Multiple periodic solutions of scalar second order differential equations

Alessandro Fonda and Luca Ghirardelli

Abstract. We prove multiplicity of periodic solutions for a scalar second order differential equation with an asymmetric nonlinearity, thus generalizing previous results by Lazer and McKenna [5] and Del Pino, Manasevich and Murua [2]. The main improvement lies in the fact that we do not require any differentiability condition on the nonlinearity. The proof is based on the use of the Poincaré-Birkhoff Fixed Point Theorem.

1 Introduction

In 1987, Lazer and McKenna [5] provided a multiplicity result for the periodic problem

$$\begin{cases} x'' + g(x) = s(1 + h(t))\,, \\ x(0) = x(T)\,, \quad x'(0) = x'(T)\,. \end{cases} \tag{1}$$

They assumed $g : \mathbb{R} \to \mathbb{R}$ to be a C^1-function, $h : \mathbb{R} \to \mathbb{R}$ a "small" continuous and T-periodic function, and s a "large" real parameter. In 1992, their result was slightly generalized by Del Pino, Manasevich and Murua [2], who proved the following.

Theorem 1 *Assume that the limits*

$$\lim_{x\to-\infty} g'(x) = \nu\,, \qquad \lim_{x\to+\infty} g'(x) = \mu\,,$$

exist and that there are two positive integers k, m such that

$$\left(\frac{2\pi(k-1)}{T}\right)^2 < \nu < \left(\frac{2\pi k}{T}\right)^2 \le \left(\frac{2\pi m}{T}\right)^2 < \mu < \left(\frac{2\pi(m+1)}{T}\right)^2.$$

Let $n \ge 0$ be an integer such that

$$\frac{T}{n+1} < \frac{\pi}{\sqrt{\mu}} + \frac{\pi}{\sqrt{\nu}} < \frac{T}{n}\,. \tag{2}$$

There are two positive constants h_0 and s_0 such that, if

$$\|h\|_\infty \le h_0 \quad \textit{and} \quad |s| \ge s_0\,,$$

then problem (1) has at least $2(m-n)+1$ solutions for positive s, and at least $2(n-k)+1$ solutions for negative s.

(For convenience, if n is equal to zero, in this paper we agree that $\frac{T}{n}$ is $+\infty$, so that the last inequality in (2) is trivially satisfied.) The proof was carried out by the use of the Poincaré-Birkhoff Fixed Point Theorem, in its more general version due to W. Ding [3]. Later on, further generalizations of Theorem 1 were given in [1, 6, 7, 8].

In this paper, we consider the more general problem

$$\begin{cases} x'' + g(t,x) = sw(t)\,, \\ x(0) = x(T)\,, \quad x'(0) = x'(T)\,. \end{cases} \tag{3}$$

Here, $g : [0,T] \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, and $w : [0,T] \to \mathbb{R}$ is assumed to be integrable on its domain. We will focus on the case of a positive parameter s. Analogous results can be obtained for a negative s.

This problem has been already considered in [8], where Zanini and Zanolin assumed $g(t,x)$ to be differentiable in x, with continuous partial derivative $\frac{\partial g}{\partial x}(t,x)$. They assumed the existence of the limits

$$\lim_{x\to-\infty} \frac{g(t,x)}{x} = b(t)\,, \qquad \lim_{x\to+\infty} \frac{\partial g}{\partial x}(t,x) = a(t)\,,$$

Moreover, they asked that the only solution of

$$\begin{cases} x'' + a(t)x = w(t)\,, \\ x(0) = x(T)\,, \quad x'(0) = x'(T) \end{cases}$$

has to be strictly positive. Estimating the rotation numbers associated to the equations

$$x'' + a(t)x = 0\,, \qquad x'' + a(t)x^+ - b(t)x^- = 0\,,$$

they were able to obtain a generalization of Theorem 1. Again, the proof was based on the Poincaré-Birkhoff Theorem.

In the following, we will consider problem (3) without any differentiability assumption on the function g. However, in order to guarantee uniqueness for the associated Cauchy problems, we assume $g(t,x)$ to be locally Lipschitz continuous with respect to x. We will prove the following generalization of Theorem 1 with a positive parameter s.

Theorem 2 *Let the following hypotheses hold.*

(i) There are two positive numbers ν_1, ν_2 such that

$$\nu_1 \le \liminf_{x\to-\infty} \frac{g(t,x)}{x} \le \limsup_{x\to-\infty} \frac{g(t,x)}{x} \le \nu_2\,, \tag{4}$$

uniformly for almost every $t \in [0,T]$.

(ii) *There is a function* $a(t)$ *such that*

$$\lim_{x\to+\infty} \frac{g(t,x)}{x} = a(t), \tag{5}$$

uniformly for almost every $t \in [0,T]$.

(iii) *There are two positive numbers* μ_1, μ_2 *and an integer* $m \geq 0$ *such that, for almost every* $t \in [0,T]$,

$$\left(\frac{2\pi m}{T}\right)^2 < \mu_1 \leq a(t) \leq \mu_2 < \left(\frac{2\pi(m+1)}{T}\right)^2. \tag{6}$$

Moreover, the only solution of

$$\begin{cases} x'' + a(t)x = w(t)\,, \\ x(0) = x(T), \quad x'(0) = x'(T) \end{cases} \tag{7}$$

is strictly positive.

(iv) *There is an integer* $n \geq 0$ *such that*

$$\frac{T}{n+1} < \frac{\pi}{\sqrt{\mu_2}} + \frac{\pi}{\sqrt{\nu_2}} \leq \frac{\pi}{\sqrt{\mu_1}} + \frac{\pi}{\sqrt{\nu_1}} < \frac{T}{n}\,. \tag{8}$$

Then, there is a $s_0 \geq 0$ *such that, for every* $s \geq s_0$*, problem (3) has at least* $2|m-n|+1$ *solutions.*

The proof of Theorem 2 is given in Section 2. After a suitable change of variables, we apply the Poincaré-Birkhoff Theorem in the phase-plane, by estimating the number of rotations both of the solutions having a small amplitude and of those with a large amplitude. The difference from the proofs in [1, 2, 6, 7, 8] lies in the fact that we are able to avoid the use of the linearized equation, thus not needing any differentiability assumption on the function g.

Notice that assumption (*iii*) holds, e.g., if $a(t)$ is a constant satisfying (6), and $w(t)$ is nearly equal to 1, so that Theorem 2 is indeed a generalization of Theorem 1, for positive s. (The case of negative s can be obtained by a change of variable in (3).) However, if $a(t)$ is not constant, condition (6) is not sufficient to guarantee that the solution of (7) is positive, even if $w(t)$ is constantly equal to 1. This will be shown in Section 3 (see Remark 6), where possible extensions of Theorem 2 will be discussed, as well.

2 Proof of the main result

In this section, we provide a proof of Theorem 2. We will always assume $s \geq 1$. Let us first recall the regularity assumptions on the function $g(t,x)$, i.e., the Carathéodory conditions, with local Lipschitz continuity in x. Briefly,

- $g(\cdot, x)$ is integrable on $[0,T]$, for every $x \in \mathbb{R}$,
- for every $R > 0$ there is a $\ell_R \in L^1(0,T)$ such that, if $u, v \in [-R, R]$, then

$$|g(t,u) - g(t,v)| \le \ell_R(t)|u - v| \,, \qquad \text{for a.e. } t \in [0,T] \,.$$

Recall that, by (5) and (6), we have that $a \in L^\infty(0,T)$, while $w \in L^1(0,T)$. In the sequel, we will denote by $\|\cdot\|_p$ the usual norm in $L^p(0,T)$.

Lemma 1 *There are three positive constants ε_0, c_0 and C_0 such that, if $h \in L^1(0,T)$ and $\gamma \in L^\infty(0,T)$ satisfy*

$$\|h\|_1 \le \varepsilon_0 \,, \qquad \|\gamma - a\|_\infty \le \varepsilon_0 \,,$$

then the linear problem

$$\begin{cases} z'' + \gamma(t) z = w(t) + h(t) \,, \\ z(0) = z(T), \quad z'(0) = z'(T) \end{cases}$$

has a unique solution z, and $c_0 \le z(t) \le C_0$ for every $t \in [0,T]$.

Proof We will take ε_0 such that

$$0 < \varepsilon_0 < \min\Big\{\mu_1 - \Big(\frac{2\pi m}{T}\Big)^2, \, \Big(\frac{2\pi(m+1)}{T}\Big)^2 - \mu_2\Big\}. \tag{9}$$

Then, using (6), if $\|\gamma - a\|_\infty \le \varepsilon_0$, we have that

$$\Big(\frac{2\pi m}{T}\Big)^2 < \mu_1 - \varepsilon_0 \le \gamma(t) \le \mu_2 + \varepsilon_0 < \Big(\frac{2\pi(m+1)}{T}\Big)^2.$$

So, we can define the resolvent $\mathcal{R}_\gamma : L^1(0,T) \to C([0,T])$, which associates to every function $v \in L^1([0,T])$ the unique solution $z = \mathcal{R}_\gamma(v)$ of

$$\begin{cases} z'' + \gamma(t) z = v(t) \,, \\ z(0) = z(T), \quad z'(0) = z'(T) \,. \end{cases}$$

We know that $\mathcal{R}_\gamma$ is a linear and bounded operator, i.e.,

$$\mathcal{R}_\gamma \in \mathcal{L}(L^1(0,T), C([0,T])) \,,$$

and we denote by $\|\mathcal{R}_\gamma\|_{\mathcal{L}}$ its norm:

$$\|\mathcal{R}_\gamma\|_{\mathcal{L}} = \sup\{\|\mathcal{R}_\gamma(v)\|_\infty : \|v\|_1 = 1\} \,.$$

Since $\mathcal{R}_a(w) > 0$, there are two positive constants c_1 and C_1 such that

$$c_1 \le \mathcal{R}_a(w)(t) \le C_1 \,,$$

for every $t \in [0,T]$. If $\|h\|_1$ is small enough,

$$\|\mathcal{R}_a(h)\|_\infty \le \|\mathcal{R}_a\|_{\mathcal{L}} \, \|h\|_1 \le \tfrac{1}{4} c_1 \,,$$

so that

$$\tfrac{3}{4}c_1 \le \mathcal{R}_a(w+h)(t) \le C_1 + \tfrac{1}{4}c_1 \,, \tag{10}$$

for every $t \in [0,T]$. We will assume $\|h\|_1 \le 1$. Let $\varepsilon_1 > 0$ be such that $\varepsilon_1(\|w\|_1 + 1) \le \frac{1}{4}c_1$. Let

$$U = \{\gamma \in L^\infty([0,T]) : \|\gamma - a\|_\infty \le \varepsilon_0\} \,.$$

Since the function $\gamma \mapsto \mathcal{R}_\gamma$ is continuous from U, as a subset of $L^1([0,T])$, to $\mathcal{L}(L^1(0,T), C([0,T]))$, taking $\|\gamma - a\|_\infty$ small enough, we have

$$\|\mathcal{R}_\gamma - \mathcal{R}_a\|_{\mathcal{L}} \le \varepsilon_1 \,.$$

In particular,

$$\|\mathcal{R}_\gamma(w+h) - \mathcal{R}_a(w+h)\|_\infty \le \varepsilon_1\|w+h\|_1 \le \varepsilon_1(\|w\|_1 + 1) \le \tfrac{1}{4}c_1 \,. \tag{11}$$

Hence, if $\|h\|_1$ and $\|\gamma - a\|_\infty$ are small enough, by (10) and (11),

$$\tfrac{1}{2}c_1 \le \mathcal{R}_\gamma(w+h)(t) \le C_1 + \tfrac{1}{2}c_1 \,,$$

for every $t \in [0,T]$. Setting $c_0 = \frac{1}{2}c_1$ and $C_0 = C_1 + \frac{1}{2}c_1$, the lemma is thus proved. ∎

Having in mind (8), we will assume that the constant $\varepsilon_0 > 0$ provided by Lemma 1, besides satisfying (9), is so small that $\mu_1 - \varepsilon_0 > 0$, $\nu_1 - \varepsilon_0 > 0$, and

$$\frac{T}{n+1} < \frac{\pi}{\sqrt{\mu_2 + \varepsilon_0}} + \frac{\pi}{\sqrt{\nu_2 + \varepsilon_0}} \le \frac{\pi}{\sqrt{\mu_1 - \varepsilon_0}} + \frac{\pi}{\sqrt{\nu_1 - \varepsilon_0}} < \frac{T}{n} \,. \tag{12}$$

Lemma 2 *Let $\varepsilon_0 > 0$ be as above. We can write the function g as*

$$g(t,x) = \tilde{a}(t,x)x^+ - b(t,x)x^- + r(t,x) \,,$$

where $\tilde{a}$, b, $r : [0,T] \times \mathbb{R} \to \mathbb{R}$ are Carathéodory functions such that, for almost every $t \in [0,T]$ and every $x \in \mathbb{R}$,

$$a(t) - \varepsilon_0 \le \tilde{a}(t,x) \le a(t) + \varepsilon_0 \,, \tag{13}$$

$$\nu_1 - \varepsilon_0 \le b(t,x) \le \nu_2 + \varepsilon_0 \,, \tag{14}$$

and $r(t,x)$ is bounded: there is a $\tilde{r} \in L^1(0,T)$ such that, for almost every $t \in [0,T]$ and every $x \in \mathbb{R}$,

$$|r(t,x)| \le \tilde{r}(t) \,. \tag{15}$$

Proof Using (5), we can find $R_+ > 0$ such that, for almost every $t \in [0,T]$,

$$x \geq R_+ \quad \Rightarrow \quad a(t) - \varepsilon_0 \leq \frac{g(t,x)}{x} \leq a(t) + \varepsilon_0 \,.$$

We define

$$\tilde{a}(t,x) = \begin{cases} \dfrac{g(t,x)}{x} & \text{if } x > R_+ \,, \\[2ex] \dfrac{g(t,R_+)}{R_+} & \text{if } x \leq R_+ \,. \end{cases}$$

Similarly, using (4), let $R_- < 0$ be such that

$$x \leq R_- \quad \Rightarrow \quad \nu_1 - \varepsilon_0 \leq \frac{g(t,x)}{x} \leq \nu_2 + \varepsilon_0 \,.$$

We define

$$b(t,x) = \begin{cases} \dfrac{g(t,x)}{x} & \text{if } x < R_- \,, \\[2ex] \dfrac{g(t,R_-)}{R_-} & \text{if } x \geq R_- \,. \end{cases}$$

Finally, let

$$r(t,x) = g(t,x) - \tilde{a}(t,x)x^+ + b(t,x)x^- \,.$$

Since $r(t,x) = 0$ for $x \notin [R_-, R_+]$, the proof is easily completed. ■

We now introduce a change of variable. In (3), we set

$$z(t) = \frac{1}{s}\, x(t) \,.$$

We thus have that (3) is equivalent to the periodic problem

$$\begin{cases} z'' + \dfrac{g(t,sz)}{s} = w(t) \,, \\[2ex] z(0) = z(T) \,, \quad z'(0) = z'(T) \,. \end{cases} \tag{16}$$

Lemma 3 *There is a $\bar{s}_1 \geq 1$ such that, for every $s \geq \bar{s}_1$, problem (16) has a solution z_s which satisfies*

$$c_0 \leq z_s(t) \leq C_0 \,, \tag{17}$$

for every $t \in [0,T]$, where c_0, C_0 are the positive constants given by Lemma 1.

Proof Using Lemma 2, the differential equation in (16) can also be written as

$$z'' + \tilde{a}(t,sz)z^+ - b(t,sz)z^- = w(t) - \frac{r(t,sz)}{s} \,. \tag{18}$$

We look for a positive T-periodic solution of (18). If such a solution exists, it satisfies

$$z'' + \tilde{a}(t, sz)z = w(t) - \frac{r(t, sz)}{s}\,. \tag{19}$$

Viceversa, a positive solution of (19) verifies (18). By (6), (9) and (13),

$$\Big(\frac{2\pi m}{T}\Big)^2 < \mu_1 - \varepsilon_0 \le \tilde{a}(t, sz) \le \mu_2 + \varepsilon_0 < \Big(\frac{2\pi(m+1)}{T}\Big)^2,$$

for almost every $t \in [0, T]$, every $s \ge 1$, and every $z \in \mathbb{R}$. Hence, by a well-known nonresonance result which goes back to [4], there is a T-periodic solution $z_s(t)$ of (19), for any $s \ge 1$. We want to see that, for s large enough, such a solution $z_s(t)$ must be positive.

Notice that $z_s(t)$ solves the linear equation

$$z'' + \tilde{a}(t, sz_s(t))z = w(t) - \frac{r(t, sz_s(t))}{s}\,. \tag{20}$$

By (13) and (15), setting

$$\bar{s}_1 = \frac{1}{\varepsilon_0}\|\tilde{r}\|_1\,,$$

for every $s \ge \bar{s}_1$ we have

$$\|\tilde{a}(\cdot, sz_s(\cdot)) - a(\cdot)\|_\infty \le \varepsilon_0\,, \qquad \Big\|\frac{r(\cdot, sz_s(\cdot))}{s}\Big\|_1 \le \varepsilon_0\,.$$

By Lemma 1, for $s \ge \bar{s}_1$, equation (20) has a unique T-periodic solution, which therefore must coincide with z_s, and this solution satisfies (17). ■

We now perform another change of variables. In (16), we set

$$y(t) = z(t) - z_s(t)\,.$$

We thus obtain the problem

$$\begin{cases} y'' + \dfrac{g(t, s(y + z_s(t))) - g(t, sz_s(t))}{s} = 0\,, \\ y(0) = y(T)\,, \quad y'(0) = y'(T)\,. \end{cases} \tag{21}$$

Notice that the constant $y = 0$ is a solution to (21).

Lemma 4 *The following limit exists, uniformly, for almost every $t \in [0, T]$ and every $y \in \big[-\frac{1}{2}c_0, \frac{1}{2}c_0\big]$:*

$$\lim_{s \to +\infty} \frac{g(t, s(y + z_s(t))) - g(t, sz_s(t))}{s} = a(t)y\,.$$

Proof By (17), we have

$$\lim_{s\to+\infty} \frac{g(t, s(y+z_s(t)))}{s(y+z_s(t))} = a(t)\,,$$

and

$$\lim_{s\to+\infty} \frac{g(t, sz_s(t))}{sz_s(t)} = a(t)\,,$$

uniformly for almost every $t \in [0,T]$ and every $y \in \left[-\frac{1}{2}c_0, \frac{1}{2}c_0\right]$. Hence, given $\varepsilon > 0$ there is a $s_\varepsilon \geq \bar{s}_1$ such that, for every $s \geq s_\varepsilon$, almost every $t \in [0,T]$, and every $y \in \left[-\frac{1}{2}c_0, \frac{1}{2}c_0\right]$,

$$\left|\frac{g(t, s(y+z_s(t)))}{s(y+z_s(t))} - a(t)\right| < \frac{\varepsilon}{3C_0}\,,$$

and

$$\left|\frac{g(t, sz_s(t))}{sz_s(t)} - a(t)\right| < \frac{\varepsilon}{3C_0}\,,$$

so that

$$\begin{aligned}
&\left|\frac{g(t, s(y+z_s(t))) - g(t, sz_s(t))}{s} - a(t)y\right| = \\
&\quad = \left|\frac{g(t, s(y+z_s(t)))}{s} - a(t)(y+z_s(t)) + a(t)z_s(t) - \frac{g(t, sz_s(t))}{s}\right| \\
&\quad \leq \left|\frac{g(t, s(y+z_s(t)))}{s} - a(t)(y+z_s(t))\right| + \left|a(t)z_s(t) - \frac{g(t, sz_s(t))}{s}\right| \\
&\quad \leq \left|\frac{g(t, s(y+z_s(t)))}{s(y+z_s(t))} - a(t)\right||y+z_s(t)| + \left|\frac{g(t, sz_s(t))}{sz_s(t)} - a(t)\right||z_s(t)| \\
&\quad < \frac{\varepsilon}{3C_0}(|y+z_s(t)| + |z_s(t)|) < \varepsilon\,.
\end{aligned}$$

∎

In order to apply the Poincaré-Birkhoff Theorem, we need to consider the Cauchy problem

$$\begin{cases} y'' + \dfrac{g(t, s(y+z_s(t))) - g(t, sz_s(t))}{s} = 0\,, \\ y(0) = y_1\,, \\ y'(0) = y_2\,. \end{cases} \tag{22}$$

In the following, it will be convenient to extend by T-periodicity all the functions defined on $[0,T]$. Using Lemmas 2 and 3, the function

$$\tilde{g}_s(t,y) = \frac{g(t, s(y+z_s(t))) - g(t, sz_s(t))}{s}$$

can be written as

$$\tilde{g}_s(t,y) = \tilde{a}_s(t,y)y^+ - b_s(t,y)y^- + r_s(t,y)\,,$$

where

$$\tilde{a}_s(t,y) = \tilde{a}(t, s(y+z_s(t)))\,, \qquad b_s(t,y) = b(t, s(y+z_s(t)))\,,$$

so that, for almost every t and every y,

$$a(t) - \varepsilon_0 \le \tilde{a}_s(t,y) \le a(t) + \varepsilon_0\,, \tag{23}$$

$$\nu_1 - \varepsilon_0 \le b_s(t,y) \le \nu_2 + \varepsilon_0\,. \tag{24}$$

Moreover, since

$$0 \le (y+z_s(t))^+ - y^+ \le z_s(t) \le C_0\,,$$
$$-C_0 \le -z_s(t) \le (y+z_s(t))^- - y^- \le 0\,,$$

the function $r_s(t,y)$ is bounded by a L^1-function, independently of $s \ge 1$, i.e.,

$$|r_s(t,y)| \le \tilde{R}(t)\,, \tag{25}$$

for almost every t, with

$$\tilde{R}(t) = (2\mu_2 + \nu_2 + 3\varepsilon_0)C_0 + 2\tilde{r}(t)\,.$$

In particular, by (6), for $s \ge 1$ we have

$$|\tilde{g}_s(t,y)| \le \tilde{C}|y| + \tilde{R}(t)\,, \tag{26}$$

for almost every t and every y, with

$$\tilde{C} = \max\{\mu_2, \nu_2\} + \varepsilon_0\,.$$

Hence, $\tilde{g}_s(t,y)$ has at most linear growth in y and, being also locally Lipschitz continuous in y, the solution to (22) is unique and globally defined. Hence, in particular, the Poincaré map is well defined. Moreover, since the differential equation has the constant solution $y = 0$, then, by uniqueness, if $(y_1, y_2) \neq (0,0)$, the solution of (22) is such that

$$(y(t), y'(t)) \neq (0,0)\,,$$

for every $t \in \mathbb{R}$. It is then possible to use polar coordinates

$$(y(t), y'(t)) = \rho(t)(\cos\theta(t), \sin\theta(t))\,,$$

leading us to the system

$$\begin{cases} \rho' = \rho\cos\theta\sin\theta - \tilde{g}_s(t, \rho\cos\theta)\sin\theta\,, \\[2mm] \theta' = -\dfrac{1}{\rho}\,\tilde{g}_s(t,\rho\cos\theta)\cos\theta - \sin^2\theta\,. \end{cases} \tag{27}$$

Lemma 5 *There are three positive constants δ, r and $\bar{s}_2$, with $\delta < r < \frac{1}{2}c_0$ and $\bar{s}_2 \geq \bar{s}_1$, such that, for every $s \geq \bar{s}_2$, if (y_1, y_2) satisfies*

$$\sqrt{y_1^2 + y_2^2} = r\,,$$

then the solution to (22) satisfies

$$\delta \leq \sqrt{y(t)^2 + y'(t)^2} \leq \tfrac{1}{2}c_0\,,$$

for every $t \in [0, T]$.

Proof Define

$$r = \tfrac{1}{8}c_0\, e^{-(1+\|a\|_\infty)T}\,,$$

and set $\varepsilon = T^{-1}r$. Consider the first equation in (27), and assume $\rho(0) = \sqrt{y_1^2 + y_2^2} = r$. Notice that $r < \frac{1}{2}c_0$. We first prove that, for s large enough, $\rho(t) \leq \frac{1}{2}c_0$, for every $t \in [0, T]$. We have two possibilities: either, $\rho(t) < \frac{1}{2}c_0$ for every $t > 0$; or, there is a $t_s > 0$ such that $\rho(t) < \frac{1}{2}c_0$ for every $t \in [0, t_s[$, and $\rho(t_s) = \frac{1}{2}c_0$. We need to analize this second situation.

By Lemma 4, there is a $s_\varepsilon \geq \bar{s}_1$ such that, for every $s \geq s_\varepsilon$, almost every $t \in [0, T]$ and every $y \in \left[-\frac{1}{2}c_0, \frac{1}{2}c_0\right]$,

$$|\tilde{g}_s(t, y) - a(t)y| \leq \varepsilon. \tag{28}$$

Let us prove that, if $s \geq s_\varepsilon$, then $t_s > T$. Using (28), for almost every $t \in [0, t_s]$ we have

$$\rho'(t) \leq \rho(t) + a(t)\rho(t) + \varepsilon \leq (1 + \|a\|_\infty)\rho(t) + \varepsilon\,,$$

so that, integrating,

$$\rho(t) \leq \rho(0) + \varepsilon t + (1 + \|a\|_\infty) \int_0^t \rho(\tau)\, d\tau\,.$$

By Gronwall Inequality, we get

$$\rho(t) \leq (\rho(0) + \varepsilon t_s)\, e^{(1+\|a\|_\infty)t}\,,$$

for every $t \in [0, t_s]$. Assume by contradiction that $t_s \leq T$. Then,

$$\rho(t_s) \leq (r + \varepsilon T)\, e^{(1+\|a\|_\infty)T} = 2r\, e^{(1+\|a\|_\infty)T} = \tfrac{1}{4}c_0\,,$$

against the definition of t_s. We have thus proved that $\rho(t) < \frac{1}{2}c_0$, for every $t \in [0, T]$.

Define now

$$\delta = \tfrac{1}{4}r\, e^{-(1+\|a\|_\infty)T} = \tfrac{1}{32}c_0\, e^{-2(1+\|a\|_\infty)T}\,,$$

and assume that $\sqrt{y_1^2 + y_2^2} = r$. In order to prove that $\sqrt{y(t)^2 + y'(t)^2} \geq \delta$ for every $t \in [0, T]$, we consider a time-inversion in (22), by a change of variable. Set $\eta(\upsilon) = y(T-\upsilon)$, so that $\eta(T) = y_1$ and $\eta'(T) = y_2$. Assume by contradiction that there is a $\upsilon_0 \in [0,T]$ such that $\sqrt{\eta(\upsilon_0)^2 + \eta(\upsilon_0)^2} < \delta$. Set $\eta_1 = \eta(\upsilon_0)$, and $\eta_2 = \eta'(\upsilon_0)$. Arguing as in the first part of the proof, we can see that the solution of

$$\begin{cases} \eta''(\upsilon) + \tilde{g}_s(T-\upsilon, \eta(\upsilon)) = 0\,, \\ \eta(\upsilon_0) = \eta_1\,, \\ \eta'(\upsilon_0) = \eta_2\,, \end{cases} \tag{29}$$

with $s \geq s_\varepsilon$, verifies

$$\sqrt{\eta(\upsilon)^2 + \eta'(\upsilon)^2} \leq 2\delta\, e^{(1+\|a\|_\infty)T} = \tfrac{1}{2} r\,,$$

for every $\upsilon \in [\upsilon_0, \upsilon_0 + T]$. We thus get a contradiction with the fact that $\sqrt{\eta(T)^2 + \eta'(T)^2} = \sqrt{y_1^2 + y_2^2} = r$. ∎

Define the set

$$A := \Big\{ (\alpha, \beta) \in \mathbb{R}^2 : \delta \leq \sqrt{\alpha^2 + \beta^2} \leq \tfrac{1}{2} c_0 \Big\},$$

and consider, for every $s \geq \bar{s}_2$, the Carathéodory function $f_s : \mathbb{R} \times A \to \mathbb{R}$ defined by

$$f_s(t, \alpha, \beta) = \frac{-\tilde{g}_s(t,\alpha)\,\alpha - \beta^2}{\alpha^2 + \beta^2}\,.$$

Let $y_s(t)$ be a solution of (22) with $\sqrt{y_1^2 + y_2^2} = r$. By Lemma 4,

$$(y_s(t), y_s'(t)) \in A\,,$$

for every $t \in [0,T]$. Passing to polar coordinates

$$(y_s(t), y_s'(t)) = \rho_s(t)(\cos\theta_s(t), \sin\theta_s(t))\,,$$

we have that $\delta \leq \rho_s(t) \leq \frac{1}{2} c_0$, for every $t \in [0,T]$, and the angular function verifies

$$\theta_s' = -f_s(t, \rho_s \cos\theta_s, \rho_s \sin\theta_s)\,.$$

Since, by Lemma 4,

$$\lim_{s \to +\infty} f_s(t, \alpha, \beta) = \frac{-a(t)\alpha^2 - \beta^2}{\alpha^2 + \beta^2}\,,$$

uniformly for almost every $t \in \mathbb{R}$ and every $(\alpha, \beta) \in A$, we see that

$$\lim_{s \to +\infty} \theta_s(t) = \vartheta(t)\,, \quad \text{uniformly in } t \in [0,T]\,, \tag{30}$$

where $\vartheta(t)$ satisfies

$$\vartheta' = -a(t)\cos^2\vartheta - \sin^2\vartheta\,.$$

Using (6), we have that

$$\frac{-\vartheta'(t)}{\mu_2 \cos^2 \vartheta(t) + \sin^2 \vartheta(t)} \leq 1 \leq \frac{-\vartheta'(t)}{\mu_1 \cos^2 \vartheta(t) + \sin^2 \vartheta(t)} ,$$

for almost every $t \in \mathbb{R}$. We want to estimate the time needed for a solution to rotate around the origin. Let $t_0 < t_1$ be such that $\vartheta(t_1) = \vartheta(t_0) - 2\pi$. Integrating over $[t_0, t_1]$, since

$$\int_0^{2\pi} \frac{d\theta}{\mu_i \cos^2 \theta + \sin^2 \theta} = \frac{2\pi}{\sqrt{\mu_i}} ,$$

for $i = 1, 2$, we get

$$\frac{2\pi}{\sqrt{\mu_2}} \leq t_1 - t_0 \leq \frac{2\pi}{\sqrt{\mu_1}} .$$

Using (6) and (30), we conclude that there is a $s_0 \geq \bar{s}_2$ such that, for any $s \geq s_0$, the solution $y_s(t)$, when considered in the phase plane, must rotate clockwise around the origin more than m times and less than $m+1$ times, when t varies in $[0, T]$.

We will now provide an estimate for the solutions having a large amplitude.

Lemma 6 *For every $D > 0$ there is a $\xi_D > D$ such that, if $\sqrt{y_1^2 + y_2^2} \geq \xi_D$ and $s \geq 1$, then the solution of (22) satisfies $\sqrt{y(t)^2 + y'(t)^2} > D$, for every $t \in [0, T]$.*

Proof Consider, as in the proof of Lemma 5, the function $\eta(\upsilon) = y(T - \upsilon)$, which satisfies the differential equation

$$\eta''(\upsilon) + \tilde{g}_s(T - \upsilon, \eta(\upsilon)) = 0 .$$

Let $r(\upsilon) = \rho(T - \upsilon)$ be the corresponding radial component, in the phase plane. Recalling (26), choose ξ_D so that

$$\xi_D > (D + \|\tilde{R}\|_1)\, e^{(1+\tilde{C})T} .$$

We will show that, if there is a $t_0 \in [0, T]$ for which $\rho(t_0) = \sqrt{y(t_0)^2 + y'(t_0)^2} \leq D$, then $\rho(0) = \sqrt{y_1^2 + y_2^2} < \xi_D$.

Let $t_0 \in [0, T]$ be such that $\rho(t_0) \leq D$. Setting $\upsilon_0 = T - t_0$ we have that $r(\upsilon_0) \leq D$. Using (26), from the first equation in (27) we deduce that

$$|r'(\upsilon)| \leq (1 + \tilde{C}) r(\upsilon) + \tilde{R}(\upsilon) ,$$

for almost every $\upsilon \in \mathbb{R}$, so that, integrating,

$$r(\upsilon) \leq r(\upsilon_0) + \|\tilde{R}\|_1 + (1 + \tilde{C}) \int_{\upsilon_0}^{\upsilon} r(\xi)\, d\xi ,$$

for every $\upsilon \in [\upsilon_0, \upsilon_0 + T]$. Applying the Gronwall Inequality,

$$r(\upsilon) \leq (r(\upsilon_0) + \|\tilde{R}\|_1)\, e^{(1+\tilde{C})(\upsilon - \upsilon_0)} \leq (D + \|\tilde{R}\|_1)\, e^{(1+\tilde{C})T} < \xi_D ,$$

for every $\upsilon \in [\upsilon_0, \upsilon_0 + T]$. In particular, $\rho(0) = r(T) < \xi_D$, thus completing the proof. ∎

Passing to polar coordinates, we can rewrite the second equation in (27) as

$$\theta' = \frac{-(\tilde{a}_s(t,y)y^+ - b_s(t,y)y^- + r_s(t,y))y - y'^2}{y^2 + y'^2} \,. \tag{31}$$

Let us fix $\varepsilon > 0$. Taking $D = 1/\varepsilon$ in Lemma 6, we can find a constant $R = \xi_{1/\varepsilon}$ with the property that every solution of (22) with $\sqrt{y_1^2 + y_2^2} = R$ is such that $\sqrt{y(t)^2 + y'(t)^2} \geq 1/\varepsilon$, for every $t \in [0,T]$. Since

$$\sqrt{\alpha^2 + \beta^2} \geq \frac{1}{\varepsilon} \quad \Rightarrow \quad \left| \frac{\alpha}{\alpha^2 + \beta^2} \right| \leq \varepsilon \,,$$

for such a solution we have, by (31) and (25),

$$\left| \theta'(t) - \frac{-(\tilde{a}_s(t,y(t))y^+(t) - b_s(t,y(t))y^-(t))y(t) - y'(t)^2}{y(t)^2 + y'(t)^2} \right| \leq \varepsilon \tilde{R}(t) \,, \tag{32}$$

for almost every $t \in [0,T]$.

For the solutions of

$$\begin{cases} y'' + \tilde{a}_s(t,y)y^+ - b_s(t,y)y^- = 0 \,, \\ y(0) = y_1 \,, \\ y'(0) = y_2 \,, \end{cases} \tag{33}$$

the corresponding formula for the angular function is

$$\vartheta' = \frac{-(\tilde{a}_s(t,y)y^+ - b_s(t,y)y^-)y - y'^2}{y^2 + y'^2} \,.$$

We want to estimate the time needed for a solution of (33) to rotate around the origin, in the phase plane. By (6), (23), and (24), we have that

$$\frac{-\vartheta'(t)}{(\mu_2 + \varepsilon_0)\cos^2\vartheta(t) + \sin^2\vartheta(t)} \leq 1 \leq \frac{-\vartheta'(t)}{(\mu_1 - \varepsilon_0)\cos^2\vartheta(t) + \sin^2\vartheta(t)} \,,$$

for almost every t for which $y(t) \geq 0$, and

$$\frac{-\vartheta'(t)}{(\nu_2 + \varepsilon_0)\cos^2\vartheta(t) + \sin^2\vartheta(t)} \leq 1 \leq \frac{-\vartheta'(t)}{(\nu_1 - \varepsilon_0)\cos^2\vartheta(t) + \sin^2\vartheta(t)} \,,$$

for almost every t for which $y(t) \leq 0$. Let $t_0 < t_1 < t_2$ be such that $\vartheta(t_0) = \frac{\pi}{2}$, $\vartheta(t_1) = -\frac{\pi}{2}$, and $\vartheta(t_2) = -\frac{3\pi}{2}$. Integrating over $[t_0, t_1]$, since

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{d\theta}{(\mu_i \pm \varepsilon_0)\cos^2\theta + \sin^2\theta} = \frac{\pi}{\sqrt{\mu_i \pm \varepsilon_0}} \,,$$

for $i = 1, 2$, we have that

$$\frac{\pi}{\sqrt{\mu_2 + \varepsilon_0}} \leq t_1 - t_0 \leq \frac{\pi}{\sqrt{\mu_1 - \varepsilon_0}} \,.$$

Similarly, integrating over $[t_1, t_2]$, we have

$$\frac{\pi}{\sqrt{\nu_2 + \varepsilon_0}} \leq t_2 - t_1 \leq \frac{\pi}{\sqrt{\nu_1 - \varepsilon_0}}\,.$$

Using (12), we conclude that the solutions of (33) with $\sqrt{y_1^2 + y_2^2} = R$ rotate clockwise around the origin, in the phase plane, more than n times and less than $n+1$ times, when t varies in $[0, T]$. By (32), taking ε small enough, the same conclusion holds for the solutions of (22), as well, for every $s \geq 1$.

We are now ready to apply the Poincaré-Birkhoff Theorem, in the version of [3]. We know that the Poincaré map is an area-preserving homeomorphism. We have seen that there are two positive constants r, R, with $r < R$, with the following property: taking $s \geq s_0$, when t varies from 0 to T, the solutions of (22) with $\sqrt{y_1^2 + y_2^2} = r$ rotate clockwise around the origin, in the phase plane, more than m times and less than $m+1$ times, and the solutions of (22) with $\sqrt{y_1^2 + y_2^2} = R$ rotate clockwise around the origin, in the phase plane, more than n times and less than $n+1$ times.

Taking the composition of the Poincaré map with a counter-clockwise rotation of angle $2\pi k$, with

$$k = \min\{m, n\} + 1\,,\ \min\{m, n\} + 2\,, \ldots,\ \min\{m, n\} + |m - n|\,,$$

we have a map satisfying all the hypotheses of the Poincaré-Birkhoff Theorem. We thus obtain $|m-n|$ pairs of T-periodic solutions for (21), which rotate clockwise, in the phase plane, $k = \min\{m, n\} + 1\,,\ \min\{m, n\} + 2\,, \ldots,\ \min\{m, n\} + |m - n|$ times around the origin, respectively, in the period time T. Recalling the zero solution, we thus get $2|m - n| + 1$ distinct solutions. Those solutions generate, by the change of variables we have made, $2|m - n| + 1$ distinct solutions of (3).

3 Final remarks

In this section, we provide some remarks on Theorem 2 and its possible extensions.

Remark 1 It can be worth noticing that the annulus over which we apply the Poincaré-Birkhoff Theorem has radii r and R which do not depend on s, provided that $s \geq s_0$. This is a novelty with respect to the previously quoted papers.

Remark 2 Clearly enough, the roles of $+\infty$ and $-\infty$ can be exchanged, without affecting our results. This can be done by a simple change of variable in the main equation (3).

Remark 3 The assumptions of Theorem 2 can be weakened, along the lines of [8]. Instead of (6), we can simply ask that, when t varies in $[0,T]$, the solutions of

$$x'' + a(t)x = 0 \tag{34}$$

rotate clockwise around the origin, in the phase plane, more than m times and less than $m+1$ times. Also, (4) can be weakened to

$$\nu_1(t) \le \liminf_{x\to-\infty} \frac{g(t,x)}{x} \le \limsup_{x\to-\infty} \frac{g(t,x)}{x} \le \nu_2(t)\,,$$

and, instead of (8), we can ask that, for every function $b(t)$ satisfying $\nu_1(t) \le b(t) \le \nu_2(t)$, the solutions of

$$x'' + a(t)x^+ - b(t)x^- = 0 \tag{35}$$

rotate clockwise around the origin, in the phase plane, more than n times and less than $n+1$ times, as t varies from 0 to T. Here, we assume $\nu_1, \nu_2 \in L^\infty(0,T)$.

More precisely, we can distinguish two cases. In case $n < m$, we just need the solutions of (34) to rotate more than m times, and those of (35) to rotate less than $n+1$ times. In case $m < n$, the solutions of (34) must rotate less than $m+1$ times, and those of (35) more than n times.

Remark 4 The assumption that $g(t,x)$ has to be locally Lipschitz continuous in x can be avoided, at the expense of loosing quite a lot of the periodic solutions. Indeed, if $g(t,x)$ is not locally Lipschitz continuous in x, we can approximate it by a sequence $g_n(t,x)$ of functions which are smooth in x. For each of these, and for each $k = \min\{m,n\}+1\,,\ \min\{m,n\}+2\,,\dots,\ \min\{m,n\}+|m-n|$, we find a pair of T-periodic solutions, which rotate clockwise k times around the origin, in the period time T. However, passing to the limit, the two solutions corresponding to each k could converge to the same solution of (3). The conclusion of Theorem 2 would thus lead to only $|m-n|+1$ solutions, instead of the desired $2|m-n|+1$. We do not know how to overcome this difficulty.

Remark 5 It is possible to deal with the problem of the existence of subharmonic solutions, i.e., of periodic solutions having as minimal period a multiple of T. The same techniques used to prove Theorem 2 can be adapted to this situation, following the lines of [6].

Remark 6 To conclude, let us provide an example where the function $a(t)$ satisfies (6), but the T-periodic solution of

$$x'' + a(t)x = 1 \tag{36}$$

has no definite sign. For simplicity, let $T = 2\pi$ and define

$$a(t) = \begin{cases} \frac{5}{2} - \alpha & \text{if } t \in [0,\pi[\,, \\ \\ \frac{5}{2} + \alpha & \text{if } t \in [\pi, 2\pi[\,, \end{cases}$$

extended by 2π-periodicity.

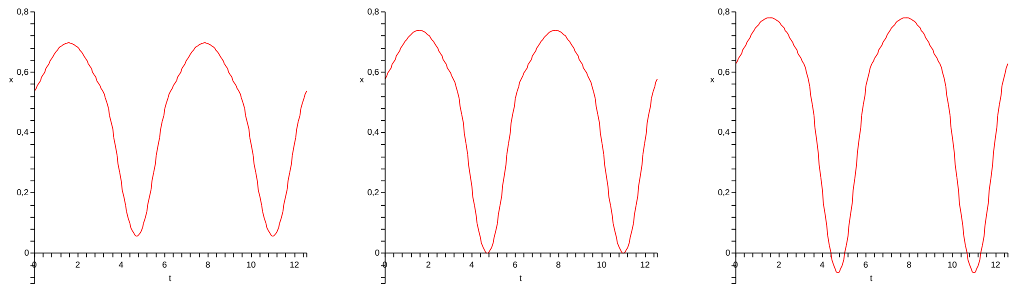

Figure 1: *The periodic solution of equation (36) when* $\alpha = 0.8$*,* 0.9 *and* 1*.*

Taking $\alpha \in [0, \frac{3}{2}[$, we have that (6) is satisfied, with $m = 1$. In this simple situation, the 2π-periodic solution of (36) can be explicitely computed. It can be seen that there is an $\alpha^* \in [0, \frac{3}{2}[$ such that, if $\alpha \in]\alpha^*, \frac{3}{2}[$, the periodic solution changes sign. Approximately, $\alpha^* = 0.9006$. In Figure 1, we have plotted the periodic solution of (36) for the values $\alpha = 0.8$, 0.9, and 1. Clearly enough, the function $a(t)$ can now be smoothed, still maintaining the same kind of behaviour for the periodic solution.

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Authors' addresses:

Alessandro Fonda
Dipartimento di Matematica e Informatica
Università di Trieste
P.le Europa 1
I-34127 Trieste
Italy
e-mail: a.fonda@units.it

Luca Ghirardelli
SISSA - International School for Advanced Studies
Via Beirut 2-4
I-34151 Trieste
Italy
e-mail: lucaghirardelli@hotmail.it

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