On a result of C.V. Coffman and W.K. Ziemer about the prescribed mean curvature equation

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Abstract

We produce a detailed proof of a result of C.V. Coffman and W.K. Ziemer [1] on the existence of positive solutions of the Dirichlet problem for the prescribed mean curvature equation

$$-\mathrm{div}\big(\nabla u/\sqrt{1+|\nabla u|^2}\big) = \lambda f(x,u) \ \text{ in } \Omega, \qquad u = 0 \ \text{ on } \partial\Omega,$$

assuming that f has a superlinear behaviour at $u = 0$.

2000 Mathematics Subject Classification: 35J25, 35J20, 35J60.
Keywords and Phrases: prescribed mean curvature equation, Dirichlet problem, positive solution, variational method, Nehari method.

1 Statements

In [1] C.V. Coffman and W.K. Ziemer proved the following result concerning the existence of positive solutions of the Dirichlet problem for the prescribed mean curvature equation

$$-\mathrm{div}\big(\nabla u/\sqrt{1+|\nabla u|^2}\big) = \lambda f(x,u) \ \text{ in } \Omega, \qquad u = 0 \ \text{ on } \partial\Omega. \tag{1}$$

A solution u of problem (1) is intended in the strong sense, namely a function $u \in W^{2,p}(\Omega)$, for some $p > N$, which satisfies the equation in (1) a.e. in Ω and the Dirichlet boundary condition pointwise on $\partial\Omega$.

Theorem 1.1. *Assume*

(h_0) *Ω is a bounded domain in $\mathbb{R}^N$ $(N \geq 2)$ with a $C^{1,1}$ boundary $\partial\Omega$;*

(h_1) *$f : \bar{\Omega} \times [0, +\infty[\to \mathbb{R}$ is continuous;*

(h_2) *there exist constants $r > 0$, $c > 1$ and $q > 2$, with $q < \frac{2N}{N-2}$ if $N \geq 3$, such that*

$$s^{q-1} \leq f(x,s) \leq cs^{q-1}$$

for every $x \in \bar{\Omega}$ and every $s \in [0,r]$;

(h_3) *there exist constants $r > 0$ and $\varepsilon > 0$ such that*

$$\frac{f(x,s)}{s^{1+\varepsilon}} < \frac{f(x,t)}{t^{1+\varepsilon}}$$

for every $x \in \bar{\Omega}$ and every $s, t \in]0, r]$, with $s < t$.

Then, for any given $p > N$, there exists $\lambda_* \in [0, +\infty[$ such that, for every $\lambda \in]\lambda_*, +\infty[$, problem (1) *has at least one non-trivial non-negative solution $u_\lambda \in H_0^1(\Omega) \cap W^{2,p}(\Omega)$ satisfying*

$$\lim_{\lambda \to +\infty} \|u_\lambda\|_{W^{2,p}(\Omega)} = 0.$$

Theorem 1.1 plays an important role in the existence theory of positive solutions for problem (1) (see e.g. [3] and the references therein). Since the rather delicate argument of the original proof was presented in [1] in a rather synthetic form and the involved ideas may be usefully exploited in other situations, we decided to produce here a detailed proof of such result. Actually we will prove the following slightly more general version of Theorem 1.1.

Theorem 1.2. *Assume*

(i_0) *Ω is a bounded domain in $\mathbb{R}^N$ $(N \geq 2)$ with a $C^{1,1}$ boundary $\partial\Omega$;*

(i_1) *$f : \Omega \times [0, +\infty[\to \mathbb{R}$ satisfies the Carathéodory conditions;*

(i_2) *there exist constants $r > 0$, $c > 1$ and $q > 2$, with $q < \frac{2N}{N-2}$ if $N \geq 3$, such that*

$$s^{q-1} \leq f(x,s) \leq cs^{q-1}$$

for a.e. $x \in \Omega$ and every $s \in [0, r]$;

(i_3) *there exist constants $r > 0$ and $\vartheta > 2$ such that*

$$\vartheta F(x,s) \leq sf(x,s)$$

for a.e. $x \in \Omega$ and every $s \in [0, r]$, where $F(x,s) = \int_0^s f(x,t)\,dt$;

(i_4) *there exists a constant $r > 0$ such that*

$$\frac{f(x,s)}{s} \leq \frac{f(x,t)}{t}$$

for a.e. $x \in \Omega$ and every $s, t \in]0, r]$, with $s < t$.

Then, for any given $p > N$, there exists $\lambda_* \in [0, +\infty[$ such that, for every $\lambda \in]\lambda_*, +\infty[$, problem (1) *has at least one solution $u_\lambda \in H_0^1(\Omega) \cap W^{2,p}(\Omega)$, satisfying $u_\lambda(x) > 0$ for every $x \in \Omega$ and $\frac{\partial u_\lambda}{\partial n}(x) < 0$ for every $x \in \partial\Omega$, n being the unit outer normal at $x \in \partial\Omega$,*

$$\int_\Omega (\sqrt{1 + |\nabla u_\lambda|^2} - 1)\,dx - \lambda \int_\Omega F(x, u_\lambda)\,dx > 0 \quad \text{and} \quad \lim_{\lambda \to +\infty} \|u_\lambda\|_{W^{2,p}(\Omega)} = 0.$$

Remark 1.1 Assumption (i_2) requires $f(\cdot, s)$ to be superlinear near $s = 0$. Assumption (i_3) is the classical Ambrosetti-Rabinowitz condition near $s = 0$. Observe that (i_2) and (i_3) yield $\vartheta \leq q$. Assumption (i_4) has a technical character and allows to implement the Nehari method. Note that assumption (h_3) implies (i_3), with $\vartheta = 1 + \varepsilon$, and (i_4).

2 Proof

The proof of Theorem 1.2 proceeds along several steps.
Step 1. A modified problem. Let $a : [0,+\infty[\to [\frac{1}{2},+\infty[$ be a strictly decreasing continuous function, such that $a(s) = (1+s)^{-1/2}$ if $s \in [0,1]$, and set $A(s) = \int_0^s a(t)\,dt$. Note that, for every $s \geq 0$,

$$\frac{1}{2} < a(s) \leq 1 \qquad \text{and} \qquad \frac{1}{2}s \leq A(s) \leq s. \tag{2}$$

We also set, for a.e. $x \in \Omega$ and every $s \in \mathbb{R}$,

$$\begin{aligned} g(x,s) &= 0 && \text{if } s < 0, \\ &= f(x,s) && \text{if } s \in [0,r[, \\ &= \frac{f(x,r)}{r^{q-1}}\, s^{q-1} && \text{if } s \geq r \end{aligned}$$

and

$$G(x,u) = \int_0^u g(x,s)\,ds,$$

where q is defined in (i_2). The function $g : \Omega \times \mathbb{R} \to [0,+\infty[$ satisfies the Carathéodory conditions and, by (i_2), there exists a constant $\kappa > 0$ such that

$$s^{q-1} \leq g(x,s) \leq \kappa s^{q-1} \tag{3}$$

for a.e. $x \in \Omega$ and every $s \in [0,+\infty[$. Further, by (i_3), we have

$$\vartheta G(x,s) \leq s g(x,s) \tag{4}$$

for a.e. $x \in \Omega$ and every $s \in [0,+\infty[$, with $\vartheta > 2$ given in (i_3). We further set, for a.e. $x \in \Omega$ and every $s \in [0,+\infty[$,

$$\begin{aligned} \gamma(x,s) &= 0 && \text{if } s \leq 0, \\ &= \frac{g(x,s)}{s} && \text{if } s > 0. \end{aligned}$$

Clearly γ satisfies the Carathéodory conditions and, by (i_4),

$$\gamma(x,s) \leq \gamma(x,t) \tag{5}$$

for a.e. $x \in \Omega$ and every $s,t \in \mathbb{R}$, with $s < t$. Let us consider the modified problem

$$-\mathrm{div}\big(a(|\nabla u|^2)\nabla u\big) = \lambda g(x,u) \ \text{ in } \Omega, \qquad u = 0 \ \text{ on } \partial\Omega. \tag{6}$$

We associate with this problem the functional $H_\lambda : H_0^1(\Omega) \to \mathbb{R}$ defined by

$$H_\lambda(u) = \frac{1}{2}\int_\Omega A(|\nabla u|^2)\,dx - \lambda \int_\Omega G(x,u)\,dx.$$

The critical points of H_λ are the weak solutions in $H_0^1(\Omega)$ of (6). Note that any such solution satisfies $u(x) \geq 0$ a.e. in Ω. Indeed, let u be a weak solution in $H_0^1(\Omega)$ of (6). Testing against $v = -u^-$ and using (2) and (3) we obtain

$$\begin{aligned} \frac{1}{2}\int_\Omega |\nabla u^-|^2\,dx &\leq \int_\Omega a(|\nabla u^-|^2)\,|\nabla u^-|^2\,dx \\ &= -\int_\Omega a(|\nabla u|^2)\,\nabla u \cdot \nabla u^-\,dx = -\lambda \int_\Omega g(x,u)u^-\,dx \leq 0. \end{aligned}$$

Therefore $u^- = 0$, i.e. $u(x) \geq 0$ a.e. in Ω.

Step 2. The Nehari manifold and its properties. For any given $\lambda > 0$ we set

$$\mathcal{N}_\lambda = \big\{u \in H^1_0(\Omega) \setminus \{0\} \mid \int_\Omega a(|\nabla u|^2)|\nabla u|^2\,dx = \lambda \int_\Omega g(x,u)u\,dx\big\}.$$

Notice that, if $u \in \mathcal{N}_\lambda$, then $u(x) > 0$ in a set of positive measure. Observe also that any non-trivial solution u of (6) belongs to $\mathcal{N}_\lambda$.
Claim. For every $w \in H^1_0(\Omega) \setminus \{0\}$, with $w(x) \geq 0$ a.e. in Ω, there is a unique $\alpha > 0$ such that $\alpha w \in \mathcal{N}_\lambda$. Take $w \in H^1_0(\Omega) \setminus \{0\}$, with $w(x) \geq 0$ a.e. in Ω, and define $h : [0, +\infty[\to \mathbb{R}$ by

$$h(t) = H_\lambda(tw) = \frac{1}{2}\int_\Omega A(t^2|\nabla w|^2)\,dx - \lambda \int_\Omega G(x,tw)\,dx.$$

Note that

$$h'(t) = \int_\Omega a(t^2|\nabla w|^2)t|\nabla w|^2\,dx - \lambda \int_\Omega g(x,tw)w\,dx$$

for all $t \geq 0$ and $h'(\alpha) = 0$ for some $\alpha > 0$ if and only if

$$\int_\Omega a(\alpha^2|\nabla w|^2)\alpha|\nabla w|^2\,dx = \lambda \int_\Omega g(x,\alpha w)w\,dx,$$

i.e. if and only if $\alpha w \in \mathcal{N}_\lambda$. Conditions (2) and (3) imply that

$$\frac{1}{4}t^2 \int_\Omega |\nabla w|^2\,dx - \frac{\lambda\kappa}{q}t^q \int_\Omega w^q\,dx \leq h(t) \leq \frac{1}{2}t^2 \int_\Omega |\nabla w|^2\,dx - \frac{\lambda}{q}t^q \int_\Omega w^q\,dx$$

for all $t \geq 0$. Therefore there exists $\delta > 0$ such that $h(t) > 0$ if $t \in]0, \delta[$ and $h(t) < 0$ if $t \in]\frac{1}{\delta}, +\infty[$. As $h(0) = 0$ we conclude that h has a maximum at some point $\alpha > 0$. Hence we have $h'(\alpha) = 0$ and $\alpha w \in \mathcal{N}_\lambda$. Let us observe that α is unique. Indeed, if $\alpha_1, \alpha_2 \in]0, +\infty[$ are critical points of h and we assume $\alpha_1 < \alpha_2$, then, as a is strictly decreasing and γ is non-decreasing, we have

$$\begin{aligned}
&\int_\Omega a(\alpha_1^2|\nabla w|^2)\alpha_1|\nabla w|^2\,dx = \lambda \textstyle\int_\Omega g(x,\alpha_1 w)w\,dx = \lambda \int_\Omega \gamma(x,\alpha_1 w)\alpha_1 w^2\,dx \\
\leq\ & \frac{\alpha_1}{\alpha_2}\lambda \int_\Omega \gamma(x,\alpha_2 w)\alpha_2 w^2\,dx = \frac{\alpha_1}{\alpha_2}\lambda \int_\Omega g(x,\alpha_2 w)w\,dx = \frac{\alpha_1}{\alpha_2}\int_\Omega a(\alpha_2^2|\nabla w|^2)\alpha_2|\nabla w|^2\,dx \\
<\ & \int_\Omega a(\alpha_1^2|\nabla w|^2)\alpha_1|\nabla w|^2\,dx,
\end{aligned}$$

which is a contradiction.

Step 3. Linear problems parametrized over the Nehari manifold. For any $u \in \mathcal{N}_\lambda$ we consider the linear problem

$$-\mathrm{div}\big(a(|\nabla u|^2)\nabla w\big) = \lambda g(x,u) \ \text{ in } \Omega, \qquad w = 0 \ \text{ on } \partial\Omega, \tag{7}$$

where, by (i_2), $g(\cdot,u) \in L^2(\Omega)$. By Lax-Milgram Theorem problem (7) has a unique solution $w \in H^1_0(\Omega)$. Moreover, since $g(x,u(x)) \geq 0$ for a.e. $x \in \Omega$ and $g(x,u(x)) > 0$ on a set of positive measure, the weak maximum principle implies that $w(x) \geq 0$ a.e. in Ω and $w(x) > 0$ on a set of positive measure.
Claim. Take $u \in \mathcal{N}_\lambda$ and let $w \in H^1_0(\Omega)$ be the solution of (7). Then there exists a unique $\alpha > 0$ such that $\alpha w \in \mathcal{N}_\lambda$. Moreover

$$H_\lambda(\alpha w) \leq H_\lambda(u) \tag{8}$$

and $H_\lambda(\alpha w) = H_\lambda(u)$ *if and only if* u *is a solution of* (6). According to the claim in Step 2 there exists a unique $\alpha > 0$ such that $\alpha w \in \mathcal{N}_\lambda$. Let us prove that $H_\lambda(\alpha w) \le H_\lambda(u)$. Since A is concave in $[0, +\infty[$, for every $s_1, s_2 \in [0, +\infty[$ we have

$$A(s_2) - A(s_1) \le a(s_1)(s_2 - s_1).$$

Since, by (5), the function $G(x, \sqrt{\cdot})$ is convex in $[0, +\infty[$, for all $x \in \bar{\Omega}$ and every $s_1, s_2 \in [0, +\infty[$ we have

$$G(x, \sqrt{s_2}) - G(x, \sqrt{s_1}) \ge \frac{1}{2}\gamma(x, \sqrt{s_1})(s_2 - s_1).$$

Setting $v = \alpha w$ we get

$$\begin{aligned} 2\big(H_\lambda(v) - H_\lambda(u)\big) &= \int_\Omega \big(A(|\nabla v|^2) - A(|\nabla u|^2)\big)\,dx - 2\lambda \int_\Omega \big(G(x, v) - G(x, u)\big)\,dx \\ &\le \int_\Omega a(|\nabla u|^2)\big(|\nabla v|^2 - |\nabla u|^2\big)\,dx - \lambda \int_\Omega \gamma(x, u)\big(v^2 - u^2\big)\,dx \\ &= \alpha\lambda \int_\Omega g(x, u)v\,dx - \lambda \int_\Omega \gamma(x, u)v^2\,dx \\ &= \lambda\big(\alpha \int_\Omega \gamma(x, u)uv\,dx - \int_\Omega \gamma(x, u)v^2\,dx\big). \end{aligned}$$

By Schwarz inequality we get

$$\begin{aligned} \alpha\lambda \int_\Omega \gamma(x, u)u^2\,dx &= \int_\Omega a(|\nabla u|^2)\nabla v \nabla u\,dx \\ &\le \big(\int_\Omega a(|\nabla u|^2)|\nabla v|^2\,dx\big)^{1/2}\big(\int_\Omega a(|\nabla u|^2)|\nabla u|^2\,dx\big)^{1/2} \\ &= \big(\alpha\lambda \int_\Omega \gamma(x, u)uv\,dx\big)^{1/2}\big(\lambda \int_\Omega \gamma(x, u)u^2\,dx\big)^{1/2}. \end{aligned} \tag{9}$$

Therefore we have

$$\big(\alpha \int_\Omega \gamma(x, u)u^2\,dx\big)^{1/2} \le \big(\int_\Omega \gamma(x, u)uv\,dx\big)^{1/2} \tag{10}$$

and hence, using Schwarz inequality again,

$$\begin{aligned} \big(\alpha \int_\Omega \gamma(x, u)u^2\,dx\big)^{1/2}\big(\int_\Omega \gamma(x, u)uv\,dx\big)^{1/2} &\le \int_\Omega \gamma(x, u)uv\,dx \\ &\le \big(\int_\Omega \gamma(x, u)u^2\,dx\big)^{1/2}\big(\int_\Omega \gamma(x, u)v^2\,dx\big)^{1/2}. \end{aligned}$$

This yields

$$\alpha \int_\Omega \gamma(x, u)uv\,dx \le \int_\Omega \gamma(x, u)v^2\,dx$$

and therefore

$$H_\lambda(v) - H_\lambda(u) \le \frac{1}{2}\lambda\big(\alpha \int_\Omega \gamma(x, u)uv\,dx - \int_\Omega \gamma(x, u)v^2\,dx\big) \le 0.$$

On the other hand, if $H_\lambda(v) = H_\lambda(u)$, then

$$\alpha \int_\Omega \gamma(x, u)uv\,dx = \int_\Omega \gamma(x, u)v^2\,dx$$

and

$$\alpha \int_\Omega \gamma(x,u)uv\,dx = \big(\alpha \int_\Omega \gamma(x,u)u^2\,dx\big)^{1/2}\big(\int_\Omega \gamma(x,u)uv\,dx\big)^{1/2}.$$

This implies that

$$\alpha \int_\Omega \gamma(x,u)u^2\,dx = \int_\Omega \gamma(x,u)uv\,dx$$

and, from (9),

$$\int_\Omega a(|\nabla u|^2)\nabla v\nabla u\,dx = \big(\int_\Omega a(|\nabla u|^2)|\nabla v|^2\,dx\big)^{1/2}\big(\int_\Omega a(|\nabla u|^2)|\nabla u|^2\,dx\big)^{1/2}.$$

Therefore u and v are proportional and hence there is $t>0$ such that $w=tu$. This means that u is a solution of

$$-\text{div}\,\big(a(|\nabla u|^2)\nabla u\big) = \frac{\lambda}{t}g(x,u) \ \text{ in } \Omega, \qquad u=0 \ \text{ on } \partial\Omega$$

and hence, as $u\in\mathcal{N}_\lambda$, $t=1$ and u is a solution of (6).

Step 4. Estimates on the functional H_λ on $\mathcal{N}_\lambda$. For all $u\in\mathcal{N}_\lambda$ we have

$$\frac{1}{2}\frac{\vartheta-2}{2\vartheta}\|u\|^2_{H^1_0(\Omega)} \le \frac{1}{2}\frac{\vartheta-2}{2\vartheta}\int_\Omega a(|\nabla u|^2)|\nabla u|^2\,dx \le H_\lambda(u) \le \frac{1}{2}\|u\|^2_{H^1_0(\Omega)}, \tag{11}$$

ϑ being defined in (i_3). Indeed we have

$$H_\lambda(u) = \frac{1}{2}\int_\Omega A(|\nabla u|^2)\,dx - \lambda\int_\Omega G(x,u)\,dx \le \frac{1}{2}\int_\Omega |\nabla u|^2\,dx$$

and, by (4) and (2),

$$\begin{aligned} H_\lambda(u) &\ge \frac{1}{2}\int_\Omega a(|\nabla u|^2)|\nabla u|^2\,dx - \frac{\lambda}{\vartheta}\int_\Omega g(x,u)\,dx \\ &= \big(\tfrac{1}{2}-\tfrac{1}{\vartheta}\big)\textstyle\int_\Omega a(|\nabla u|^2)|\nabla u|^2\,dx \ge \dfrac{1}{2}\dfrac{\vartheta-2}{2\vartheta}\|u\|^2_{H^1_0(\Omega)}. \end{aligned}$$

We also notice that H_λ is bounded away from zero on $\mathcal{N}_\lambda$. Indeed, for all $u\in\mathcal{N}_\lambda$, using (3), (2) and the embedding of $H^1_0(\Omega)$ into $L^q(\Omega)$, we get

$$\begin{aligned} \int_\Omega a(|\nabla u|^2)|\nabla u|^2\,dx &= \lambda\int_\Omega g(x,u)u\,dx \le \lambda\kappa\int_\Omega u^q\,dx \\ &\le \lambda\rho\big(\int_\Omega |\nabla u|^2\,dx\big)^{q/2} \le \lambda\rho 2^{q/2}\big(\int_\Omega a(|\nabla u|^2)|\nabla u|^2\,dx\big)^{q/2}, \end{aligned}$$

for some constant $\rho>0$, and hence

$$\int_\Omega a(|\nabla u|^2)|\nabla u|^2\,dx \ge \big(2^{q/2}\lambda\rho\big)^{-\frac{2}{q-2}}.$$

By (11) we conclude

$$H_\lambda(u) \ge \frac{1}{2}\frac{\vartheta-2}{2\vartheta}\big(2^{q/2}\rho\big)^{-\frac{2}{q-2}}\lambda^{-\frac{2}{q-2}}. \tag{12}$$

Step 5. Estimate on the coefficients α (defined in Step 2).

Claim. For all $u \in \mathcal{N}_\lambda$, if w is a solution of (7) and α is such that $\alpha w \in \mathcal{N}_\lambda$, then

$$\alpha \leq \sqrt{\frac{4\vartheta}{\vartheta - 2}}. \tag{13}$$

By inequality (10) we get, setting $v = \alpha w$,

$$\alpha\Big(\int_\Omega a(|\nabla u|^2)|\nabla u|^2\,dx\Big)^{1/2} \leq \Big(\int_\Omega a(|\nabla u|^2)|\nabla v|^2\,dx\Big)^{1/2}. \tag{14}$$

As, by (11) and (2),

$$H_\lambda(u) \leq \int_\Omega a(|\nabla u|^2)|\nabla u|^2\,dx,$$

we obtain, using (14), (2) and (11),

$$\begin{aligned}\alpha\big(H_\lambda(u)\big)^{\frac{1}{2}} &\leq \alpha\Big(\int_\Omega a(|\nabla u|^2)|\nabla u|^2\,dx\Big)^{\frac{1}{2}} \leq \Big(\int_\Omega a(|\nabla u|^2)|\nabla v|^2\,dx\Big)^{\frac{1}{2}} \\ &\leq \Big(\int_\Omega |\nabla v|^2\,dx\Big)^{\frac{1}{2}} \leq \Big(2\int_\Omega a(|\nabla v|^2)|\nabla v|^2\,dx\Big)^{\frac{1}{2}} \\ &\leq \Big(\frac{8\vartheta}{\vartheta-2}H_\lambda(u)\Big)^{\frac{1}{2}},\end{aligned}$$

Step 6. The map T_λ. In case $N = 2$, we set $\eta_0 = 1$. In case $N \geq 3$, we set

$$\eta_0 = \frac{2^*}{q-1} = \frac{2N}{N-2}\frac{1}{q-1}.$$

If $\eta_0 \geq \frac{N}{2}$, we set $l = 0$.
Claim. If $\eta_0 < \frac{N}{2}$, there exists an integer $l \geq 1$, depending only on N and q, and real numbers $\eta_1, \eta_2, \ldots, \eta_l$ defined by

$$\eta_i^* = \frac{N\eta_i}{N - 2\eta_i} \qquad \text{and} \qquad \eta_{i+1} = \frac{\eta_i^*}{q-1},$$

for all $i = 0, \ldots, l-1$, such that $\eta_0 < \eta_1 < \cdots < \eta_{l-1} < N/2 \leq \eta_l$. As $q < \frac{2N}{N-2}$ we obtain

$$\frac{N}{(N-2\eta_0)(q-1)} > 1,$$

so that we can pick $\varepsilon > 0$ such that

$$1 + \varepsilon < \frac{N}{(N-2\eta_0)(q-1)}.$$

Observe that

$$\eta_1 - \eta_0 = \eta_0\Big(\frac{N}{(N-2\eta_0)(q-1)} - 1\Big) > \varepsilon\eta_0.$$

By a recursive argument we obtain

$$\eta_{i+1} - \eta_i = \eta_i\Big(\frac{N}{(N-2\eta_i)(q-1)} - 1\Big) > \eta_0\Big(\frac{N}{(N-2\eta_0)(q-1)} - 1\Big) > \varepsilon\eta_0,$$

which proves the claim.

Fix $p > N$ and let $u \in W^{2,p}(\Omega) \cap \mathcal{N}_\lambda$. We define by induction a finite sequence $(u_n)_{0 \le n \le l+2}$, with $u_n \in W^{2,p}(\Omega) \cap \mathcal{N}_\lambda$, as follows: let w_{n+1} be the solution of the linear problem

$$\begin{cases} -\mathrm{div}\left(a(|\nabla u_n|^2)\nabla w\right) = \lambda g(x, u_n) & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega; \end{cases} \tag{15}$$

according to the claim in Step 2 there exists a unique α_{n+1} such that $\alpha_{n+1} w_{n+1} \in \mathcal{N}_\lambda$, hence we can define

$$u_{n+1} = \alpha_{n+1} w_{n+1} \in \mathcal{N}_\lambda. \tag{16}$$

As $u_n \in W^{2,p}(\Omega)$, we get $a(|\nabla u_n|^2) \in W^{1,p}(\Omega)$; then by the L^p-regularity theory (see [2, Ch. 9.5] and in particular the note at p. 241) we have $u_{n+1} \in W^{2,p}(\Omega)$. Moreover, by (8), we have

$$H_\lambda(u_{n+1}) \le H_\lambda(u_0), \tag{17}$$

for all n. Let us define the mapping $T_\lambda : \mathcal{N}_\lambda \cap W^{2,p}(\Omega) \to \mathcal{N}_\lambda \cap W^{2,p}(\Omega)$ by $T_\lambda(u) = u_{l+2}$.

Step 7. Norm estimates on T_λ. Fix an arbitrary $\kappa_0 > 0$ (a suitable value of κ_0 will be chosen in Step 8). For any $u_0 \in W^{2,p}(\Omega)$ let $u_1, \dots, u_{l+2} = T_\lambda(u)$ be defined as in Step 6. Assume

$$\|u_0\|_{W^{2,p}(\Omega)} \le 1 \quad \text{and} \quad H_\lambda(u_0) \le \kappa_0 \lambda^{-\frac{2}{q-2}}.$$

In the following argument the symbols $\kappa_1, \kappa_2, \dots$ will denote various constants independent of λ. Suppose $l > 1$. Then, by (11), we obtain

$$\|u_0\|_{H^1_0(\Omega)} \le \kappa_1 \lambda^{-\frac{1}{q-2}}.$$

By the embedding of $H^1_0(\Omega)$ into $L^{2^*}(\Omega)$ we get

$$\|u_0\|_{L^{2^*}(\Omega)} \le \kappa_2 \lambda^{-\frac{1}{q-2}}.$$

By (3) we obtain

$$\|\lambda g(\cdot, u_0)\|_{L^{\eta_0}(\Omega)} \le \lambda\kappa \|u_0^{q-1}\|_{L^{\eta_0}(\Omega)} = \lambda\kappa \|u_0\|^{q-1}_{L^{2^*}(\Omega)} \le \kappa_3 \lambda^{-\frac{1}{q-2}}.$$

As $u_0 \in W^{2,p}(\Omega)$ we have $a(|\nabla u_0|^2) \in W^{1,p}(\Omega)$; by the L^p-regularity theory [2, Ch. 9.5] we have $u_1 \in W^{2,p}(\Omega)$ and

$$\|u_1\|_{W^{2,\eta_0}(\Omega)} \le \kappa_4 \lambda^{-\frac{1}{q-2}}.$$

We can assume λ to have been chosen so large that

$$\kappa_4 \lambda^{-\frac{1}{q-2}} \le 1.$$

By the embedding of $W^{2,\eta_0}(\Omega)$ into $L^{\eta_0^*}(\Omega)$ we get

$$\|u_1\|_{L^{\eta_0^*}(\Omega)} \le \kappa_5 \lambda^{-\frac{1}{q-2}}.$$

Applying recursively the same argument to $u_1, u_2, \dots, u_l$, we finally obtain

$$\|\lambda g(\cdot, u_{l-1})\|_{L^{\eta_{l-1}}(\Omega)} \le \lambda\kappa \|u_{l-1}^{q-1}\|_{L^{\eta_{l-1}}(\Omega)} = \lambda\kappa \|u_{l-1}\|^{q-1}_{L^{\eta^*_{l-1}}(\Omega)} \le \kappa_6 \lambda^{-\frac{1}{q-2}},$$

$$\|u_l\|_{W^{2,\eta_{l-1}}(\Omega)} \le \kappa_7 \lambda^{-\frac{1}{q-2}}, \qquad \|u_l\|_{L^{\eta^*_{l-1}}(\Omega)} \le \kappa_8 \lambda^{-\frac{1}{q-2}},$$

and

$$\|\lambda g(\cdot, u_l)\|_{L^{\eta_{l-1}}(\Omega)} \le \kappa_9 \lambda^{-\frac{1}{q-2}}, \qquad \|u_{l+1}\|_{W^{2,\eta_l}(\Omega)} \le \kappa_{10} \lambda^{-\frac{1}{q-2}}.$$

By the embedding of $W^{2,\eta_l}(\Omega)$ into $L^{p(q-1)}(\Omega)$ we also obtain

$$\|u_{l+1}\|_{L^{p(q-1)}(\Omega)} \le \kappa_{11} \lambda^{-\frac{1}{q-2}},$$

$$\|\lambda g(u_{l+1})\|_{L^p(\Omega)} \le \lambda\kappa \|u_{l+1}^{q-1}\|_{L^p(\Omega)} = \lambda\kappa \|u_{l+1}\|^{q-1}_{L^{p(q-1)}(\Omega)} \le \kappa_{12} \lambda^{-\frac{1}{q-2}}$$

and finally

$$\|T_\lambda(u)\|_{W^{2,p}(\Omega)} = \|u_{l+2}\|_{W^{2,p}(\Omega)} \le \kappa_{13} \lambda^{-\frac{1}{q-2}}. \tag{18}$$

We can assume λ to have been chosen so large that $\|u_{n+1}\|_{W^{2,\eta_n}(\Omega)} \le 1$, for each $n = 1, \dots, l$, and $\|u_{l+2}\|_{W^{2,p}(\Omega)} \le 1$.

Step 8. The set S_λ. Fix any $\hat{w} \in C_c^\infty(\bar{\Omega}) \setminus \{0\}$, with $\hat{w}(x) \ge 0$ in Ω. According to the claim in Step 2 there exists $\hat{\alpha}$ such that $\hat{\alpha}\hat{w} \in \mathcal{N}_\lambda$. We have, by (2) and (3),

$$\hat{\alpha}^2 \int_\Omega |\nabla \hat{w}|^2 \, dx \ge \int_\Omega a(|\nabla \hat{\alpha}\hat{w}|^2)|\nabla \hat{\alpha}\hat{w}|^2 \, dx = \lambda \int_\Omega g(x, \hat{\alpha}\hat{w})\hat{\alpha}\hat{w} \, dx \ge \lambda \hat{\alpha}^q \int_\Omega |\hat{w}|^q \, dx,$$

so that

$$\hat{\alpha} \le \lambda^{-\frac{1}{q-2}} \|\hat{w}\|^{\frac{2}{q-2}}_{H_0^1(\Omega)} \; \|\hat{w}\|^{-\frac{q}{q-2}}_{L^q(\Omega)}.$$

At the beginning of Step 7 we fixed an arbitrary constant $\kappa_0 > 0$. Now we choose

$$\kappa_0 = \frac{1}{2} \|\hat{w}\|^{\frac{2q}{q-2}}_{H_0^1(\Omega)} \; \|\hat{w}\|^{-\frac{2q}{q-2}}_{L^q(\Omega)}. \tag{19}$$

We also set

$$m_1 = \max\left\{\kappa_{13}, \, \|\hat{w}\|_{W^{2,p}(\Omega)} \|\hat{w}\|^{\frac{2}{q-2}}_{H_0^1(\Omega)} \; \|\hat{w}\|^{-\frac{q}{q-2}}_{L^q(\Omega)}\right\} \tag{20}$$

with κ_{13} defined in Step 7. Set $\hat{u} = \hat{\alpha}\hat{w}$, then

$$\|\hat{u}\|_{W^{2,p}(\Omega)} \le m_1 \lambda^{-\frac{1}{q-2}} \tag{21}$$

and, by (11),

$$H_\lambda(\hat{u}) \le \kappa_0 \lambda^{-\frac{2}{q-2}}. \tag{22}$$

We define the set

$$S_\lambda = \left\{u \in \mathcal{N}_\lambda \cap W^{2,p}(\Omega) \mid \|u\|_{W^{2,p}(\Omega)} \le m_1 \lambda^{-\frac{1}{q-2}}, \, H_\lambda(u) \le \kappa_0 \lambda^{-\frac{2}{q-2}}\right\}.$$

Notice that, due to (21) and (22), S_λ is not empty. Moreover, by choosing $\lambda \ge m_1^{q-2}$, we guarantee, for all $u \in S_\lambda$,

$$\|u\|_{W^{2,p}(\Omega)} \le 1 \tag{23}$$

and, by (17),

$$H_\lambda(T_\lambda(u)) \le \kappa_0 \lambda^{-\frac{2}{q-2}}. \tag{24}$$

Notice finally that, by (24), (23), (18), (19) and (20), T_λ maps the set S_λ into itself.

Step 9. Existence of a positive solution of (6) *for all large* λ. Pick $\lambda \geq m_1^{q-2}$ so that (23) holds. We will show that H_λ has minimum in S_λ. By (12) H_λ is bounded from below in S_λ. Let $(u_k)_k$ be a minimizing sequence in S_λ. As $(u_k)_k$ is bounded in $W^{2,p}(\Omega)$ and $W^{2,p}(\Omega)$ is reflexive, there exists a subsequence of $(u_k)_k$, we still denote by $(u_k)_k$, weakly convergent in $W^{2,p}(\Omega)$ and, hence strongly convergent in $C^1(\bar{\Omega})$, to a function $\bar{u} \in W^{2,p}(\Omega)$. As $\mathcal{N}_\lambda$ is closed in $C^1(\bar{\Omega})$, H_λ is continuous with respect to the $C^1(\bar{\Omega})$ convergence and

$$\|\bar{u}\|_{W^{2,p}(\Omega)} \leq \liminf_{n\to+\infty} \|u_n\|_{W^{2,p}(\Omega)},$$

we have $\bar{u} \in S_\lambda$ and $H_\lambda(\bar{u}) = \min_{S_\lambda} H_\lambda(u)$. Recalling the recursive definitions given in Step 6 of $\bar{u}_0 = \bar{u}, \bar{u}_1, \ldots, \bar{u}_{l+2} = T_\lambda(\bar{u})$, we obtain, by (17),

$$H_\lambda(\bar{u}) \leq H_\lambda(T_\lambda(\bar{u})) = H_\lambda(\bar{u}_{l+2}) \leq \cdots \leq H_\lambda(\bar{u}_1) \leq H_\lambda(\bar{u})$$

and hence, in particular,

$$H_\lambda(\bar{u}) = H_\lambda(\bar{u}_1).$$

By the claim in Step 3 we conclude that $\bar{u}$ is a solution of (6). As a limit of non-negative functions, $\bar{u}$ is also non-negative. Further, by (12), $\bar{u}$ is non-trivial.

Step 10. Existence of a positive solution $u_\lambda \in W^{2,p}(\Omega)$ *of* (1) *for all large* λ. Set

$$\lambda_* = \left(\frac{m_1}{\min\{1, r\}}\right)^{q-2},$$

where m_1 is defined in Step 8, and fix $\lambda \in]\lambda_*, +\infty[$. Let $u_\lambda \in W^{2,p}(\Omega)$ be the solution of (6) whose existence is proved in Step 9. Since $\|\nabla u_\lambda\|_\infty \leq 1$ and $\|u_\lambda\|_\infty \leq r$, u_λ is a non-trivial non-negative solution of (1) too. The strong maximum principle and the Hopf boundary lemma imply that $u(x) > 0$ in Ω and $\frac{\partial u}{\partial n}(x) < 0$ on $\partial\Omega$. Finally, estimates (11) and (21) yield

$$\int_\Omega (\sqrt{1 + |\nabla u_\lambda|^2} - 1)\, dx - \lambda \int_\Omega F(x, u_\lambda)\, dx > 0$$

and

$$\lim_{\lambda\to+\infty} \|u_\lambda\|_{W^{2,p}(\Omega)} = 0.$$

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