\right)$ $=\mathcal{J}(v)$. Arguing as above we see that

$$
\lim _{k \rightarrow+\infty} \int_{0}^{T} f(t, u)\left(w_{k}-u\right) d t=\int_{0}^{T} f(t, u)(w-u) d t
$$

and using again Proposition 2.3 we conclude that

$$
\mathcal{J}(v)-\mathcal{J}(u) \geq \int_{0}^{T} f(t, u)(v-u) d t
$$

thus showing that $u$ is a solution of the $T$-periodic problem associated with (1).

We show now that the two-sided bound on $f$ required by $\left(h_{12}\right)$ can be replaced by a one-sided bound, as expressed by $\left(h_{13}\right)$, or $\left(h_{14}\right)$.

Theorem 4.3. Assume $\left(h_{1}\right),\left(h_{2}\right)$, $\left(h_{11}\right)$ and either
$\left(h_{13}\right)$ there exists a measurable function $k:[0, T] \rightarrow \mathbb{R}$ such that $\left\|k^{+}\right\|_{L^{1}}<2$ and $f(t, s) \leq k(t)$ for a.e. $t \in[0, T]$ and every $s \in \mathbb{R}$,
or
$\left(h_{14}\right)$ there exists a measurable function $k:[0, T] \rightarrow \mathbb{R}$ such that $\left\|k^{-}\right\|_{L^{1}}<2$ and $f(t, s) \geq k(t)$ for a.e. $t \in[0, T]$ and every $s \in \mathbb{R}$.
Then the T-periodic problem associated with (1) has at least one solution $u \in B V(0, T)$.
Proof. The proof is similar to that of Theorem 4.2. Let us assume $\left(h_{13}\right)$, in case of $\left(h_{14}\right)$ the proof is the same. For each $n$ define $g_{n}$ as in 26). By $\left(h_{13}\right)$ we have

$$
g_{n}(t, s, \xi) \leq \frac{1}{\varepsilon_{n}} k^{+}(t)
$$

for a.e. $t \in[0, T]$, every $s \in \mathbb{R}$ and every $\xi \in \mathbb{R}$. By [14, pp. 173-174], for each $n$ there exist a solution $u_{n} \in W^{2,1}(0, T)$ of the $T$-periodic problem associated with (25) satisfying (27) for some $t_{n}^{\prime}, t_{n}^{\prime \prime} \in[0, T]$. Taking $v=u_{n}-\underset{[0, T]}{\operatorname{ess} \inf } u_{n}$ as a test function in 28 we get

$$
\begin{aligned}
\int_{0}^{T} f\left(t, u_{n}\right)\left(u_{n}-\underset{[0, T]}{\operatorname{ess} \inf } u_{n}\right) d t & =\varepsilon_{n} \int_{0}^{T} u_{n}^{\prime 2} d t+\int_{0}^{T} u_{n}^{\prime 2} / \sqrt{1+u_{n}^{\prime 2}} d t \\
& \geq \int_{0}^{T}\left|u_{n}^{\prime}\right| d t-c T
\end{aligned}
$$

$c$ being defined by (30). Assumption $\left(h_{13}\right)$ yields the existence of a constant $\left.\vartheta \in\right] 0,1[$, independent of $n$, such that

$$
\begin{aligned}
\int_{0}^{T}\left|u_{n}^{\prime}\right| d t-c T & \leq \int_{0}^{T} f\left(t, u_{n}\right)\left(u_{n}-\underset{[0, T]}{\operatorname{essinf}} u_{n}\right) d t \leq \int_{0}^{T} k^{+}(t)\left(u_{n}-\underset{[0, T]}{\operatorname{ess} \inf } u_{n}\right) d t \\
& \leq\left\|k^{+}\right\|_{L^{1}}\left(\underset{[0, T]}{\operatorname{esssup}} u_{n}-\underset{[0, T]}{\operatorname{ess} \inf } u_{n}\right) d t \leq 2 \vartheta\left(\underset{[0, T]}{\operatorname{ess} \sup } u_{n}-\underset{[0, T]}{\operatorname{ess} \inf } u_{n}\right) d t .
\end{aligned}
$$

By Proposition 2.7 we get, for all $n$,

$$
\left\|u_{n}^{\prime}\right\|_{L^{1}} \leq \frac{c T}{1-\vartheta}
$$

To complete the proof we proceed as in the proof of Theorem 4.2
Examples. We produce here two sample applications of the previous existence theorems; they can be compared with some statements obtained in [8, Sections 3, 4] and in [29, Section 3], but are independent of them.

Example 4.1. Assume that $f:[0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous and satisfies either $\left(h_{12}\right)$, or $\left(h_{13}\right)$, or $\left(h_{14}\right)$. Suppose that there exist $a, b \in \mathbb{R}$ such that

$$
f(t, a) f(t, b)<0
$$

in $[0, T]$. Then the $T$-periodic problem associated with (1) has at least one solution.

This statement follows by combining Proposition 3.3 with Theorem 4.2 or Theorem 4.3 Of course, if $a \leq b$, then we do not need to assume $\left(h_{12}\right)$, or $\left(h_{13}\right)$, or $\left(h_{14}\right)$.

Example 4.2. Assume that $f:[0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ and $e:[0, T] \rightarrow \mathbb{R}$ are continuous. Set $g=f+e$ and $\bar{e}=\frac{1}{T} \int_{0}^{T} e d t$. Suppose that there exists a constant $c>0$ such that

$$
(g(t, s)-\bar{e}) \operatorname{sgn}(s) \leq 0
$$

for all $t \in[0, T]$ and every $s$ with $|s| \geq c$. Assume finally that $\left(h_{4}\right)$ or $\left(h_{6}\right)$ hold. Then the $T$-periodic problem associated with (1) has at least one solution.

This statement follows by combining Proposition 3.4 and Proposition 3.5, or Proposition 3.6 and Proposition 3.7, with Theorem 4.1.

Remark 4.3 We point out that a condition like $\left(h_{4}\right)$ cannot be avoided in order to get the conclusion in Example 4.2. This is a direct consequence of the following non-existence result.

A non-existence result. We conclude this section by producing a non-existence result which shows the sharpness of some of the assumptions previously considered; it improves and specifies a similar statement obtained in [29, Proposition 3.2].
Proposition 4.4. Fix $\rho, \sigma \in \mathbb{R}_{0}^{+}$such that $\frac{1}{\rho}+\frac{1}{\sigma}<\frac{T}{2}$ and set $\tau=\frac{\sigma}{\rho+\sigma} T$. Then there exists $\gamma \in L^{1}(0, T)$ such that for every $e \in L^{1}(0, T)$, with $\frac{1}{\tau} \int_{0}^{\tau} e d t=-\rho$ and $\frac{1}{T-\tau} \int_{\tau}^{T} e d t=\sigma$ (and hence $\int_{0}^{T} e d t=0$ ), and for every $g:[0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ satisfying $\left(h_{1}\right)$, with $|g(t, s)| \leq \gamma(t)$ for a.e. $t \in[0, T]$ and every $s \in \mathbb{R}$, the periodic problem associated with $\mathbb{1} 1$, where $f(t, s)=$ $g(t, s)-e(t)$, has no solution.
Proof. Let $w \in B V(0, T)$ be given by $w(t)=1$, if $t \in[0, \tau[$, and $w(t)=-1$, if $t \in] \tau, T]$. Take any $u \in B V(0, T)$ and compute, for $k \in \mathbb{R}_{0}^{+}$,

$$
\begin{aligned}
\mathcal{J}(k w) & -\int_{0}^{T}(g(t, u)-e) k w d t \\
& \leq T+4 k+k \int_{0}^{\tau} e d t-k \int_{\tau}^{T} e d t+k \int_{0}^{T}|\gamma| d t \\
& =T+4 k-k \rho \tau-k \sigma(T-\tau)+k\|\gamma\|_{L^{1}} \\
& =T+4 k-2 k \frac{\rho \sigma}{\rho+\sigma} T+k\|\gamma\|_{L^{1}} \\
& =T+2 k T\left(\frac{2}{T}-\frac{\rho \sigma}{\rho+\sigma}+\frac{1}{2 T}\|\gamma\|_{L^{1}}\right) .
\end{aligned}
$$

Clearly, the last term tends to $-\infty$ as $k \rightarrow+\infty$, provided that $\left.\|\gamma\|_{L^{1}} \in\right] 0, \frac{2 T \rho \sigma}{\rho+\sigma}-4[$. Therefore $u$ is not a solution of the $T$-periodic problem associated with 11 , where $f(t, s)=$ $g(t, s)-e(t)$.
Corollary 4.5. Fix $\mu, \nu \in \mathbb{R}_{0}^{+}$such that $\frac{1}{\sqrt{\mu}}+\frac{1}{\sqrt{\nu}}=\sqrt{T}$. Then there exist $\gamma \in L^{1}(0, T)$ and $e \in C^{\infty}([0, T])$, with $\int_{0}^{T} e d t=0$, essinf $e<-\mu$ and $\underset{[0, T]}{\operatorname{ess} \sup } e>\nu$, such that for every $g:[0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ satisfying $\left(h_{1}\right)$, with $|g(t, s)| \leq \gamma(t)$ for a.e. $t \in[0, T]$ and every $s \in \mathbb{R}$, the periodic problem associated with (11), where $f(t, s)=g(t, s)-e(t)$, has no solution.
Proof. Pick $\eta>1$ such that $\frac{1}{\mu}+\frac{1}{\nu}<\eta \frac{T}{2}$ and set $\rho=\eta \mu$ and $\sigma=\eta \nu$. Then Proposition 4.4 yields the conclusion. Note that $\frac{1}{\mu}+\frac{1}{\nu}=\frac{T}{2}$ if and only if $\mu=\nu=\frac{4}{T}$ : in this case we can take $\eta$ as close to 1 as we want.

## 5 Regularity results

We prove in this section the regularity of the bounded variation solutions of the $T$-periodic problem associated with (1), when additional regularity and monotonicity conditions on the right-hand side $f$ of the equation are assumed. We will use here a regularity result for solutions of the Dirichlet problem associated with (1) that we proved in 30 .

Theorem 5.1. Assume $\left(h_{1}\right)$. Let $u \in B V(0, T)$ be a non-constant solution of the $T$-periodic problem associated with (1) and suppose that

Then $u$ is the unique solution of the T-periodic problem associated with (1), having range in $[\operatorname{ess} \inf u, \operatorname{ess} \sup u]$, and the $T$-periodic extension of $u$ onto $\mathbb{R}$ belongs to $C^{3}(\mathbb{R})$.
$[0, T] \quad[0, T]$
Proof. To prove uniqueness, let us assume that $u_{1}$ and $u_{2}$ are both solutions of the $T$-periodic problem associated with (1), having range in $\underset{[0, T]}{\operatorname{essinf}} u$, $\underset{[0, T]}{\operatorname{ess} \sup } u]$. Testing $\left[9\right.$, with $u=u_{1}$, against $u_{2}$ and (9), with $u=u_{2}$, against $u_{1}$, we obtain

$$
\mathcal{J}\left(u_{2}\right)-\mathcal{J}\left(u_{1}\right) \geq \int_{0}^{T} f\left(t, u_{1}\right)\left(u_{2}-u_{1}\right) d t
$$

and

$$
\mathcal{J}\left(u_{1}\right)-\mathcal{J}\left(u_{2}\right) \geq \int_{0}^{T} f\left(t, u_{2}\right)\left(u_{1}-u_{2}\right) d t
$$

Summing the two inequalities we get

$$
\int_{0}^{T}\left(f\left(t, u_{1}\right)-f\left(t, u_{2}\right)\right)\left(u_{2}-u_{1}\right) d t \leq 0
$$

By $\left(h_{15}\right)$ we conclude that $u_{1}=u_{2}$ a.e. in $[0, T]$.
To prove that $u \in C^{3}(] 0, T[)$, we shall use a regularity result for solutions of the Dirichlet problem associated with (1) proved in [30. To this aim we first show that $u$ is a solution of a suitably defined Dirichlet problem.
Claim. Let $u \in B V(0, T)$ be a solution of the T-periodic problem associated with (1). Then, $u$ is a solution of the Dirichlet problem

$$
\begin{equation*}
\left.-\left(v^{\prime} / \sqrt{1+v^{\prime 2}}\right)^{\prime}=f(t, v) \text { in }\right] 0, T\left[, \quad v(0)=u\left(0^{+}\right), v(T)=u\left(T^{-}\right)\right. \tag{35}
\end{equation*}
$$

We recall that $v \in B V(0, T)$ is a solution of (35) if

$$
\begin{aligned}
\int_{0}^{T} \sqrt{1+|D w|^{2}}+ & \left|w\left(0^{+}\right)-u\left(0^{+}\right)\right|+\left|w\left(T^{-}\right)-u\left(T^{-}\right)\right| \\
& \quad-\left(\int_{0}^{T} \sqrt{1+|D v|^{2}}+\left|v\left(0^{+}\right)-u\left(0^{+}\right)\right|+\left|v\left(T^{-}\right)-u\left(T^{-}\right)\right|\right) \\
\geq & \int_{0}^{T} f(t, v)(w-v) d t
\end{aligned}
$$

for all $w \in B V(0, T)$. Fix any $w \in B V(0, T)$. Using the elementary inequality

$$
|a-b|-|c-d| \leq|a-c|+|b-d|,
$$

which holds for all $a, b, c, d \in \mathbb{R}$, and the fact that $u$ satisfies (9) with $v=w$, we see that

$$
\begin{aligned}
& \int_{0}^{T} \sqrt{1+}+|D w|^{2} \\
& \quad\left|w\left(0^{+}\right)-u\left(0^{+}\right)\right|+\left|w\left(T^{-}\right)-u\left(T^{-}\right)\right|-\int_{0}^{T} \sqrt{1+|D u|^{2}} \\
& \geq \int_{0}^{T} \sqrt{1+|D w|^{2}}+\left|w\left(T^{-}\right)-w\left(0^{+}\right)\right|-\int_{0}^{T} \sqrt{1+|D u|^{2}}-\left|u\left(T^{-}\right)-u\left(0^{+}\right)\right| \\
& \geq \int_{0}^{T} f(t, u)(w-u) d t
\end{aligned}
$$

Hence, $u$ is a solution of 35) and the claim is proved.
As $u$ is a solution of the Dirichlet problem (35), by [30, Corollary 1.6] we have that $u \in C^{3}(] 0, T[)$.

To prove that the $T$-periodic extension $\hat{u}$ of $u$ onto $\mathbb{R}$ belongs to $C^{3}(\mathbb{R})$, we first notice that, by Remark 3.2 the restriction $\tilde{u}$ to $\left[\frac{1}{2} T, \frac{3}{2} T\right]$ of $\hat{u}$ is a solution of the $T$-periodic problem associated with (1] on $\left[\frac{1}{2} T, \frac{3}{2} T\right]$. Then we repeat the previous argument and we conclude that $\tilde{u} \in C^{3}(] \frac{1}{2} T, \frac{3}{2} T[)$. Hence $\hat{u}$ belongs to $C^{3}(\mathbb{R})$.

The following statement yields the regularity of solutions of the $T$-periodic problem associated with (1) in the presence of a lower solution $\alpha$ and of an upper solution $\beta$ such that $\alpha<\beta$. This result should be compared with [8, Theorem 2], where the further condition
$\left(h_{16}\right)$ there exists $c \in C^{0}([0, T])$ such that $\left\|c^{-}\right\|_{L^{1}}<\frac{1}{2}$ (respectively $\left\|c^{+}\right\|_{L^{1}}<\frac{1}{2}$ ) and $f(t, s) \geq$ $c(t)$ (respectively $f(t, s) \leq c(t)$ ) for every $t \in[0, T]$ and every $s \in \mathbb{R}$
was needed, but no additional regularity, besides continuity, nor monotonicity were assumed on $f$.
Corollary 5.2. Assume $\left(h_{1}\right),\left(h_{8}\right)$ and
$\left(h_{17}\right) f \in C^{1}(\mathbb{R} \times[-\rho, \rho])$ and $\frac{\partial f}{\partial s}(t, s)<0$ in $\mathbb{R} \times[-\rho, \rho]$, with $\rho>\max \left\{\left\|\alpha_{i}\right\|_{L^{\infty}},\left\|\beta_{j}\right\|_{L^{\infty}}\right.$ : $i=1, \ldots, m ; j=1, \ldots, n\}$.

Then the T-periodic problem associated with (1) has a unique solution $u$, having range in $[-\rho, \rho]$, and the $T$-periodic extension of $u$ onto $\mathbb{R}$ belongs to $C^{3}(\mathbb{R})$.
Proof. By Theorem 4.1 and Remark 4.1 there exists a solution $u$ of the $T$-periodic problem associated with (1) such that $\alpha \leq u \leq \beta$. The uniqueness can be verified exactly as in the proof of Theorem 5.1. Theorem 5.1 finally yields the regularity of $u$.

Example 5.1. From Proposition 3.3 and Corollary 5.2 we easily deduce that the sinecurvature equation

$$
\begin{equation*}
-\left(u^{\prime} / \sqrt{1+u^{\prime 2}}\right)=A \sin u-e(t) \tag{36}
\end{equation*}
$$

has, for any given $A \in \mathbb{R}$ and $e \in C_{T}^{1}(\mathbb{R})$ satisfying $\|e\|_{L^{\infty}}<A$, exactly one (classical) $T$ periodic solution $u \in C^{3}(\mathbb{R})$, with $\frac{\pi}{2}<\min _{[0, T]} u \leq \max _{[0, T]} u<\frac{3 \pi}{2}$. A related result was obtained in
[6. Corollary 1, Example 1], but there the additional condition $\left\|(e-A)^{-}\right\|_{L^{\infty}}<\frac{1}{2}$ was required. The solvability of the $T$-periodic problem associated with the sine-curvature equation (36) under different conditions has been investigated in 28.

Example 5.2. The singular curvature equation

$$
\begin{equation*}
-\left(u^{\prime} / \sqrt{1+u^{\prime 2}}\right)=u^{-p}-e(t) \tag{37}
\end{equation*}
$$

has, for any given $p>0$ and $e \in C_{T}^{1}(\mathbb{R})$ satisfying $\bar{e}>0$ and $\left(h_{4}\right)$ (or $\left(h_{6}\right)$ ), exactly one (classical) $T$-periodic solution $u \in C^{3}(\mathbb{R})$ such that $\min _{[0, T]} u>0$. This can be deduced from Proposition 3.3 Proposition 3.5 and Corollary 5.2 observing that $\lim _{s \rightarrow 0^{+}}\left(s^{-p}-\|e\|_{L^{\infty}}\right)=+\infty$ and $\lim _{s \rightarrow+\infty}\left(s^{-p}-\bar{e}\right)<0$. Related results have been obtained in [8, Example 2] under different conditions. The solvability of the Dirichlet problem associated with equation (37) has been investigated in [10].

## 6 Stability results

In this section we discuss how certain stability properties of the solutions of the $T$-periodic problem associated with (1) can be detected by the use of lower and upper solutions.
Order stability. We introduce the following concept of stability, adapted to the present setting from [20, Chapter I].

- We say that a solution $u$ of the $T$-periodic problem associated with (1) is order stable (respectively strictly order stable) from below if there exists a sequence $\left(\alpha_{n}\right)_{n}$ of lower solutions (respectively proper lower solutions) such that, for each $n, \alpha_{n}<\alpha_{n+1}$ and $\lim _{n \rightarrow+\infty} \alpha_{n}=u$ in $L^{q}(0, T)$ for some $q>1$.
- We say that a solution $u$ of the $T$-periodic problem associated with (1) is order stable (respectively strictly order stable) from above if there exists a sequence $\left(\beta_{n}\right)_{n}$ of upper solutions (respectively proper upper solutions) such that, for each $n, \beta_{n}>\beta_{n+1}$ and $\lim _{n \rightarrow+\infty} \beta_{n}=u$ in $L^{q}(0, T)$ for some $q>1$.
The following order stability results hold. We point out that our conclusions are obtained without assuming any additional regularity condition, like, e.g., Lipschitz continuity, on $f$, as it is usually required in the semilinear case in order to associate with the considered problem an order preserving operator (see, e.g., [1, 20]).
Theorem 6.1. Assume ( $h_{1}$ ) and $\left(h_{7}\right)$. Let $v \in B V(0, T)$ be a solution of the T-periodic problem associated with $\sqrt{17}$. Suppose that there exists a $B V$-lower solution $\alpha$ of the $T$-periodic problem associated with (1) such that $v>\alpha$ and there is no solution $u$ of the $T$-periodic problem associated with (1) satisfying $\alpha \leq u<v$. Then $v$ is strictly order stable from below.

Proof. Assume that $\alpha=\tilde{\alpha}_{1} \vee \cdots \vee \tilde{\alpha}_{m}$, where, for each $i=1, \ldots, m$, $\tilde{\alpha}_{i}$ satisfies

$$
\mathcal{J}\left(\tilde{\alpha}_{i}+z\right)-\mathcal{J}\left(\tilde{\alpha}_{i}\right) \geq \int_{0}^{T} f\left(t, \tilde{\alpha}_{i}\right) z d t
$$

for all $z \in B V(0, T)$ with $z \leq 0$. Fix $\rho>\max _{i=1, \ldots, m}\left\|\tilde{\alpha}_{i}\right\|_{L^{\infty}}$. By [16, Lemma 2.1] (see also [15, Proposition 2.3]) there exists a function $h:[0, T] \times[-\rho, \rho] \times[-\rho, \rho] \rightarrow \mathbb{R}$ such that

- $h$ satisfies the Carathéodory conditions, i.e., for a.e. $t \in[0, T], h(t, \cdot, \cdot):[-\rho, \rho] \times$ $[-\rho, \rho] \rightarrow \mathbb{R}$ is continuous and, for every $(s, r) \in[-\rho, \rho] \times[-\rho, \rho], h(\cdot, s, r):[0, T] \rightarrow \mathbb{R}$ is measurable;
- there exists $\gamma \in L^{p}(0, T)$ such that $|h(t, s, r)| \leq \gamma(t)$ for a.e. $t \in[0, T]$ and every $(s, r) \in[-\rho, \rho] \times[-\rho, \rho] ;$
- $h(t, \cdot, r):[-\rho, \rho] \rightarrow \mathbb{R}$ is strictly increasing for a.e. $t \in[0, T]$ and every $r \in[-\rho, \rho]$;
- $h(t, s, \cdot):[-\rho, \rho] \rightarrow \mathbb{R}$ is strictly decreasing for a.e. $t \in[0, T]$ and every $s \in[-\rho, \rho]$;
- $h(t, s, r)=-h(t, r, s)$ for a.e. $t \in[0, T]$ and every $(s, r) \in[-\rho, \rho] \times[-\rho, \rho]$;
- for a.e. $t \in[0, T]$ and every $(s, r) \in[-\rho, \rho] \times[-\rho, \rho]$, with $r<s$, we have

$$
\begin{equation*}
|f(t, s)-f(t, r)|<h(t, s, r) . \tag{38}
\end{equation*}
$$

Let us consider the equation

$$
\begin{equation*}
-\left(u^{\prime} / \sqrt{1+u^{\prime 2}}\right)^{\prime}+h(t, u, \alpha)=f(t, \alpha) \tag{39}
\end{equation*}
$$

A solution $u$ of the $T$-periodic problem associated with 39 is a function $u \in B V(0, T)$ satisfying $\|u\|_{L^{\infty}} \leq \rho$ and

$$
\begin{equation*}
\mathcal{J}(u+z)-\mathcal{J}(u) \geq \int_{0}^{T}(f(t, \alpha)-h(t, u, \alpha)) z d t \tag{40}
\end{equation*}
$$

for all $z \in B V(0, T)$.
Claim. The T-periodic problem associated with (39) has a unique solution $\alpha_{1}$, satisfying $\alpha<\alpha_{1}<v$, which is a proper BV-lower solution of the T-periodic problem associated with (1).

We first show that the $T$-periodic problem associated with 39 has at most one solution. Suppose that both $u_{1}$ and $u_{2}$ are solutions of the $T$-periodic problem associated with 39 . Then we have

$$
\mathcal{J}\left(u_{2}\right)-\mathcal{J}\left(u_{1}\right) \geq \int_{0}^{T}\left(f(t, \alpha)-h\left(t, u_{1}, \alpha\right)\right)\left(u_{2}-u_{1}\right) d t
$$

and

$$
\mathcal{J}\left(u_{1}\right)-\mathcal{J}\left(u_{2}\right) \geq \int_{0}^{T}\left(f(t, \alpha)-h\left(t, u_{2}, \alpha\right)\right)\left(u_{1}-u_{2}\right) d t
$$

Since the function $h(t, \cdot, r):[-\rho, \rho] \rightarrow \mathbb{R}$ is strictly increasing for a.e. $t \in[0, T]$ and every $r \in[-\rho, \rho]$, we conclude that

$$
0 \geq \int_{0}^{T}\left(h\left(t, u_{1}, \alpha\right)-h\left(t, u_{2}, \alpha\right)\right)\left(u_{1}-u_{2}\right) d t \geq 0
$$

and hence $u_{1}=u_{2}$.
Next we prove that the $T$-periodic problem associated with 39 has a solution. Let us verify that $\alpha=\tilde{\alpha}_{1} \vee \cdots \vee \tilde{\alpha}_{m}$ is a $B V$-lower solution of the $T$-periodic problem associated with (39), that is, for each $j=1, \ldots, m$,

$$
\mathcal{J}\left(\tilde{\alpha}_{j}+z\right)-\mathcal{J}\left(\tilde{\alpha}_{j}\right) \geq \int_{0}^{T}\left(f(t, \alpha)-h\left(t, \tilde{\alpha}_{j}, \alpha\right)\right) z d t
$$

for all $z \in B V(0, T)$ with $z \leq 0$. Indeed, as $\tilde{\alpha}_{j} \leq \alpha$, we have, by (38),

$$
f\left(\cdot, \tilde{\alpha}_{j}\right) \leq f(\cdot, \alpha)+h\left(\cdot, \alpha, \tilde{\alpha}_{j}\right)=f(\cdot, \alpha)-h\left(\cdot, \tilde{\alpha}_{j}, \alpha\right)
$$

and, hence, as $\alpha$ is a $B V$-lower solution of the $T$-periodic problem associated with (1),

$$
\mathcal{J}\left(\tilde{\alpha}_{j}+z\right)-\mathcal{J}\left(\tilde{\alpha}_{j}\right) \geq \int_{0}^{T} f\left(t, \tilde{\alpha}_{j}\right) z d t \geq \int_{0}^{T}\left(f(t, \alpha)-h\left(t, \tilde{\alpha}_{j}, \alpha\right)\right) z d t
$$

Similarly, we verify that $v$ is an upper solution of the $T$-periodic problem associated with (39), that is

$$
\mathcal{J}(v+z)-\mathcal{J}(v) \geq \int_{0}^{T}(f(t, \alpha)-h(t, v, \alpha)) z d t
$$

for all $z \in B V(0, T)$ with $z \geq 0$. Indeed, as $v>\alpha$, we have, by (38),

$$
\begin{equation*}
f(\cdot, v)>f(\cdot, \alpha)-h(\cdot, v, \alpha) \tag{41}
\end{equation*}
$$

and, hence, as $v$ is a solution of the $T$-periodic problem associated with (1),

$$
\mathcal{J}(v+z)-\mathcal{J}(v) \geq \int_{0}^{T} f(t, v) z d t \geq \int_{0}^{T}(f(t, \alpha)-h(t, v, \alpha)) z d t
$$

Theorem 4.1 and Remark 4.1 yield the existence of a solution $\alpha_{1}$ of the $T$-periodic problem associated with (39) such that $\alpha \leq \alpha_{1} \leq v$.

Let us prove that $\alpha<\alpha_{1}$. Suppose, by contradiction, that $\alpha_{1}=\alpha$. Then we get

$$
\mathcal{J}(\alpha+z)-\mathcal{J}(\alpha) \geq \int_{0}^{T}\left(f(t, \alpha)-h\left(t, \alpha_{1}, \alpha\right)\right) z d t=\int_{0}^{T} f(t, \alpha) z d t
$$

for all $z \in B V(0, T)$, i.e., $\alpha$ is a solution of the $T$-periodic problem associated with (1), thus contradicting the assumption that $\alpha$ is a proper $B V$-lower solution.

Let us prove that $\alpha_{1}<v$. Suppose, by contradiction, that $\alpha_{1}=v$. Then, testing 40) against $z=-1$ and against $z=1$, we obtain

$$
\int_{0}^{T}(f(t, \alpha)-h(t, v, \alpha)) d t=0
$$

Similarly, testing

$$
\mathcal{J}(v+z)-\mathcal{J}(v) \geq \int_{0}^{T} f(t, v) z d t
$$

against $z=-1$ and against $z=1$, we get

$$
\int_{0}^{T} f(t, v) d t=0
$$

Hence, we have

$$
\int_{0}^{T}(f(t, v)-f(t, \alpha)+h(t, v, \alpha)) d t=0
$$

thus contradicting 41).

Finally, we observe that $\alpha_{1}$ is a $B V$-lower solution of the $T$-periodic problem associated with (1). Indeed, if $z \in B V(0, T)$, with $z \leq 0$, we have, using (38),

$$
\mathcal{J}\left(\alpha_{1}+z\right)-\mathcal{J}\left(\alpha_{1}\right) \geq \int_{0}^{T}\left(f(t, \alpha)-h\left(t, \alpha_{1}, \alpha\right)\right) z d t \geq \int_{0}^{T} f\left(t, \alpha_{1}\right) z d t
$$

As the $T$-periodic problem associated with (1) has no solutions $u$, with $\alpha<u<v$, we conclude that $\alpha_{1}$ is proper.

We now recursively define a sequence $\left(\alpha_{n}\right)_{n}$, where $\alpha_{0}=\alpha, \alpha_{1}$ has been constructed in the above claim and, for every $n \geq 1, \alpha_{n+1}$ is the unique solution of the $T$-periodic problem associated with

$$
-\left(u^{\prime} / \sqrt{1+u^{\prime 2}}\right)^{\prime}+h\left(t, u, \alpha_{n}\right)=f\left(t, \alpha_{n}\right)
$$

that is

$$
\mathcal{J}\left(\alpha_{n+1}+z\right)-\mathcal{J}\left(\alpha_{n+1}\right) \geq \int_{0}^{T}\left(f\left(t, \alpha_{n}\right)-h\left(t, \alpha_{n+1}, \alpha_{n}\right)\right) z d t
$$

for all $z \in B V(0, T)$. Arguing as above we see that the sequence $\left(\alpha_{n}\right)_{n}$ is well-defined and, for each $n, \alpha_{n}$ is a proper $B V$-lower solution of the $T$-periodic problem associated with (1), satisfying $\alpha_{n}<\alpha_{n+1}<v$.

Let us verify that $\left(\alpha_{n}\right)_{n}$ converges to $v$ in $L^{q}(0, T)$, where $q=\frac{p}{p-1}$. Since the sequence $\left(\alpha_{n}\right)_{n}$ is increasing and uniformly bounded in $L^{\infty}(0, T)$, it converges a.e. in $[0, T]$ and in $L^{q}(0, T)$ to some function $\tilde{v}$, with $\alpha<\tilde{v} \leq v$. Since there exists $\gamma \in L^{p}(0, T)$ such that, for all $n$,

$$
\left|h\left(t, \alpha_{n+1}(t), \alpha_{n}(t)\right)\right| \leq \gamma(t)
$$

and

$$
\lim _{n \rightarrow+\infty} h\left(t, \alpha_{n+1}(t), \alpha(t)\right)=h(t, \tilde{v}(t), \tilde{v}(t))=0
$$

a.e. in $[0, T]$, we get

$$
\lim _{n \rightarrow+\infty} h\left(\cdot, \alpha_{n+1}, \alpha_{n}\right)=0
$$

in $L^{p}(0, T)$. By $\left(h_{7}\right)$ we also have

$$
\lim _{n \rightarrow+\infty} f\left(\cdot, \alpha_{n}\right)=f(\cdot, \tilde{v})
$$

in $L^{p}(0, T)$. Fix $w \in B V(0, T)$. Since, for each $n$,

$$
\mathcal{J}(w)-\mathcal{J}\left(\alpha_{n+1}\right) \geq \int_{0}^{T}\left(f\left(t, \alpha_{n}\right)-h\left(t, \alpha_{n+1}, \alpha_{n}\right)\right)\left(w-\alpha_{n+1}\right) d t
$$

passing to the limit and using Proposition 2.3. we conclude that

$$
\mathcal{J}(w)-\mathcal{J}(\tilde{v}) \geq \int_{0}^{T} f(t, \tilde{v})(w-\tilde{v}) d t
$$

Accordingly, $\tilde{v}$ is a solution of the $T$-periodic problem associated with (1), satisfying $\alpha \leq \tilde{v} \leq$ $v$, and hence $\tilde{v}=v$.

In a completely similar way we can prove the following symmetric result.

Theorem 6.2. Assume $\left(h_{1}\right)$ and $\left(h_{7}\right)$. Let $w \in B V(0, T)$ be a solution of the T-periodic problem associated with (1). Suppose that there exists a $B V$-upper solution $\beta$ of the $T$-periodic problem associated with (1) such that $w<\beta$ and there is no solution $u$ of the $T$-periodic problem associated with (1) satisfying $w<u \leq \beta$. Then, $w$ is strictly order stable from above.

Combining Theorem 6.1 and Theorem 6.2 yields the order stability of the minimum and the maximum solutions of the $T$-periodic problem associated with (1), lying between a pair of lower and upper solutions $\alpha$ and $\beta$, with $\alpha \leq \beta$.

Corollary 6.3. Assume $\left(h_{1}\right),\left(h_{7}\right)$ and $\left(h_{8}\right)$. Suppose further that $\alpha$ and $\beta$ are proper lower and upper solutions, respectively, of the T-periodic problem associated with (1). Then the minimum solution $v$ and the maximum solution $w$ in $[\alpha, \beta]$ of the $T$-periodic problem associated with (1) are strictly order stable from below and strictly order stable from above, respectively.

We stress that we cannot expect in the context of Corollary 6.3 , even in the case where all solutions between $\alpha$ and $\beta$ are regular, the existence of a Lyapunov stable solution. This is the content of the following result.

Proposition 6.4. Assume ( $h_{1}$ ). Let $\alpha$ and $\beta$ be respectively a $W^{2,1}$-lower and a $W^{2,1}$-upper solution, with $m=n=1$, of the T-periodic problem associated with (1). Suppose that $\alpha$ and $\beta$ are proper and that $\alpha<\beta$. Finally, assume that $\left(h_{17}\right)$ holds. Then the $T$-periodic problem associated with (1) has a unique classical solution $u$, with $\alpha<u<\beta$ in $[0, T]$. Further, $u$ is strictly order stable from below and from above, but is Lyapunov unstable in the past and in the future.
Proof. Proposition 3.1, Proposition 3.2, Corollary 5.2 and Corollary 6.3 imply that the $T$ periodic problem associated with (1) has a unique classical solution $u$, with $\alpha<u<\beta$, which is strictly order stable from below and from above.

Let us prove that $u$ is Lyapunov unstable in the past and in the future. We first note that $u$ is a $T$-periodic solution of the equation

$$
-u^{\prime \prime}=f(t, u)\left(1+u^{\prime 2}\right)^{\frac{3}{2}} .
$$

Next we take $R>\max \left\{\left\|u^{\prime}\right\|_{L^{\infty}},\left\|\alpha^{\prime}\right\|_{L^{\infty}},\left\|\beta^{\prime}\right\|_{L^{\infty}}\right\}$ and we define a function $k: \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ by setting, for every $(t, s) \in \mathbb{R}^{2}$,

$$
k(t, s, r)= \begin{cases}f(t, s)\left(1+r^{2}\right)^{\frac{3}{2}} & \text { if }|r| \leq R, \\ f(t, s)\left(1+R^{2}\right)^{\frac{3}{2}} & \text { if }|r|>R .\end{cases}
$$

Note that $k$ satisfies the Bernstein-Nagumo condition. Obviously, $\alpha$ is a proper lower solution, $\beta$ is a proper upper solution and $u$ is a solution of the $T$-periodic problem associated with the equation

$$
\begin{equation*}
-u^{\prime \prime}=k\left(t, u, u^{\prime}\right) . \tag{42}
\end{equation*}
$$

Pick $t_{1}, t_{2} \in[0, T]$ such that $\alpha\left(t_{1}\right)<u\left(t_{1}\right)$ and $u\left(t_{2}\right)<\beta\left(t_{2}\right)$. Theorem 3.1 in 33 implies that, for any $u_{0} \in\left[\alpha\left(t_{1}\right), u\left(t_{1}\right)\left[\right.\right.$ (respectively, $\left.\left.u_{0} \in\right] u\left(t_{2}\right), \beta\left(t_{2}\right)\right]$ ), there are solutions $\left.\left.u_{l}:\right]-\infty, t_{1}\right] \rightarrow$ $\mathbb{R}$ and $u_{r}:\left[t_{1},+\infty[\rightarrow \mathbb{R}\right.$ of 42), with

$$
\left.\left.u_{l}\left(t_{1}\right)=u_{0}=u_{r}\left(t_{1}\right), \quad \alpha(t) \leq u_{l}(t) \leq u(t) \text { in }\right]-\infty, t_{1}\right], \quad \alpha(t) \leq u_{r}(t) \leq u(t) \text { in }\left[t_{1},+\infty[\right.
$$

(respectively,

$$
\left.\left.u_{l}\left(t_{2}\right)=u_{0}=u_{r}\left(t_{2}\right), u(t) \leq u_{l}(t) \leq \beta(t) \text { in }\right]-\infty, t_{2}\right], u(t) \leq u_{r}(t) \leq \beta(t) \text { in }\left[t_{2},+\infty[)\right.
$$

satisfying

$$
\begin{aligned}
\lim _{t \rightarrow-\infty}\left(u_{l}(t)-u(t)\right)=0, & \lim _{t \rightarrow-\infty}\left(u_{l}^{\prime}(t)-u^{\prime}(t)\right)=0, \\
\lim _{t \rightarrow+\infty}\left(u_{r}(t)-u(t)\right)=0, & \lim _{t \rightarrow+\infty}\left(u_{r}^{\prime}(t)-u^{\prime}(t)\right)=0 .
\end{aligned}
$$

Hence, it follows that $u$ is Lyapunov unstable in the past and in the future as a solution of (42) and therefore as a solution of (36), as it is immediately checked.

Example 6.1. We know from Example 5.1 that the sine-curvature equation (36) has, for any given $A \in \mathbb{R}$ and $e \in C_{T}^{1}(\mathbb{R})$ satisfying $\|e\|_{L^{\infty}}<A$, exactly one $T$-periodic solution $u \in C^{3}(\mathbb{R})$, with $\frac{\pi}{2}<\min _{[0, T]} u \leq \max _{[0, T]} u<\frac{3 \pi}{2}$. Proposition 6.4 implies that $u$ is strictly order stable from below and from above, but is Lyapunov unstable in the past and in the future.

Example 6.2. We observed in Example 5.2 that the singular curvature equation (37) has, for any given $p>0$ and $e \in C_{T}^{1}(\mathbb{R})$ satisfying $\bar{e}>0$ and $\left(h_{4}\right)$ (or $\left(h_{6}\right)$ ), exactly one $T$-periodic solution $u \in C^{3}(\mathbb{R})$ such that $\min _{[0, T]} u>0$. By the strict monotonicity we see that there exists a constant $\varepsilon>0$ such that $\alpha=u-\varepsilon$ and $\beta=u+\varepsilon$ are respectively a proper $W^{2,1}$-lower and a proper $W^{2,1}$-upper solution. Accordingly, Proposition 6.4 implies that $u$ is strictly order stable from below and from above, but is Lyapunov unstable in the past and in the future.

## References

[1] H. Amann, Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces, SIAM Rev., 18 (1976), 620-709.
[2] L. Ambrosio, N. Fusco and D. Pallara, "Functions of bounded variation and free discontinuity problems", Oxford University Press, New York, 2000.
[3] G. Anzellotti, The Euler equation for functionals with linear growth, Trans. Amer. Math. Soc., 290 (1985), 483-501.
[4] P. Benevieri, J.M. do Ó and E.S. de Medeiros, Periodic solutions for nonlinear systems with mean curvature-like operators, Nonlinear Anal. T.M.A., 65 (2006), 1462-1475.
[5] P. Benevieri, J.M. do Ó and E.S. de Medeiros, Periodic solutions for nonlinear equations with mean curvature-like operators, Appl. Math. Lett., 20 (2007), 484-492.
[6] C. Bereanu, P. Jebelean and J. Mawhin, Periodic solutions of pendulum-like perturbations of singular and bounded $\phi$-Laplacians, J. Dynam. Differential Equations, 22 (2010), 463-471.
[7] C. Bereanu and J. Mawhin, Boundary value problems with non-surjective $\phi$-Laplacian and one-side bounded nonlinearity, Adv. Differential Equations, 11 (2006), 35-60.
[8] C. Bereanu and J. Mawhin, Periodic solutions of nonlinear perturbations of $\phi$-Laplacians with possibly bounded $\phi$, Nonlinear Anal. T.M.A., 68 (2008), 1668-1681.
[9] C. Bereanu and J. Mawhin, Multiple periodic solutions of ordinary differential equations with bounded nonlinearities and $\phi$-Laplacian, Nonlinear Differential Equations Appl., 15 (2008), 159-168.
[10] D. Bonheure, P. Habets, F. Obersnel, and P. Omari, Classical and non-classical positive solutions of a prescribed curvature equation with singularities, Rend. Istit. Mat. Univ. Trieste, 39 (2007), 63-85.
[11] D. Bonheure, P. Habets, F. Obersnel and P. Omari, Classical and non-classical solutions of a prescribed curvature equation, J. Differential Equations, 243 (2007), 208-237.
[12] M. Burns and M. Grinfeld, Steady state solutions of a bi-stable quasi-linear equation with saturating flux, European J. Appl. Math., 22 (2011), 317-331.
[13] M. Carriero, G. Dal Maso, A. Leaci, and E. Pascali, Relaxation of the nonparametric Plateau problem with an obstacle, J. Math. Pures Appl., 67 (1988), 359-396.
[14] C. De Coster and P. Habets, "Two-Point Boundary Value Problems: Lower and Upper Solutions", Elsevier, Amsterdam, 2006.
[15] C. De Coster, F. Obersnel, and P. Omari, A qualitative analysis, via lower and upper solutions, of first order periodic evolutionary equations with lack of uniqueness, Handbook of differential equations: ordinary differential equations, Vol. III, 203-339, Handb. Differ. Equ., Elsevier/North-Holland, Amsterdam, 2006.
[16] C. De Coster and P. Omari, Unstable periodic solutions of a parabolic problem in the presence of non-well-ordered lower and upper solutions, J. Funct. Anal., 175 (2000), 52-88.
[17] E.N. Dancer and R. Ortega, The index of Lyapunov stable fixed points in two dimensions, J. Dynam. Differential Equations, 6 (1994), 631-637.
[18] R. Finn, "Equilibrium capillary surfaces", Springer-Verlag, New York, 1986.
[19] E. Giusti, "Minimal Surfaces and Functions of Bounded Variations", Birkhäuser, Basel, 1984.
[20] P. Hess, "Periodic-parabolic boundary value problems and positivity", Wiley, New York, 1991.
[21] A. Kurganov and P. Rosenau, On reaction processes with saturating diffusion, Nonlinearity, 19 (2006), 171-193.
[22] E. M. Landesman, A.C. Lazer, Nonlinear perturbations of linear elliptic boundary value problems at resonance, J. Math. Mech., 19 (1969/1970), 609-623.
[23] A.C. Lazer, D.E. Leach, Bounded perturbations of forced harmonic oscillators at resonance, Ann. Mat. Pura Appl., 82 (1969), 49-68.
[24] V.K. Le, On a sub-supersolution method for the prescribed mean curvature problem, Czech. Math. J., 58 (2008), 541-560.
[25] J. Mawhin, Resonance problems for some non-autonomous ordinary differential equations, CIME course "Stability and bifurcation for non-autonomous differential equations", Cetraro, 2011, http://php.math.unifi.it/users/cime/.
[26] F. Obersnel, Classical and non-classical sign changing solutions of a one-dimensional autonomous prescribed curvature equation, Adv. Nonlinear Stud., 7 (2007), 1-13.
[27] F. Obersnel and P. Omari, Existence and multiplicity results for the prescribed mean curvature equation via lower and upper solutions, Differential Integral Equations, 22 (2009), 853-880.
[28] F. Obersnel and P. Omari, Multiple bounded variation solutions of a periodically perturbed sine-curvature equation, Commun. Contemp. Math., 13 (2011) 863-883.
[29] F. Obersnel and P. Omari, The periodic problem for curvature-like equations with asymmetric perturbations, J. Differential Equations, 251 (2011), 1923-1971.
[30] F. Obersnel and P. Omari, Existence, regularity and boundary behaviour of bounded variation solutions of a one-dimensional capillarity equation, Discrete Contin. Dyn. Syst., in press (available at http://www.dmi.units.it/pubblicazioni/Quaderni_Matematici/ 615_2011.pdf).
[31] H. Pan and R. Xing, Time maps and exact multiplicity results for one-dimensional prescribed mean curvature equations. II, Nonlinear Anal. T.M.A., 74 (2011) 3751-3768.
[32] G.M. Troianiello, "Elliptic Differential Equations and Obstacle Problems", Plenum Press, New York, 1987.
[33] A.J. Ureña, Dynamics of periodic second-order equations between an ordered pair of lower and upper solutions, Adv. Nonlinear Stud., 11 (2011), 675-694.


[^0]:    *Research supported by G.N.A.M.P.A., in the frame of the project "Soluzioni periodiche di alcune classi di equazioni differenziali ordinarie", and by M.I.U.R., in the frame of the P.R.I.N. "Equazioni differenziali ordinarie e applicazioni".

