Multiple periodic solutions of Hamiltonian systems in the plane

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Abstract. Our aim is to prove a multiplicity result for periodic solutions of Hamiltonian systems in the plane, by the use of the Poincaré-Birkhoff Fixed Point Theorem. Our main theorem generalizes previous results obtained for scalar second order equations by Lazer and McKenna [6] and Del Pino, Manasevich and Murua [2].

1 Introduction

We consider the periodic problem

$$\begin{cases} J\dot{u} = \nabla H(u) + \nabla F(t,u) + s v_0(t)\,, \\ u(0) = u(T)\,. \end{cases} \tag{1}$$

Here, $H : \mathbb{R}^2 \to \mathbb{R}$ is a continuously differentiable function, with Lipschitz continuous gradient. It is positively homogeneous of degree 2, i.e.,

$$H(\alpha u) = \alpha^2 H(u)\,, \qquad \text{for every } \alpha > 0 \text{ and } u \in \mathbb{R}^2, \tag{2}$$

and positive, i.e.,

$$H(u) > 0\,, \qquad \text{for every } u \in \mathbb{R}^2 \backslash \{0\}\,. \tag{3}$$

The function $F : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ is assumed to be differentiable in $u \in \mathbb{R}^2$ with gradient $\nabla F(t,u)$ satisfying the following Carathéodory-type conditions, with locally Lipschitz continuity in u:

- $\nabla F(\cdot\,, u)$ is integrable on $[0,T]$, for every $u \in \mathbb{R}^2$,
- for every $R > 0$ there is a $\ell_R \in L^1(0,T)$ such that, if $u, v \in B(0,R)$, then

$$\|\nabla F(t,u) - \nabla F(t,v)\| \le \ell_R(t)\|u - v\|\,, \qquad \text{for a.e. } t \in [0,T]\,.$$

Moreover, ∇F has a sublinear growth, i.e.,

$$\lim_{\|u\| \to +\infty} \frac{\nabla F(t,u)}{\|u\|} = 0\,, \quad \text{uniformly for a.e. } t \in [0,T]\,. \tag{4}$$

The number s is a large positive parameter, and $v_0 : [0,T] \to \mathbb{R}$ is an integrable function.

Notice that, assuming (2) and (3), all the solutions of the autonomous system

$$J\dot{u} = \nabla H(u) \tag{5}$$

are periodic with the same minimal period, so that the origin is an isochronous center. Such a situation has been discussed in [4].

Let us state our multiplicity result.

Theorem 1 *Let the following assumptions hold.*

(i) The function H satisfies (2) and (3).

(ii) There are a function $w_0 : \mathbb{R} \to \mathbb{R}^2$, solution of

$$\begin{cases} J\dot{w} = \nabla H(w) + v_0(t)\,, \\ w(0) = w(T)\,, \end{cases} \tag{6}$$

a constant $r_0 > 0$ and two positive definite symmetric matrices A_1 and A_2 such that, setting

$$\mathcal{B}_{r_0} = \{w_0(t) + x :\ t \in [0,T],\ \|x\| \le r_0\}\,,$$

one has that $0 \notin \mathcal{B}_{r_0}$ and, for every $u, v \in \mathcal{B}_{r_0}$,

$$\langle A_1(u-v)|u-v\rangle \ \le\ \langle \nabla H(u) - \nabla H(v)|u-v\rangle\ \le \langle A_2(u-v)|u-v\rangle\,. \tag{7}$$

Moreover, defining

$$\sigma_1 = \frac{2\pi}{\sqrt{\det A_1}}\,, \qquad \sigma_2 = \frac{2\pi}{\sqrt{\det A_2}}\,,$$

there is an integer m for which

$$m < \frac{T}{\sigma_1} \le \frac{T}{\sigma_2} < m+1\,. \tag{8}$$

(iii) Denoting by τ the period of the solutions to (5), there is an integer n such that

$$n < \frac{T}{\tau} < n+1\,. \tag{9}$$

(iv) The function $\nabla F(t,u)$ satisfies (4).

Then, there is a $s_0 > 0$ such that, for every $s \ge s_0$, problem (1) has at least $2|n-m|+1$ solutions.

Assumption (iii) guarantees that there is at least one solution $w_0(t)$ to problem (6), cf. [4]. In assumption (ii) we ask that this solution does not touch the origin, so that there is a $r_0 > 0$ for which $0 \notin \mathcal{B}_{r_0}$, and that condition (7) holds in $\mathcal{B}_{r_0}$. Notice that, by the positive homogeneity of ∇H, this is equivalent to assuming the existence of a cone, containing the orbit of $w_0(t)$ in its interior, over which (7) holds. The interesting case is when this cone does not coincide with the whole plane.

In order to illustrate a consequence of the above result, let us consider the scalar periodic problem

$$\begin{cases} x'' + g(t,x) = s(1 + h(t)) \,, \\ x(0) = x(T) \,,\ x'(0) = x'(T) \,, \end{cases} \tag{10}$$

where $g : [0,T] \times \mathbb{R} \to \mathbb{R}$ satisfies the Carathéodory conditions with locally Lipschitz continuity in x, and $h \in L^1(0,T)$. In the sequel, we denote by $\| \cdot \|_p$ the usual norm in $L^p(0,T)$.

Corollary 1 *Assume that the limits*

$$\lim_{x\to-\infty} \frac{g(t,x)}{x} = \nu \,, \qquad \lim_{x\to+\infty} \frac{g(t,x)}{x} = \mu$$

exist, uniformly for almost every $t \in [0,T]$, and that there are two positive integers k, m such that

$$\left(\frac{2\pi(k-1)}{T}\right)^2 < \nu < \left(\frac{2\pi k}{T}\right)^2 \le \left(\frac{2\pi m}{T}\right)^2 < \mu < \left(\frac{2\pi(m+1)}{T}\right)^2 .$$

Let n be a positive integer such that

$$\frac{T}{n+1} < \frac{\pi}{\sqrt{\mu}} + \frac{\pi}{\sqrt{\nu}} < \frac{T}{n} \,.$$

Then, there are two positive constants h_0 and s_0 such that, if

$$\|h\|_1 \le h_0 \quad \text{and} \quad |s| \ge s_0 \,,$$

then problem (10) has at least $2(m-n)+1$ solutions for positive s, and at least $2(n-k)+1$ solutions for negative s.

Notice indeed that, under the assumptions of Corollary 1, we can write

$$g(t,x) = \mu x^+ - \nu x^- + f(t,x) \,,$$

with

$$\lim_{|x|\to\infty} \frac{f(t,x)}{x} = 0 \,, \qquad \text{uniformly for a.e. } t \in [0,T] \,.$$

The scalar equation can then be written as

$$\begin{cases} -y' = \mu x^+ - \nu x^- + f(t,x) - s(1+h(t)) \,, \\ x' = y \,, \end{cases}$$

which is of the form (1), with

$$H(x,y) = \tfrac{1}{2} \left[\mu (x^+)^2 + \nu (x^-)^2 + y^2\right] .$$

The assumptions (i), (iii), and (iv) of Theorem 1 are readily verified, with $\tau = \frac{\pi}{\sqrt{\mu}} + \frac{\pi}{\sqrt{\nu}}$. Concerning (ii), notice that, if $\|h\|_1$ is small enough, the T-periodic solution of

$$\begin{cases} x'' + \mu x = 1 + h(t)\,, \\ x(0) = x(T)\,,\; x'(0) = x'(T) \end{cases}$$

is positive. Hence, (7) holds on the half-plane $\{(x, y) : x > 0\}$, with $A_1 = A_2 = \mu I$, where I denotes the identity matrix. Theorem 1 then gives the conclusion when s is positive. The case when s is negative can be led back to the above by a change of variable in (10).

Corollary 1 generalizes the results by Lazer and McKenna [6] and Del Pino, Manasevich and Murua [2]. In those papers, the function g was assumed to be only dependent on x, continuously differentiable, with

$$\lim_{x\to-\infty} g'(x) = \nu\,, \qquad \lim_{x\to+\infty} g'(x) = \mu\,.$$

Later on, further generalizations were given in [1, 7, 8, 9], but the differentiability of g was always required. A further generalization of Corollary 1 for the scalar equation was recently obtained in [5].

As a direct consequence of Theorem 1, in the case when $v_0(t)$ is constant, we have the following.

Corollary 2 *Let the function H satisfy (2) and (3). Assume that $v_0(t) = v_0$ for every t and there is a vector $w_0 \neq 0$ at which H is twice continuously differentiable, with positive definite hessian matrix $H''(w_0)$, such that*

$$\nabla H(w_0) = -v_0\,.$$

Set

$$\sigma = \frac{2\pi}{\sqrt{\det H''(w_0)}}\,,$$

and let m be an integer such that

$$m < \frac{T}{\sigma} < m + 1\,.$$

Denoting by τ the period of the solutions to (5), let n be an integer such that

$$n < \frac{T}{\tau} < n + 1\,.$$

Let the function $\nabla F(t, u)$ satisfy (4). Then, there is a $s_0 > 0$ such that, for every $s \geq s_0$, problem (1) has at least $2|n - m| + 1$ solutions.

The above corollary generalizes [4, Theorem 6], where the simpler case $\nabla F(t, u) = e(t)$ was considered.

2 Proof of Theorem 1

In this section, we provide a proof for Theorem 1. Let us make in (1) the change of variables

$$\lambda = \frac{1}{s}\,, \qquad y = \lambda u - w_0\,. \tag{11}$$

Moreover, let $f : [0,T] \times \mathbb{R}^2 \times [0,+\infty[\to \mathbb{R}^2$ be the function defined by

$$f(t,y;\lambda) = \begin{cases} \lambda \nabla F\left(t, \frac{1}{\lambda}(y + w_0(t))\right) & \text{if } \lambda > 0\,, \\ 0 & \text{if } \lambda = 0\,. \end{cases}$$

We thus have that, for $\lambda \in \,]0,+\infty[$, problem (1) is equivalent to

$$\begin{cases} J\dot{y} = \nabla H(y + w_0(t)) - \nabla H(w_0(t)) + f(t,y;\lambda)\,, \\ y(0) = y(T)\,. \end{cases} \tag{12}$$

Let $B_\infty(0,r_0)$ denote the open ball in $L^\infty(0,T)$, centered in 0, with radius r_0 given by assumption (ii), and let $\overline{B}_\infty(0,r_0)$ be its closure. We would like to show that problem (12) has a solution y_λ in $\overline{B}_\infty(0,r_0)$, for $\lambda > 0$ small enough. Notice that, since $0 \notin \mathcal{B}_{r_0}$, by (4),

$$\lim_{\lambda \to 0} f(t,y;\lambda) = 0\,, \qquad \text{uniformly for } y \in \overline{B}(0,r_0) \text{ and a.e. } t \in [0,T]\,. \tag{13}$$

We then start analyzing the case when $\lambda = 0$.

Lemma 1 *The problem*

$$\begin{cases} J\dot{y} = \nabla H(y + w_0(t)) - \nabla H(w_0(t))\,, \\ y(0) = y(T) \end{cases} \tag{14}$$

has no nontrivial solutions y in $\overline{B}_\infty(0,r_0)$.

Proof Clearly, the constant 0 is a solution of (14). Assume by contradiction that there is a nonzero solution y such that $\|y(t)\| \le r_0$, for every $t \in [0,T]$. By the uniqueness of the solutions to Cauchy problems, it has to be $y(t) \neq 0$ for every $t \in [0,T]$. Passing to polar coordinates

$$y(t) = \rho(\cos(\theta(t)), \sin(\theta(t)))\,,$$

we have

$$-\theta' = \frac{\langle \nabla H(y + w_0(t)) - \nabla H(w_0(t)) \,|\, y \rangle}{\|y\|^2}\,.$$

Using (7), we see that

$$\frac{\langle A_1 y \,|\, y\rangle}{\|y\|^2} \le -\theta' \le \frac{\langle A_2 y \,|\, y\rangle}{\|y\|^2}\,.$$

Hence, the angular coordinate $\theta(t)$ of $y(t)$ can be compared with the angular coordinates $\theta_1(t)$ and $\theta_2(t)$ of the solutions $y_1(t)$ and $y_2(t)$ of the linear systems

$$J\dot{y}_1 = A_1 y_1 \,, \qquad J\dot{y}_2 = A_2 y_2 \,,$$

respectively, having the same initial conditions. Recalling assumption (8), these solutions rotate clockwise around the origin more than m times and less that $m+1$ times, as t varies from 0 to T. By the above, we have

$$\theta_2(t) \leq \theta(t) \leq \theta_1(t) \,,$$

for every $t \in [0, T]$. So, even $y(t)$ rotates clockwise around the origin more than m times and less that $m+1$ times, as t varies in $[0, T]$, and we get a contradiction with the fact that $y(t)$ is T-periodic. ■

Let us define the linear operator $L : D(L) \subset C([0, T]) \to L^1(0, T)$ by

$$D(L) = \{u \in W^{1,1}(0, T) : u(0) = u(T)\} \,, \qquad Lu = J\dot{u} \,.$$

The Nemytzkii operator $N_\lambda : C([0, T]) \to L^1(0, T)$ is defined by

$$(N_\lambda y)(t) = \nabla H(y(t) + w_0(t)) - \nabla H(w_0(t)) + f\,(t, y(t); \lambda) \;.$$

Let us fix a constant σ, not belonging to the spectrum of L, and define the operator $\Phi : C([0, T]) \times [0, 1] \;\to C([0, T])$ by

$$\Phi(y, \lambda) = (L - \sigma I)^{-1}(N_\lambda y - \sigma y) \,.$$

It is a completely continuous operator. Problem (12) with $\lambda \in [0, 1]$ is then equivalent to the fixed point problem

$$\Phi(y, \lambda) = y \,. \tag{15}$$

Notice that, by (13),

$$\lim_{\lambda \to 0} \Phi(y; \lambda) = \Phi(y; 0) \,, \qquad \text{uniformly for } y \in \overline{B}_\infty(0, r_0) \,. \tag{16}$$

As a consequence of Lemma 1, if $\lambda = 0$, there is no solution of (15) on the boundary of $B_\infty(0, r_0)$. We will now see that this is true also for sufficiently small λ.

Lemma 2 *There is a* $\lambda_0 > 0$ *such that*

$$\Phi(y, \lambda) \neq y \,, \qquad \text{for every } (y, \lambda) \in \partial B_\infty(0, r_0) \times [0, \lambda_0] \,.$$

Proof Assume by contradiction that there are a sequence $(\lambda_n)_n$ in $[0, 1]$ and a sequence $(y_n)_n$ in $\partial B_\infty(0, r_0)$ such that $\lambda_n \to 0$ and $\Phi(y_n, \lambda_n) = y_n$. By (16),

$$\lim_{n \to +\infty} \|\Phi(y_n, 0) - y_n\|_\infty = \lim_{n} \|\Phi(y_n, 0) - \Phi(y_n, \lambda_n)\|_\infty = 0 \,. \tag{17}$$

Since $(y_n)_n$ is bounded and $\Phi(\cdot\,,0)$ is completely continuous, there is a subsequence $(y_{n_k})_k$ and a $\bar{y} \in C([0,T])$ such that

$$\lim_{n\to+\infty} \Phi(y_{n_k},0) = \bar{y}\,.$$

It then follows from (17) that $y_{n_k} \to \bar{y}$ uniformly, so that

$$\bar{y} \in \partial B_\infty(0,r_0) \quad \text{ and } \quad \Phi(\bar{y},0) = \bar{y}\,,$$

in contradiction with Lemma 1. ■

Lemma 3 *The topological degree*

$$\deg(\Phi(\cdot,0) - I,\, B_\infty(0,r_0))$$

is different from 0.

<u>Proof</u> Consider, for $\xi \in [0,1]$, the problem

$$\begin{cases} J\dot{y} = \xi(\nabla H(y + w_0(t)) - \nabla H(w_0(t))) + (1-\xi)A_1 y\,, \\ y(0) = y(T)\,. \end{cases} \tag{18}$$

Using the argument in the proof of Lemma 1, it is possible to show that (18) has no nontrivial solutions in $\overline{B}_\infty(0,r_0)$. Hence, by homotopy invariance,

$$\deg(\Phi(\cdot,0) - \mathrm{I}\,,\, B_\infty(0,r_0)) = \deg((L-\sigma I)^{-1}(A_1 - \sigma I) - \mathrm{I}\,,\, B_\infty(0,r_0))\,.$$

This last degree is not zero, since the operator involved is linear and invertible. ■

By Lemma 2, we have that

$$\deg(\Phi(\cdot,\lambda) - \mathrm{I}\,,\, B_\infty(0,r_0)) = \deg(\Phi(\cdot,0) - \mathrm{I}\,,\, B_\infty(0,r_0))\,,$$

for every $\lambda \in [0,\lambda_0]$. By Lemma 3, this degree is different from 0. We conclude that, for every $\lambda \in [0,\lambda_0]$, there is a solution of (12) in $\overline{B}_\infty(0,r_0)$. We will denote by y_λ such a solution.

Lemma 4 *We have that*

$$\lim_{\lambda\to 0} \|y_\lambda\|_\infty = 0\,.$$

<u>Proof</u> By contradiction, assume that there is an $\epsilon > 0$, a sequence $(\lambda_n)_n$ in $[0,1]$ and a sequence $(t_n)_n$ in $[0,T]$ such that $\lambda_n \to 0$, and

$$\|y_{\lambda_n}(t_n)\| \geq \epsilon\,, \qquad \text{for every } n \in \mathbb{N}\,.$$

Since $y_{\lambda_n} \in \overline{B}_\infty(0,r_0)$, passing to subsequences we will have

$$\lim_{n\to+\infty} t_n = \bar{t}, \qquad \lim_{n\to+\infty} y_{\lambda_n}(t_n) = \bar{y}\,,$$

for some $\bar{t} \in [0,T]$ and $\bar{y} \in \overline{B}(0,r_0)$, with $\|\bar{y}\| \geq \epsilon$. Let $\bar{y}(t)$ be the unique solution to the Cauchy problem

$$\begin{cases} J\dot{y} = \nabla H(y + w_0(t)) - \nabla H(w_0(t))\,, \\ y(\bar{t}) = \bar{y}\,. \end{cases} \tag{19}$$

By (13) and the continuous dependence, $y_{\lambda_n}(t) \to \bar{y}(t)$, uniformly in $t \in [0,T]$, so that $\bar{y} \in \overline{B}_\infty(0,r_0)$ and $\bar{y}(0) = \bar{y}(T)$. Hence, $\bar{y}(t)$ is a nontrivial solution of (14) in $\overline{B}_\infty(0,r_0)$, in contradiction with Lemma 1. ■

We now make in (12) the change of variable

$$z = y - y_\lambda\,, \tag{20}$$

thus obtaining the equivalent problem

$$\begin{cases} J\dot{z} = \nabla H(z + y_\lambda(t) + w_0(t)) - \nabla H(y_\lambda(t) + w_0(t)) + \\ \qquad + f(t, z + y_\lambda(t); \lambda) - f(t, y_\lambda(t); \lambda)\,, \\ z(0) = z(T)\,. \end{cases} \tag{21}$$

Notice that the constant 0 is a solution of (21). In order to simplify the notation, let

$$\begin{aligned} g(t,z;\lambda) = \nabla H(z + y_\lambda(t) + w_0(t)) - \nabla H(y_\lambda(t) + w_0(t)) + \\ + f(t, z + y_\lambda(t); \lambda) - f(t, y_\lambda(t); \lambda)\,. \end{aligned}$$

With the aim of applying the Poincaré-Birkhoff Theorem, we need to consider the Cauchy problem

$$\begin{cases} J\dot{z} = g(t,z;\lambda)\,, \\ z(0) = z_0\,. \end{cases} \tag{22}$$

In the following, it will be convenient to extend by T-periodicity all the functions defined on $[0,T]$. Since $g(t,z;\lambda)$ has at most linear growth and is locally Lipschitz continuous in z, the solution to (22) is unique and globally defined. Hence, the Poincaré map is well defined.

By (13) and Lemma 4,

$$\begin{aligned} \lim_{\lambda \to 0} g(t,z;\lambda) = \nabla H(z + w_0(t)) - \nabla H(w_0(t))\,, \\ \text{uniformly for } z \in \overline{B}\left(0, \tfrac{1}{2} r_0\right) \text{ and a.e. } t \in [0,T]\,. \end{aligned} \tag{23}$$

Let us first study the limit case.

Lemma 5 *There are two positive constants $\tilde{r}_0$, and $\bar{r}$, with $2\tilde{r}_0 < \bar{r} < \frac{1}{2} r_0$, such that, if z_0 is verifies*

$$\|z_0\| = \bar{r}\,,$$

then the solution to the Cauchy problem

$$\left\{ \begin{array}{l} J\dot{z} = \nabla H(z + w_0(t)) - \nabla H(w_0(t))\,, \\ z(0) = z_0 \end{array} \right. \tag{24}$$

satisfies

$$2\tilde{r}_0 \le \|z(t)\| \le \tfrac{1}{2} r_0\,, \qquad \text{for every } t \in [0,T]\,.$$

<u>Proof</u> As already seen in Lemma 1, it is possible to use polar coordinates

$$z(t) = \rho(t)(\cos\theta(t), \sin\theta(t))\,,$$

leading us to the system

$$\left\{ \begin{array}{l} \rho' = -\Big\langle \nabla H(t, z + w_0(t)) - \nabla H(w_0(t)) \,\Big|\, (-\sin\theta, \cos\theta) \Big\rangle\,, \\ \\ \theta' = -\dfrac{1}{\rho} \Big\langle H(t, z + w_0(t)) - \nabla H(w_0(t)) \,\Big|\, (\cos\theta, \sin\theta) \Big\rangle\,. \end{array} \right. \tag{25}$$

Define

$$\bar{r} = \tfrac{1}{2} r_0\, e^{-LT}\,.$$

Consider the first equation in (25), and assume $\rho(0) = \|z_0\| = \bar{r}$. Then, using the fact that ∇H is Lipschitz continuous,

$$\rho'(t) \le \|\nabla H(t, z(t) + w_0(t)) - \nabla H(w_0(t))\| \le L\rho(t)\,,$$

so that

$$\rho(t) \le \rho(0) e^{Lt} \le \bar{r} e^{LT} = \tfrac{1}{2} r_0\,, \qquad \text{for every } t \in [0,T]\,.$$

Define now

$$\tilde{r}_0 = \tfrac{1}{2} \bar{r}\, e^{-LT},$$

and assume that $\|z_0\| = \bar{r}$. In order to prove that $\|z(t)\| \ge 2\tilde{r}_0$ for every $t \in [0,T]$, we consider a time-inversion in (24), by a change of variable. Set $\eta(\upsilon) = z(T - \upsilon)$, so that $\eta(T) = z_0$. Assume by contradiction that there is a $t_0 \in [0,T]$ such that $\|z(t_0)\| < 2\tilde{r}_0$. Set $\upsilon_0 = T - t_0$ and $\eta_0 = z(t_0)$. Arguing as in the first part of the proof, we can see that the solution of

$$\left\{ \begin{array}{l} J\dot{\eta}(\upsilon) = -\nabla H(\eta + w_0(T - \upsilon)) + \nabla H(w_0(T - \upsilon))\,, \\ \eta(\upsilon_0) = \eta_0\,, \end{array} \right.$$

verifies

$$\|\eta(\upsilon)\| \le \|\eta(\upsilon_0)\| e^{L(\upsilon - \upsilon_0)} < 2\tilde{r}_0 e^{LT} = \bar{r}\,, \qquad \text{for every } \upsilon \in [\upsilon_0, \upsilon_0 + T]\,.$$

We thus get a contradiction with the fact that $\|\eta(T)\| = \bar{r}$. ∎

Lemma 6 *Let $\bar{r} > 0$ be as in Lemma 5. Then, there is a $\lambda_1 \in]0, \lambda_0]$ such that every solution of (22) with $\|z_0\| = \bar{r}$ and $\lambda \in [0, \lambda_1]$ rotates clockwise around the origin more than m times and less that $m + 1$ times, as t varies from 0 to T.*

<u>Proof</u> By Lemma 5, the solutions of (24) with $\|z_0\| = \bar{r}$ belong to $\overline{B}_\infty(0, r_0)$. Hence, as already seen in the proof of Lemma 1, they rotate clockwise around the origin more than m times and less that $m + 1$ times, as t varies from 0 to T. By (23), the solutions of (22) with $\|z_0\| = \bar{r}$ remain close to those of (24). In particular, for λ small enough, any solution of (22) is such that

$$\tilde{r}_0 \leq \|z(t)\| \leq r_0 \,, \qquad \text{for every } t \in [0, T] \,,$$

and, being close to the solution of (24), it rotates clockwise around the origin more than m times and less that $m + 1$ times, as well, when t varies from 0 to T. Since $\partial B(0, \bar{r})$ is compact, there is a $\lambda_1 \in]0, \lambda_0]$ such that, if $\lambda \in [0, \lambda_1]$, all the solutions of (24) starting from $\partial B(0, \bar{r})$ must behave as above. ∎

We now need to estimate the number of rotations of the solutions of (22) when $\|z_0\|$ is large. In the following, the parameter $\lambda \in]0, \lambda_1]$ will be considered as fixed. Recalling the change of variables (11) and (20), we have set

$$z(t) = \lambda u(t) - w_0(t) - y_\lambda(t) \,.$$

Let $\varphi(t)$ be the solution of (5) such that

$$H(\varphi(t)) = \tfrac{1}{2} \,, \qquad \text{for every } t \in \mathbb{R} \,. \tag{26}$$

Recall that $\varphi(t)$ has minimal period τ. It rotates around the origin with a star-shaped orbit. Therefore, we can use some kind of generalized polar coordinates, setting

$$u(t) = \frac{1}{\delta} r(t) \varphi(t + \vartheta(t)) \,, \tag{27}$$

for some $\delta > 0$ to be fixed. More precisely, since we are dealing with the Cauchy problem (22), we set

$$z(t) = \frac{\lambda}{\delta} r(t) \varphi(t + \vartheta(t)) - w_0(t) - y_\lambda(t) \,. \tag{28}$$

Substitution in the differential equation leads to

$$\begin{aligned} r' J\varphi(t + \vartheta) +{}& r(1 + \theta') J\dot{\varphi}(t + \vartheta) = \\ &= \nabla H(r\varphi(t + \vartheta)) + \delta \, \nabla F\left(t, \frac{1}{\delta}(r\varphi(t + \vartheta))\right) + \frac{\delta}{\lambda} v_0(t) \,. \end{aligned}$$

Using (26) and the Euler Identity, we then get the system

$$\begin{cases} r' = -\delta \Big\langle \nabla F\Big(t, \dfrac{r\varphi(t+\vartheta)}{\delta}\Big) + \dfrac{1}{\lambda} v_0(t) \,\Big|\, \dot{\varphi}(t + \vartheta) \Big\rangle, \\[2ex] \vartheta' = \dfrac{\delta}{r} \Big\langle \nabla F\Big(t, \dfrac{r\varphi(t+\vartheta)}{\delta}\Big) + \dfrac{1}{\lambda} v_0(t) \,\Big|\, \varphi(t + \vartheta) \Big\rangle. \end{cases} \tag{29}$$

Lemma 7 *There is a $\bar{R}_\lambda > \bar{r}$ such that, if $\|z_0\| = \bar{R}_\lambda$, every solution of (22) with $\|z_0\| = \bar{R}_\lambda$ rotates clockwise around the origin more than n times and less that $n+1$ times, as t varies from 0 to T.*

Proof By (4), one has

$$\lim_{\delta \to 0} \delta \Big\langle \nabla F\Big(t, \frac{r\varphi(t+\vartheta)}{\delta}\Big) \,\Big|\, \dot{\varphi}(t+\vartheta) \Big\rangle = 0\,,$$

$$\lim_{\delta \to 0} \frac{\delta}{r} \Big\langle \nabla F\Big(t, \frac{r\varphi(t+\vartheta)}{\delta}\Big) \,\Big|\, \varphi(t+\vartheta) \Big\rangle = 0\,,$$

uniformly for $\vartheta \in \mathbb{R}$, r varying on compact subsets of $]0,+\infty[$, and for almost every $t \in [0,T]$. Denote by $(r(t;\vartheta_0,\delta), \vartheta(t;\vartheta_0,\delta))$ the solution of (29) satisfying

$$r(0) = 1\,, \qquad \vartheta(0) = \vartheta_0 \in [0,\tau]\,.$$

Then, $r'(\cdot\,;\vartheta_0,\delta) \to 0$ and $\vartheta'(\cdot\,;\vartheta_0,\delta) \to 0$ in $L^1(0,T)$, uniformly in ϑ_0, as $\delta \to 0$. Then,

$$\lim_{\delta \to 0} r(t;\vartheta_0,\delta) = 1\,, \qquad \lim_{\delta \to 0} \vartheta(t;\vartheta_0,\delta) = \vartheta_0\,,$$

uniformly in $(t,\vartheta_0) \in [0,T] \times [0,\tau]$.

By (9), the function $\varphi(\cdot + \vartheta_0)$ rotates clockwise around the origin more than n times and less than $n+1$ times, as t varies from 0 to T. Recalling (28), since y_λ and w_0 are bounded, we deduce that there is a $\bar{\delta} > 0$ such that, fixing $\delta \in]0,\bar{\delta}]$ and setting

$$\bar{R}_\lambda = \frac{\lambda}{\delta}\, \|\varphi\|_\infty + \|w_0\|_\infty + \|y_\lambda\|_\infty\,,$$

the solutions of (22) with $\|z_0\| \geq \bar{R}_\lambda$ must rotate clockwise around the origin more than n times and less than $n+1$ times, as well, as t varies from 0 to T.

■

We are now ready to apply the Poincaré-Birkhoff Theorem, in the version of [3]. We know that the Poincaré map is an area-preserving homeomorphism. We have seen in Lemma 6 and Lemma 7 that, if $\lambda \in]0,\lambda_1]$, there are two positive constants $\bar{r}$, $\bar{R}_\lambda$, with $\bar{r} < \bar{R}_\lambda$, having the following property: when t varies in $[0,T]$, the solutions of (22) with $\|z_0\| = \bar{r}$ rotate clockwise around the origin more than m times and less than $m+1$ times, and the solutions of (22) with $\|z_0\| = \bar{R}_\lambda$ rotate clockwise around the origin more than n times and less than $n+1$ times.

Taking the composition of the Poincaré map with a counter-clockwise rotation of angle $2\pi k$, with

$$k = \min\{m,n\} + 1\,,\ \min\{m,n\} + 2\,, \ldots,\ \min\{m,n\} + |m-n|\,,$$

we have a map satisfying all the hypotheses of the Poincaré-Birkhoff Theorem. We thus obtain $|m-n|$ pairs of T-periodic solutions for (21), which rotate clockwise, respectively, $k = \min\{m,n\}+1\,,\ \min\{m,n\}+2\,,\ldots,\ \min\{m,n\}+|m-n|$ times around the origin, in the period time T. Recalling the zero solution, we thus get $2|m-n|+1$ distinct solutions of (21). Those solutions generate, by the change of variables we have made, $2|m-n|+1$ distinct solutions of (1).

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