Grothendieck's counterexample to the Generalized Hodge Conjecture

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Introduction

At page 214 of of his book [4] Hodge stated a problem which later on became known as the "Generalized Hodge Conjecture". We will recall here for the reader convenience the particular case of it we are interested in.

Let X be a projective, smooth and connected threefold over $\mathbb{C}$. Moreover, let $S \subset X$ be a closed algebraic surface. Then there is a canonical map of vector spaces

$$(1) \qquad H_3(S, \mathbb{Q}) \to H_3(X, \mathbb{Q})$$

and it is certainly of great interest for the study of the algebraic geometry of X to be able to characterize the subspace of $H_3(X, \mathbb{Q})$ generated by the images of all such maps, when S varies inside X. By the way, notice that it would be better to work with homology with integral coefficients, as Hodge did. It was subsequently realized that the problems raised by Hodge have, in general, negative answers in this case (see [1]).

Hodge found a *necessary* condition for a class in $H_3(X, \mathbb{Q})$ to belong to such a subspace, and we are going to explain it. For simplicity we will assume S smooth. If $[\Gamma] \in H_3(S, \mathbb{Q})$, we can assume that the singular 3-cycle Γ is a linear combination of $\mathscr{C}^\infty$ singular 3-simplexes. The image of $[\Gamma]$ in the map (1) can be still represented by the cycle Γ. Now, for every closed $(3, 0)$-form α on X, Hodge remarked that

$$(2) \qquad \int_\Gamma \alpha \, = \, 0$$

To prove this, notice that the form which is actually integrated here is the pull-back of α to the various singular simplexes of Γ. An intermediate step in this pull-back procedure is the pull-back of α to S. But α contains too many dz's to be supported by a surface, hence

$$\alpha_{|S} \, \equiv \, 0$$

and (2) is proved.

At this point it is quite natural to ask whether the vanishing of the integral (2) for any choice of the closed $(3,0)$-form α on X is also a sufficient condition for a class $[\Gamma] \in H_3(X, \mathbb{Q})$ to be in the image of some map (1). This is Hodge's problem referred at the beginning.

This problem is even more natural if we put it in a historical perspective. In fact, let S be a projective, smooth and connected surface over $\mathbb{C}$, and let $C \subset S$ be a closed, algebraic curve. Then, the same argument used above shows that for any closed $(2,0)$-form α on S we have

$$(3) \qquad \int_C \alpha = 0$$

And it was a great breakthrough when Lefschetz proved that, conversely, if $[\Gamma] \in H_2(S, \mathbb{Q})$ is such that (3) is satisfied for every α as above, then $[\Gamma]$ is in the image of some canonical map $H_2(C, \mathbb{Q}) \to H_2(S, \mathbb{Q})$, where $C \subset S$ is an algebraic curve, not necessarily irreducible ([5]).

To progress further, Grothendieck had the idea to translate everything from homology to cohomology [3]. The device for this is the Poincaré duality isomorphism $PD : H_r(X, \mathbb{C}) \to H^{6-r}(X, \mathbb{C})$ which works as follows. Fix $[\Gamma] \in H_r(X, \mathbb{C})$, and let $i : \Gamma \to X$ denote the inclusion (cum grano salis because Γ is a cycle). Then we have a well defined $\mathbb{C}$-linear map

$$\lambda_{[\Gamma]} : H^r(X, \mathbb{C}) \to \mathbb{C} \qquad \text{given by} \qquad [\omega] \mapsto \int_\Gamma i^*\omega$$

Therefore, thanks to the canonical perfect pairing

$$(4) \quad \Psi : H^r(X, \mathbb{C}) \times H^{6-r}(X, \mathbb{C}) \to \mathbb{C} \qquad ([\omega], [\omega']) \mapsto \int_X \omega \wedge \omega'$$

there is one and only one $[\xi] \in H^{6-r}(X, \mathbb{C})$ such that

$$\lambda_{[\Gamma]} = \Psi(\_, [\xi])$$

or, in more down-to-earth terms

$$(5) \qquad \int_\Gamma i^*\omega = \int_X \omega \wedge \xi$$

for any closed r-form ω. The class $[\xi] \in H^{6-r}(X, \mathbb{C})$ is called the Poincaré dual of $[\Gamma] \in H_r(X, \mathbb{Q})$.

The attentive reader had certainly noticed that, to introduce Poincaré duality as above, one represents cohomology classes à la de Rham, i.e. by mean

of closed forms. This forces us to use cohomology with *complex* coefficients. How to deal in this set-up with *rational* cohomology classes?

Recall that, if η is a closed differential s-form on X, then a *period* of η is any complex number

$$\int_\Gamma \eta$$

where Γ is a s-cycle with *integral* coefficients. Then, η represents a class in $H^s(X, \mathbb{Q})$ ($\subset H^s(X, \mathbb{C})$) iff all its periods are in $\mathbb{Q}$ ([9], pp. 34-35).

With these last preparations, we have at hand everything we need to translate in cohomological terms Hodge's necessary condition.

First of all, the image in the Poincaré duality map of the sum of all the images of the possible maps (1) is customary denoted nowadays by $N^1 H^3(X, \mathbb{Q})$, and is a particular instance of the coniveau filtration on the rational cohomology of X. Actually, the standard definition of the spaces of the coniveau filtration is rather different, we will give it in a moment. For the comparison of the two definitions the reader is referred to [7].

Moreover, let $[\Gamma] \in H_3(X, \mathbb{Q})$ be such that (2) is satisfied for every closed $(3, 0)$-form α on X. Then by (5)

$$\int_X \alpha \wedge \xi = 0$$

for any α, i.e.

$$PD([\Gamma]) = [\xi] \in H^{3,0}(X)^\perp$$

where the orthogonal subspace is taken with respect to the canonical perfect pairing (4). But it is easily computed that

(6) $\qquad H^{3,0}(X)^\perp = H^{1,2}(X) \oplus H^{2,1}(X) \oplus H^{3,0}(X) =: F^1 H^3(X, \mathbb{C})$

where $F^1 H^3(X, \mathbb{C})$ is a subspace of the Hodge filtration of $H^3(X, \mathbb{C})$. Therefore we have

(7) $\qquad\qquad N^1 H^3(X, \mathbb{Q}) \subset F^1 H^3(X, \mathbb{C}) \cap H^3(X, \mathbb{Q})$

and Hodge problem amounts to ask if, actually, the above inclusion is an equality.

The usual definition of the coniveau filtration spaces is based on the concept of Gysin map. Given a proper map $f : S \to X$, where S is a smooth surface, the corresponding Gysin map $f_* : H^1(S, \mathbb{Q}) \to H^3(X, \mathbb{Q})$ is defined as the composition

$$H^1(S, \mathbb{Q}) \xrightarrow{PD} H_3(S, \mathbb{Q}) \xrightarrow{can.} H_3(X, \mathbb{Q}) \xrightarrow{PD} H^3(X, \mathbb{Q})$$

Then it is customary to set

$$N^1 H^3(X, \mathbb{Q}) := \sum_{f \ as \ above} Im(f_*)$$

The advantage of the coomological translation is that f_* *is a map of rational Hodge structures* ([8], 7.3.2), hence $Im(f_*)$ is a rational sub-Hodge structure of $H^3(X, \mathbb{Q})$. In particular, if we extend the scalars of f_* from $\mathbb{Q}$ to $\mathbb{C}$, we have

$$f_*(H^{0,1}(S)) \subset H^{1,2}(X) \qquad\qquad f_*(H^{1,0}(S)) \subset H^{2,1}(X)$$

and

$$\overline{f_*(H^{1,0}(S))} = f_*(H^{0,1}(S))$$

Then, by general facts on the category of (pure) rational Hodge structures, the space $N^1 H^3(X, \mathbb{Q})$ is also a rational sub-Hodge structure of $H^3(X, \mathbb{Q})$, and the above relations imply that

$$N^1 H^3(X, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C} = K^{1,2} \oplus K^{2,1}$$

where

$$K^{1,2} \subset H^{1,2}(X) \qquad K^{2,1} \subset H^{2,1}(X) \qquad \text{and} \qquad \overline{K^{1,2}} = K^{2,1}$$

Hence we can conclude that *the dimension of $N^1 H^3(X, \mathbb{Q})$ is even.*

On the other hand, Grothendieck exibited in [3] a particular abelian three-fold X for which the dimension of $H^3(X, \mathbb{Q}) \cap F^1 H^3(X, \mathbb{C})$ is odd, tush answering Hodge's question for the negative. The content of this expository paper is a thorough analysis of this example.

Grothendieck also showed how it is possible to correct the Generalized Hodge Conjecture simply by asking whether $N^1 H^3(X, \mathbb{Q})$ (instead to be equal to $H^3(X, \mathbb{Q}) \cap F^1 H^3(X, \mathbb{C})$) is the *maximal* rational sub-Hodge structure of $H^3(X, \mathbb{Q})$, which is contained into $F^1 H^3(X, \mathbb{C})$. The abelian threefold X he considers satisfies this amended Generalized Hodge Conjecture, as we will see.

1 The variety and its topology

Let E be an elliptic curve over the field of complex numbers. The projective manifolds we are interested in are the abelian threefolds

$$X := E \times E \times E = E^3$$

More precisely, X can be defined as follows. Let e_1, e_2, e_3 denote the standard basis of $\mathbb{C}^3$, and let z_1, z_2, z_3 denote the corresponding complex coordinates. Fix a complex number

(8) $\tau = u + iv$ where $u, v \in \mathbb{R}$ and $v > 0$

Then $e_1, e_2, e_3, \tau\, e_1, \tau\, e_2, \tau\, e_3$ is the (ordered) basis of a lattice $\Lambda \simeq \mathbb{Z}^6$ contained into $\mathbb{C}^3$, and we set

$$X := \mathbb{C}^3 / \Lambda$$

We will denote by $u_1, u_2, u_3, u_4, u_5, u_6$ the real coordinates in $\mathbb{C}^3$ with respect to the basis of Λ fixed above, namely

(9) $z_h = u_h + \tau\, u_{h+3}$ $h = 1, 2, 3$

Concerning the topology of X, let us consider integral homology first.
Let $I = [0, 1] \subset \mathbb{R}$, and define maps $\gamma_i : I \to \mathbb{C}^3$ by setting

$\gamma_i(t) = t\, e_i$ for $i = 1, 2, 3$ and $\gamma_i(t) = t\, \tau\, e_{i-3}$ for $i = 4, 5, 6$

If we compose these γ_i with the canonical map $\pi : \mathbb{C}^3 \to X$ we get six singular 1-cycles of X, whose classes are a basis for the free abelian group $H_1(X, \mathbb{Z})$. So inside X there are six copies of S^1, the images of the $\pi \circ \gamma_i$. We will denote them by $C_1, \ldots, C_6$. It is well known that a basis for $H_r(X, \mathbb{Z})$ is given by the classes of all the r-cycles

$$C_{i_1} \times C_{i_2} \times \ldots \times C_{i_r} \qquad \text{where} \quad 1 \le i_1 < i_2 < \ldots < i_r \le 6$$

Now we turn to the cohomology spaces of X with coefficients in $\mathbb{C}$.
A basis for $H^r(X, \mathbb{C})$ is given by the classes of the closed r-forms

(10) $d_H = du_{h_1 h_2 \ldots h_r} = du_{h_1} \wedge du_{h_2} \wedge \ldots \wedge du_{h_r}$

where $H = (h_1 h_2 \ldots h_r)$ is a multi-index, and $1 \le h_1 < h_2 < \ldots < h_r \le 6$. A straightforward computation then shows that

(11) $\displaystyle \int_{C_{i_1} \times C_{i_2} \times \ldots \times C_{i_r}} du_{h_1} \wedge du_{h_2} \wedge \ldots \wedge du_{h_r} = \delta_{i_1}^{h_1} \delta_{i_2}^{h_2} \ldots \delta_{i_r}^{h_r}$

where the δ's are Kronecker's. As was remarked in Introduction, this implies that *the classes of the forms (10) are also a basis for $H^r(X, \mathbb{Q})$ over $\mathbb{Q}$*.
For future use, let me show how goes the computation (11), at least in a particular case, for $r = 3$. Parametrize $C_2 \times C_4 \times C_5$ by first defining

$$\varphi : [0, 1]^3 \to \mathbb{C}^3 \qquad \varphi : (t_1, t_2, t_3) \mapsto t_1\, e_2 + t_2\, \tau\, e_1 + t_3\, \tau\, e_2$$

5

and then composing with the canonical map $\pi : \mathbb{C}^3 \to X$. Namely we have

$$u_1 = 0 \qquad u_2 = t_1 \qquad u_3 = 0 \qquad u_4 = t_2 \qquad u_5 = t_3 \qquad u_6 = 0$$

Actually, we can consider φ defined in an open neighborhood of $[0,1]^3$ inside $\mathbb{R}^3$, and therefore

$$\varphi^* \left(du_2 \wedge du_4 \wedge du_5 \right) = dt_1 \wedge dt_2 \wedge dt_3$$

which yields the result.

We turn now to the Hodge decomposition of the spaces $H^r(X, \mathbb{C})$, and to their relations with $H^r(X, \mathbb{Q})$. By general facts about the Hodge decomposition of an abelian variety, the Hodge diamond of X is

(12)
$$
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 3 & & 3 & & \\
 & 3 & & 9 & & 3 & \\
1 & & 9 & & 9 & & 1 \\
 & 3 & & 9 & & 3 & \\
 & & 3 & & 3 & & \\
 & & & 1 & & & \\
\end{array}
$$

For our purposes we have to use also the basis of $H^1(X, \mathbb{C})$

(13)
$$dz_1 \qquad dz_2 \qquad dz_3 \qquad d\overline{z}_1 \qquad d\overline{z}_2 \qquad d\overline{z}_3$$

Because of (9), the simple relations between the dz_h, $d\overline{z}_k$ and the du_j are

(14)
$$\boxed{dz_h = du_h + \tau\, du_{h+3}} \qquad \boxed{d\overline{z}_h = du_h + \overline{\tau}\, du_{h+3}}$$

(15)
$$\boxed{du_h = \left(\frac{1}{2} + i\, \frac{u}{2v} \right) dz_h + \left(\frac{1}{2} - i\, \frac{u}{2v} \right) d\overline{z}_h}$$

$$\boxed{du_{h+3} = -\frac{i}{2v}\, dz_h + \frac{i}{2v}\, d\overline{z}_h}$$

for any $h = 1, 2, 3$.

Finally, we can compute the various classes $PD([C_i \times C_j \times C_k])$ with respect to the basis (10) by mean of formula (5). They are

(16)
$$
\begin{array}{ll}
PD([C_1 \times C_2 \times C_3]) = -du_{456} & PD([C_2 \times C_3 \times C_4]) = du_{156} \\
PD([C_1 \times C_2 \times C_4]) = du_{356} & PD([C_2 \times C_3 \times C_5]) = -du_{146} \\
PD([C_1 \times C_2 \times C_5]) = -du_{346} & PD([C_2 \times C_3 \times C_6]) = du_{145} \\
PD([C_1 \times C_2 \times C_6]) = du_{345} & PD([C_2 \times C_4 \times C_5]) = du_{136} \\
PD([C_1 \times C_3 \times C_4]) = -du_{256} & PD([C_2 \times C_4 \times C_6]) = -du_{135} \\
PD([C_1 \times C_3 \times C_5]) = du_{246} & PD([C_2 \times C_5 \times C_6]) = du_{134} \\
PD([C_1 \times C_3 \times C_6]) = -du_{245} & PD([C_3 \times C_4 \times C_5]) = -du_{126} \\
PD([C_1 \times C_4 \times C_5]) = -du_{236} & PD([C_3 \times C_4 \times C_6]) = du_{125} \\
PD([C_1 \times C_4 \times C_6]) = du_{235} & PD([C_3 \times C_5 \times C_6]) = -du_{124} \\
PD([C_1 \times C_5 \times C_6]) = -du_{234} & PD([C_4 \times C_5 \times C_6]) = du_{123}
\end{array}
$$

2 Divisors of X

To test Hodge's and Grothendieck's guesses on X we will need a rather detailed knowledge of the divisors of this variety. We will determine in this section a set of divisors on X whose cohomology classes generate the space $H^2(X, \mathbb{Q}) \cap H^{1,1}(X)$. In particular, to apply the various results of Lefschetz's theory, we will determine an ample divisor.

The first step is the computation of $H^2(X, \mathbb{Q}) \cap H^{1,1}(X)$. To produce a basis for this $\mathbb{Q}$-module we will use the approach of [6]. Consider the closed 2-form

(17)
$$
F = \sum_{1 \le h,k \le 3} a_{hk} \, dz_h \wedge d\bar{z}_k = \sum_{1 \le s < t \le 6} b_{st} \, du_s \wedge du_t
$$

where the a_{hk} and b_{st} are all in $\mathbb{C}$. By (11), we have that $[F] \in H^2(X, \mathbb{Q})$ if and only if all the b_{st} are in $\mathbb{Q}$. Murasaki's idea is to write

(18)
$$
F = F_1 + F_2 + F_3 + F_{12} + F_{13} + F_{23}
$$

where, for every $h = 1, 2, 3$

$$
F_h := a_{hh} \, dz_h \wedge d\bar{z}_h
$$

and for every $1 \le h < k \le 3$

$$
F_{hk} := a_{hk} \, dz_h \wedge d\bar{z}_k + a_{kh} \, dz_k \wedge d\bar{z}_h
$$

Claim F is a rational class if and only if all the F_h and the F_{hk} are such.

One direction is obvious, so assume that F is rational. From (14) we get

$$F_h \;=\; -\,2iv\,a_{hh}\;du_h \wedge du_{h+3}$$

(19)   and

$$
\begin{aligned}
F_{hk} \;=\;& (\,a_{hk}-a_{kh}\,)\,du_h \wedge du_k \;+\; (\,a_{hk}\,\overline{\tau}-a_{kh}\,\tau\,)\,du_h \wedge du_{k+3} \;+\\[4pt]
&+\,(\,a_{kh}\,\overline{\tau}-a_{hk}\,\tau\,)\,du_k \wedge du_{h+3} \;+\; (\,a_{hk}-a_{kh}\,)\,\tau\,\overline{\tau}\,du_{h+3} \wedge du_{k+3}
\end{aligned}
$$

In particular, this shows that each of the six terms in (18) involves different elements of the basis $du_i \wedge du_j$ of $H^2(X,\mathbb{C})$, so that the Claim above is completely proved.

Now, the first equation of (19) yields by (17) that

$$-\,2iv\,a_{hh} \;=\; b_{h,h+3}$$

Therefore $b_{h,h+3} \in \mathbb{Q}$ if and only if

$$a_{hh} \;=\; \frac{i}{2\,v}\,r \qquad \text{where} \quad r \in \mathbb{Q}$$

In other words, for any $h = 1,2,3$

(20) $$\frac{i}{2\,v}\,dz_h \wedge d\overline{z}_h \;=\; du_h \wedge du_{h+3} \in H^2(X,\mathbb{Q}) \cap H^{1,1}(X)$$

and they are independent over $\mathbb{Q}$.

Concerning the class F_{hk}, for any fixed $1 \le h < k \le 3$, from the second relation in (19) it follows that

$$
\begin{cases}
a_{hk}-a_{kh} \;\in\; \mathbb{Q} \\[8pt]
(\,a_{hk}-a_{kh}\,)\,\tau\,\overline{\tau} \;\in\; \mathbb{Q} \\[8pt]
a_{hk}\,\overline{\tau}-a_{kh}\,\tau \;\in\; \mathbb{Q} \\[8pt]
a_{kh}\,\overline{\tau}-a_{hk}\,\tau \;\in\; \mathbb{Q}
\end{cases}
$$

The last two relations imply that

$$(\,a_{hk}-a_{kh}\,)(\,\tau+\overline{\tau}\,) \;\in\; \mathbb{Q}$$

Therefore, if $a_{hk}-a_{kh} \neq 0$, *then necessarily τ is an algebraic number over $\mathbb{Q}$, of degree $[\mathbb{Q}(\tau):\mathbb{Q}] \le 2$.* From now on we will rule out this possibility by making the

Assumption 1. *If the complex number τ is algebraic over $\mathbb{Q}$, then its degree over $\mathbb{Q}$ is ≥ 3.*

Hence if F is rational, then necessarily $a_{hk} = a_{kh} = a$ and

$$F_{hk} = a\,(\bar{\tau} - \tau)\,\big(du_h \wedge du_{k+3} + du_k \wedge du_{h+3}\big) =$$

$$= -2\,i\,a\,v\,\big(du_h \wedge du_{k+3} + du_k \wedge du_{h+3}\big)$$

We get in this way three more classes into $H^2(X, \mathbb{Q}) \cap H^{1,1}(X)$, independent over $\mathbb{Q}$, namely

$$(21) \qquad \frac{i}{2\,v}\,\big(dz_h \wedge d\bar{z}_k + dz_k \wedge d\bar{z}_h\big) = du_h \wedge du_{k+3} + du_k \wedge du_{h+3}$$

To summarize

$$dim_{\mathbb{Q}}\big(H^2(X, \mathbb{Q}) \cap H^{1,1}(X)\big) = 6$$

Let us consider the divisors now.

First of all, the abelian surface $E \times E$ can be embedded in X in a trivial way by setting, for an arbitrary $P \in E$

$$S_3 := E \times E \times P$$

The family $\{E \times E \times P\}_{P \in E}$ is a fibration of X. Moreover, if $P, Q \in E$, then $E \times E \times P$ and $E \times E \times Q$ are algebraically equivalent, hence they are homologically equivalent.

To determine the cohomology class of S_3, we remark that as a singular 4-cycle inside X we have $S_3 = C_1 \times C_2 \times C_4 \times C_5$. Then, for any closed 4-form

$$\alpha = \sum_{\#I=4} b_I\,du_I \qquad b_I \in \mathbb{Q} \quad \text{for any} \quad I$$

relation (11) implies

$$\int_{S_3} \alpha = \int_{C_1 \times C_2 \times C_4 \times C_5} \alpha = b_{(1,2,4,5)}$$

Therefore, (5) will be satisfied for every form α if we take

$$(22) \qquad \omega_3 := du_3 \wedge du_6 = \frac{i}{2\,v}\,dz_3 \wedge d\bar{z}_3$$

In other words

$$(23) \qquad PD(\,[S_3]\,) = \frac{i}{2\,v}\,dz_3 \wedge d\bar{z}_3$$

On X we have also two other obvious families of surfaces, given respectively by

$$S_1 := P \times E \times E \qquad\qquad S_2 := E \times P \times E$$

and the corresponding cohomology classes, computed as above, are

$$PD([S_1]) = \omega_1 := \frac{i}{2\,v}\, dz_1 \wedge d\bar{z}_1 \qquad PD([S_2]) = \omega_2 := \frac{i}{2\,v}\, dz_2 \wedge d\bar{z}_2$$

Consider now E embedded into $\mathbb{P}^2$ via $|3P|$. Then we have also the embedding

$$(24) \qquad X = E \times E \times E \ \lhook\joinrel\longrightarrow\ \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2 \ \lhook\joinrel\longrightarrow\ \mathbb{P}^{26}$$

where $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$ is embedded into $\mathbb{P}^{26}$ à la Segre. It is clear that the corresponding very ample divisor of X is nothing but $3\,(S_1 + S_2 + S_3)$. We will use in the sequel the ample divisor

$$(25) \qquad H := S_1 + S_2 + S_3$$

Finally

$$(26) \qquad \omega := PD([H]) = \frac{i}{2\,v}\left(dz_1 \wedge d\bar{z}_1 + z_2 \wedge d\bar{z}_2 + dz_3 \wedge d\bar{z}_3 \right)$$

To produce divisors not equivalent to the S_i's, we can try to embed $E \times E$ inside X by using the diagonal map $\Delta : E \to E \times E$ like in

$$(27) \qquad f : E \times E \ \overset{\Delta \times id_E}{\lhook\joinrel\longrightarrow}\ E \times E \times E$$

We will denote by T_3 the image of $E \times E$ in the proper map f. Two other surfaces T_1 and T_2 can be defined inside X as the images of the (proper) map

$$E \times E \ \overset{id_E \times \Delta}{\lhook\joinrel\longrightarrow}\ E \times E \times E$$

and similarly for T_2.

We will determine now the cohomology class of the divisor T_3. To compute the pull-back of forms it is better to describe the map f in (27) as follows. If $\varepsilon_1, \varepsilon_2$ is the standard basis of $\mathbb{C}^2$, we have the real basis $\varepsilon_1, \varepsilon_2, \tau\,\varepsilon_1, \tau\,\varepsilon_2$ of this space. It generates over the integers a lattice $L \subset \mathbb{C}^2$, and of course $E \times E = \mathbb{C}^2/L$. Moreover, let v_1, v_2, v_3, v_4 denote real coordinates in $\mathbb{C}^2$ with

respect to $\varepsilon_1, \varepsilon_2, \tau \varepsilon_1, \tau \varepsilon_2$. Then the map f is induced by the map $\mathbb{C}^2 \to \mathbb{C}^3$ given in real coordinates by

$$
(28) \qquad
\begin{array}{ccc}
u_1 = v_1 & u_2 = v_1 & u_3 = v_2 \\[2mm]
u_4 = v_3 & u_5 = v_3 & u_6 = v_4
\end{array}
$$

First consequences of these relations are

$$
f^* (du_1 \wedge du_2) = 0 \qquad\qquad f^* (du_4 \wedge du_5) = 0
$$

Hence, given any rational closed 4-form $\alpha = b_{1234} \, du_1 \wedge du_2 \wedge du_3 \wedge du_4 + \dots$ on X, we have

$$
f^* \alpha = - (b_{1346} + b_{1356} + b_{2346} + b_{2356}) \, dv_1 \wedge dv_2 \wedge dv_3 \wedge dv_4
$$

If we set

$$
(29) \qquad\qquad \eta_3 := du_{15} + du_{24} - du_{14} - du_{25}
$$

it is easily checked that

$$
\eta_3 \wedge \alpha = - (b_{1346} + b_{1356} + b_{2346} + b_{2356}) \, du_1 \wedge du_2 \wedge \dots \wedge du_6
$$

and we can conclude that

$$
\int_{T_3} f^* \alpha = \int_X \eta_3 \wedge \alpha
$$

for any rational closed 4-form α on X, namely that the Poincaré dual σ_3 of T_3 is represented by the closed form η_3, which can also be written as

$$
(30) \qquad \eta_3 = \frac{i}{2\,v} \left(dz_1 \wedge d\bar{z}_2 + dz_2 \wedge d\bar{z}_1 - dz_1 \wedge d\bar{z}_1 - dz_2 \wedge d\bar{z}_2 \right)
$$

because of (20) and (21).

To conclude this section, I will record some invariants of a general member $Y \in |\, 3H \,|$, computed just for fun. From the Hodge diamond (12) we get

$$
H^1(X, \mathscr{O}_X) \simeq H^2(X, \mathscr{O}_X) \simeq \mathbb{C}^3 \qquad\qquad H^3(X, \mathscr{O}_X) \simeq \mathbb{C}
$$

Moreover, by Kodaira's Vanishing Theorem

$$
H^i(X, \mathscr{O}_X(-1)) = 0 \qquad \text{for} \quad i = 0, 1, 2
$$

11

and also
$$H^i(X, \mathscr{O}_X(1)) = 0 \qquad \text{for} \quad i = 1, 2, 3$$

because of

(31)
$$\omega_X \simeq \mathscr{O}_X$$

This is just a translation of $K_X = 0$, which in turn is a straightforward consequence of $K_E = 0$. Moreover, thanks to (31), Serre's duality yields

$$H^3(X, \mathscr{O}_X(-1)) \simeq H^0(X, \mathscr{O}_X(1)) \simeq \mathbb{C}^{27}$$

From all this facts and the short exact sequence

$$0 \to \mathscr{O}_X(-1) \to \mathscr{O}_X \to \mathscr{O}_Y \to 0$$

we get finally

(32)
$$H^1(Y, \mathscr{O}_Y) \simeq \mathbb{C}^3 \qquad\qquad H^2(Y, \mathscr{O}_Y) \simeq \mathbb{C}^{29}$$

For a complete knowledge of the Hodge diamond of Y it remains to compute only $H^{1,1}(Y)$. We will do it via the topological Euler-Poincarè characteristic of Y, which is related to (32) by Noether's formula

$$\chi(\mathscr{O}_Y) = \frac{1}{12}\left(K_Y^2 + \chi_{top}(Y)\right)$$

Now, by adjunction formula

(33)
$$\omega_Y \simeq \mathscr{O}_Y(1)$$

hence

$$K_Y^2 = \underbrace{Y^3}_{inside\ X} = 162$$

and we conclude

$$H^{1,1}(Y) \simeq \mathbb{C}^{114}$$

3 Classes of $N^1 H^3(X, \mathbb{Q})$

The purpose of this section is to compute the contribution to $N^1 H^3(X, \mathbb{Q})$ of the several different surfaces on X we know.

Let i denote the inclusion $S_1 \subset X$. We start by computing the image of the Gysin map $i_* : H^1(S_1, \mathbb{Q}) \to H^3(X, \mathbb{Q})$. A basis for $H_3(S_1, \mathbb{Q})$ is given by the classes

$$[C_2 \times C_3 \times C_5] \qquad [C_2 \times C_3 \times C_6] \qquad [C_2 \times C_5 \times C_6] \qquad [C_3 \times C_5 \times C_6]$$

12

They are sent by i_* into the same classes, viewed as elements of $H_3(X, \mathbb{Q})$. Finally, by the table (16) we conclude that $i_*(H^1(S_1, \mathbb{Q}))$ is generated by

$$du_{146} \, , \ du_{145} \, , \ du_{134} \, , \ du_{124}$$

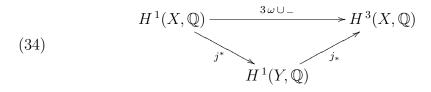Similarly, bases for the images of the Gysin maps for the surfaces S_2 and S_3 are given respectively by

$$du_{256} \, , \ du_{245} \, , \ du_{235} \, , \ du_{125}$$

and

$$du_{356} \, , \ du_{346} \, , \ du_{236} \, , \ du_{136}$$

A quick inspection shows that the twelve 3-forms above are distinct elements of the basis (10) of $H^3(X, \mathbb{Q})$; let M_0 denote the subspace they generate.

Now we will compute the image of the Gysin map for the inclusion $j : Y \subset X$ of the very ample divisor Y. By (26), $3\omega \in H^2(X, \mathbb{Q})$ is the cohomology class of Y. Then the following diagram is commutative

(34)
$$
\begin{array}{ccc}
H^1(X, \mathbb{Q}) & \xrightarrow{\ 3\omega \cup \_\ } & H^3(X, \mathbb{Q}) \\
& \searrow{\scriptstyle j^*} \quad \nearrow{\scriptstyle j_*} & \\
& H^1(Y, \mathbb{Q}) &
\end{array}
$$

In fact, we have the commutative diagram, where $[X] \in H_6(X, \mathbb{Q})$ and $[Y] \in H_4(Y, \mathbb{Q})$ are the fundamental classes of X and Y respectively

$$
\begin{array}{ccc}
H_3(Y, \mathbb{Q}) & \xrightarrow{\ j_*\ } & H_3(X, \mathbb{Q}) \\
{\scriptstyle \_ \cap [Y]}\big\uparrow & & \big\uparrow{\scriptstyle \_ \cap [X]} \\
H^1(X, \mathbb{Q}) \xrightarrow[j^*]{} H^1(Y, \mathbb{Q}) & \xrightarrow[j_*]{} & H^3(X, \mathbb{Q})
\end{array}
$$

The cohomology class 3ω and $[Y]$ are related by Poincaré duality as

$$j_*[Y] = 3\omega \cap [X]$$

Then, for every $x \in H^1(X, \mathbb{Q})$ we have by the "projection formula" and the above relation

$$(j_* j^* x) \cap [X] = j_*(j^* x \cap [Y]) = x \cap j_*[Y] =$$

$$x \cap (3\omega \cap [X]) = (x \cup 3\omega) \cap [X] = (3\omega \cup x) \cap [X]$$

where the last equality is true because the degree of ω is 2. Since the Poincaré duality map is an isomorphism, the commutativity of (34) is completely proved.

It is customary to denote by L the above map $H^1(X, \mathbb{Q}) \to H^3(X, \mathbb{Q})$ given by capping with $3\,\omega$. Since $j^* : H^1(X, \mathbb{Q}) \to H^1(Y, \mathbb{Q})$ is an isomorphism by the Weak Lefschetz Theorem, we have then

$$Im(j_*) \; = \; Im(L)$$

If α, β are closed forms, then $[\alpha] \cup [\beta] = [\alpha \wedge \beta]$. Moreover, due to the use of rational coefficients, we get the same image if we cap with ω instead of $3\,\omega$. Finally, the form (26) can be written in the base (10) as

$$\omega \; = \; du_{14} + du_{25} + du_{36}$$

Therefore

$$(35) \qquad \begin{aligned}
\omega \wedge du_1 &= -du_{125} - du_{136} \\
\omega \wedge du_2 &= -du_{125} - du_{136} \\
\omega \wedge du_3 &= -du_{125} - du_{136} \\
\omega \wedge du_4 &= -du_{125} - du_{136} \\
\omega \wedge du_5 &= -du_{125} - du_{136} \\
\omega \wedge du_6 &= -du_{125} - du_{136}
\end{aligned}$$

These six elements are already in M_0, so no further progress has been made toward the generation of $N^1 H^3(X, \mathbb{Q})$. Morally this was to be expected because of (25).

The contribution of the surfaces T_1, T_2 and T_3 to $N^1 H^3(X, \mathbb{Q})$ is not easily determined in homology, so we switch directly to cohomology.

Consider the map f defined in (27) (or (28), in coordinates). The image of f was denoted by T_3; for simplicity, we will still denote by f the inclusion of T_3 into X.

The Gysin maps induced by f appears in the following diagram, which can be shown to be commutative by the same argument used above to prove the commutativity of (34) (recall also (30))

$$(36) \qquad \begin{array}{ccc}
H^1(X, \mathbb{Q}) & \xrightarrow{\;\sigma_3 \cup -\;} & H^3(X, \mathbb{Q}) \\
& {\scriptstyle f^*} \searrow \quad \nearrow {\scriptstyle f_*} & \\
& H^1(T_3, \mathbb{Q}) &
\end{array}$$

The only relevant difference here is that f^* cannot be any more an isomorphism for the trivial reason that $H^1(X, \mathbb{Q})$ and $H^1(T_3, \mathbb{Q})$ have different dimensions (6 and 4 respectively).

Recall that the closed form η_3 representing $\sigma_3 = PD([T_3])$ was already given in (29). Then the space $Im(\sigma_3 \cup \_)$ is generated by the classes of

$$
\begin{aligned}
\eta_3 \wedge du_1 &= du_{124} - du_{125} \in M_0 \\
\eta_3 \wedge du_2 &= du_{124} - du_{125} \\
\eta_3 \wedge du_3 &= \underbrace{du_{134} + du_{235}}_{\in M_0} - du_{135} - du_{234}
\end{aligned}
$$

$$
\begin{aligned}
\eta_3 \wedge du_4 &= du_{245} - du_{145} \in M_0 \\
\eta_3 \wedge du_5 &= du_{245} - du_{145} \\
\eta_3 \wedge du_6 &= \underbrace{-du_{256} - du_{146}}_{\in M_0} + du_{246} + du_{156}
\end{aligned}
$$

These relations shows that $Im(\sigma_3 \cup \_)$ has dimension four. But this forces f^* to be onto because $dim_{\mathbb{Q}} H^1(T_3, \mathbb{Q}) = 4$, hence

$$
Im(f_*) = Im(\sigma_3 \cup \_)
$$

To summarize, the contribution of $Im(f_*)$ to the generation of the space $N^1 H^3(X, \mathbb{Q})$ is given by the classes

$$
(37) \qquad du_{135} + du_{234} \qquad\qquad du_{246} + du_{156}
$$

Similar computations can be performed for the surfaces T_1 and T_2, which contribute to the generation of $N^1 H^3(X, \mathbb{Q})$ respectively with the classes

$$
(38) \qquad du_{135} + du_{126} \qquad\qquad du_{246} + du_{345}
$$

and

$$
(39) \qquad du_{126} - du_{234} \qquad\qquad du_{156} - du_{345}
$$

Finally, it is easily seen that a basis for the subspace of $H^3(X, \mathbb{Q})$ generated by the six classes (37), (38) and (39) is given by

$$
(40) \quad du_{126} - du_{234} \qquad du_{156} - du_{345} \qquad du_{246} + du_{345} \qquad du_{135} + du_{234}
$$

and that

$$
M_0 \cap \langle du_{126} - du_{234}, du_{156} - du_{345}, du_{246} + du_{345}, du_{135} + du_{234} \rangle = 0
$$

We set

(41) $\quad M := M_0 \oplus \langle\, du_{126} - du_{234}\,,\ du_{156} - du_{345}\,,\ du_{246} + du_{345}\,,\ du_{135} + du_{234}\,\rangle$

This space M is a rational sub-Hodge structure of $H^3(X, \mathbb{Q})$, contained into $N^1 H^3(X, \mathbb{Q})$, hence into $F^1 H^3(X, \mathbb{C})$. From $dim_{\mathbb{Q}} M = 16$, we get then

(42) $$dim_{\mathbb{Q}} N^1 H^3(X, \mathbb{Q}) \geq 16$$

4 Computation of $F^1 H^3(X, \mathbb{C}) \cap H^3(X, \mathbb{Q})$

For this computation we will exploit (6) and the fact that $H^{3,0}(X)$ is isomorphic to $\mathbb{C}$, generated by the class of the closed form

$$\alpha := dz_1 \wedge dz_2 \wedge dz_3$$

In terms of the base (10) the form α is given by

(43)
$$\begin{aligned}
\alpha \;&=\; dz_1 \wedge dz_2 \wedge dz_3 \;= \\
&=\; (\, du_1 + \tau\, du_4\,) \wedge (\, du_2 + \tau\, du_5\,) \wedge (\, du_3 + \tau\, du_6\,) \;= \\
&=\; du_{123} + \tau\,(du_{126} - du_{135} + du_{234}) + \tau^2\,(du_{156} - du_{246} + du_{345}) + \tau^3\, du_{456}
\end{aligned}$$

Then, for an arbitrary $\omega = \sum_{1 \leq i < j < k \leq 6} r_{ijk}\, du_{ijk}$ where $r_{ijk} \in \mathbb{Q}$ for any i, j, k, we have that $\omega \wedge \alpha$ is

$$\left(\, r_{123}\,\tau^3 - (\, r_{234} - r_{135} + r_{126}\,)\,\tau^2 + (\, r_{345} - r_{246} + r_{156}\,)\,\tau - r_{456}\,\right) du_{123456}$$

Hence $[\omega]$ is orthogonal to $[\alpha]$ with respect to (4) if and only if

(44) $\quad r_{123}\,\tau^3 - (\, r_{234} - r_{135} + r_{126}\,)\,\tau^2 + (\, r_{345} - r_{246} + r_{156}\,)\,\tau - r_{456} \;=\; 0$

Now, by Assumption 1, if τ is not algebraic over $\mathbb{Q}$, of degree 3, this relation is satisfied only if all the coefficients in it vanish. In this case

$$dim_{\mathbb{Q}}\left(\, H^3(X, \mathbb{Q}) \cap F^1 H^3(X, \mathbb{C})\,\right) \;=\; 16$$

Therefore, by (7) and (42) we conclude

Corollary 2. *If $[\mathbb{Q}(\tau) : \mathbb{Q}] > 3$ (in particular, if τ is trascendental over $\mathbb{Q}$), then*

$$N^1 H^3(X, \mathbb{Q}) \;=\; F^1 H^3(X, \mathbb{C}) \cap H^3(X, \mathbb{Q})$$

On the other hand, Grothendieck considered the case when τ is algebraic over $\mathbb{Q}$, of degree 3. If the minimal polynomial of τ over $\mathbb{Q}$ is

$$(45) \qquad f \; = \; X^3 \, + \, \mu_1 \, X^2 \, + \, \mu_2 \, X \, + \, \mu_3 \; = \; 0 \qquad\qquad \mu_1, \mu_2, \mu_3 \in \mathbb{Q}$$

then relation (44) can be rewritten as

$$\left(\, r_{234} - r_{135} + r_{126} + \mu_1 \, r_{123} \, \right) \tau^2 - \left(\, r_{345} - r_{246} + r_{156} - \mu_2 \, r_{123} \, \right) \tau + r_{456} + \mu_3 \, r_{123} \; = \; 0$$

Since $[\mathbb{Q}(\tau) : \mathbb{Q}] = 3$, we have necessarily

$$(46) \qquad \begin{cases} r_{234} - r_{135} + r_{126} + \mu_1 \, r_{123} & = \; 0 \\ r_{345} - r_{246} + r_{156} - \mu_2 \, r_{123} & = \; 0 \\ r_{456} + \mu_3 \, r_{123} & = \; 0 \end{cases}$$

This linear system has rank 3, hence

$$dim_{\mathbb{Q}} \left(\, H^3(X, \mathbb{Q}) \cap F^1 \, H^3(X, \mathbb{C}) \, \right) \; = \; 20 - 3 \; = \; 17$$

and *the Generalized Hodge Conjecture fails in it original form*.

But in our case $N^1 H^3(X, \mathbb{Q}) = M$, the space introduced in (41), is the maximal rational sub-Hodge structure of $F^1 H^3(X, \mathbb{C})$. Hence, the Generalized Hodge Conjecture as amended by Grothendieck is still true for X.

In the remaining of this section we will try to get a better understanding of what is really going on.

Any element in $H^3(X, \mathbb{Q})$ can be represented by a closed form like

$$\begin{aligned} \omega \; = \; & r_{123} \, du_{123} + r_{126} \, du_{126} + r_{135} \, du_{135} + r_{156} \, du_{156} + \\ & r_{234} \, du_{234} + r_{246} \, du_{246} + r_{345} \, du_{345} + r_{456} \, du_{456} + \nu \end{aligned}$$

where $\nu \in M_0$ and $r_{ijk} \in \mathbb{Q}$ for every i, j, k. Moreover $\omega \in F^1 H^3(X, \mathbb{C})$ if and only if the coefficients r_{ijk} satisfy conditions (46), and in this case ω can be rewritten as

$$\begin{aligned} \omega \; = \; & r_{123} \, du_{123} + r_{126} \, du_{126} + r_{135} \, du_{135} + r_{156} \, du_{156} + \\ & + \left(\, r_{135} - r_{126} - \mu_1 \, r_{123} \, \right) du_{234} + r_{246} \, du_{246} + \\ & + \left(\, r_{246} - r_{156} + \mu_2 \, r_{123} \, \right) du_{345} - \mu_3 \, r_{123} \, du_{456} + \nu \; = \\[6pt] = \; & r_{123} \left(\, du_{123} - \mu_1 \, du_{234} + \mu_2 \, du_{345} - \mu_3 \, du_{456} \, \right) + \\ & + r_{126} \left(\, du_{126} - du_{234} \, \right) + r_{135} \left(\, du_{135} + du_{234} \, \right) + \\ & + r_{156} \left(\, du_{156} - du_{345} \, \right) + r_{246} \left(\, du_{246} + du_{345} \, \right) + \nu \end{aligned}$$

This expression of ω clarifies the role of the forms (40). In particular

$$\omega = r_{123}\left(du_{123} - \mu_1\, du_{234} + \mu_2\, du_{345} - \mu_3\, du_{456}\right) + \nu'$$

where $\nu' \in N^1 H^3(X, \mathbb{Q})$, and $[\omega] \in N^1 H^3(X, \mathbb{Q})$ if and only if $r_{123} = 0$.

Now set

$$\varphi := d_{123} - \mu_1\, d_{234} + \mu_2\, d_{345} - \mu_3\, d_{456}$$

The Poincaré dual of $[\varphi]$ can be computed with the help of the table (5); it is represented by the cycle

$$\Gamma := C_4 \times C_5 \times C_6 + \mu_1\, C_1 \times C_5 \times C_6 + \mu_2\, C_1 \times C_2 \times C_6 + \mu_3\, C_1 \times C_2 \times C_3$$

Let us set

(47)
$$\Gamma_1 := C_4 \times C_5 \times C_6 \qquad \Gamma_2 := C_1 \times C_5 \times C_6$$

$$\Gamma_3 := C_1 \times C_2 \times C_6 \qquad \Gamma_4 := C_1 \times C_2 \times C_3$$

Then (11) and (43) imply

(48)
$$\int_{\Gamma_1} \beta = \tau^3 \qquad \int_{\Gamma_2} \beta = \tau^2 \qquad \int_{\Gamma_3} \beta = \tau \qquad \int_{\Gamma_4} \beta = 1$$

and therefore *no one of the classes* $[d_{123}]$, $[d_{234}]$, $[d_{345}]$, $[d_{456}]$ *in* $H^3(X, \mathbb{C})$ *is contained into* $F^1 H^3(X, \mathbb{C})$.

Finally, since

$$\Gamma = \Gamma_1 + \mu_1\, \Gamma_2 + \mu_2\, \Gamma_3 + \mu_3\, \Gamma_4$$

we conclude

$$\int_\Gamma \beta = \tau^3 + \mu_1 \tau^2 + \mu_2 \tau + \mu_3 = 0$$

a little, great miracle! This implies $[\varphi] \in F^1 H^3(X, \mathbb{C})$, as we already know.

5   A few, simple facts about τ

First of all, we have

(49)
$$\tau\,\overline{\tau} \notin \mathbb{Q}$$

In fact, if σ denotes the unique real root of f, then clearly $\sigma \notin \mathbb{Q}$, but $\sigma\,\tau\,\overline{\tau} \in \mathbb{Q}$.

Another fact is

(50) $$u \notin \mathbb{Q} \qquad v \notin \mathbb{Q}$$

To prove this, notice that we have the factorization

$$f = (X - \sigma)\left(X^2 + (\mu_1 + \sigma)X + \mu_2 + \mu_1\sigma + \sigma^2\right)$$

hence

$$X^2 + (\mu_1 + \sigma)X + \mu_2 + \mu_1\sigma + \sigma^2 = (X - \tau)(X - \overline{\tau}) = X^2 - 2uX + u^2 + v^2$$

and we get

$$-2u = \mu_1 + \sigma \qquad\qquad u^2 + v^2 = \mu_2 + \mu_1\sigma + \sigma^2$$

Since $\sigma \notin \mathbb{Q}$ but $\mu_1 \in \mathbb{Q}$, the first relation implies $u \notin \mathbb{Q}$. Moreover, the two relations toghether yield

$$\frac{3}{4}\sigma^2 + \frac{1}{2}\mu_1\sigma + \mu_2 - \frac{1}{4}\mu_1^2 - v^2 = 0$$

If $v \in \mathbb{Q}$ then σ would be a root of a non trivial polynomial of degree two with rational coefficients, contradiction.

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