PERIOD TWO IMPLIES ANY PERIOD FOR A CLASS OF DIFFERENTIAL INCLUSIONS

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Abstract. We produce a detailed proof of a result stated in [4, Remark 3] concerning scalar time-periodic first order differential inclusions. Such a result shows that the existence of just one subharmonic implies the existence of large sets of subharmonics of all given orders.

Let us consider the differential inclusion

$$x' \in F(t,x). \tag{1}$$

We suppose that $F : \mathbb{R} \times \mathbb{R} \to 2^{\mathbb{R}}$ is a set-valued map having non-empty compact convex values, which is periodic in t with period 1 and satisfies the following conditions:

- (i) $F(\cdot, x)$ is measurable for every $x \in \mathbb{R}$,
- (ii) $F(t, \cdot)$ is upper semicontinuous for a.e.$t \in [0,1]$,
- (iii) for each $\rho > 0$ there exists $\gamma \in L^1(0,1)$ such that $|F(t,x)| \leq \gamma(t)$ for a.e. $t \in [0,1]$ and every $x \in [-\rho, \rho]$.

Solutions of (1) are locally absolutely continuous functions satisfying (1) almost everywhere. Given $n \in \mathbb{N}$, with $n > 1$, a subharmonic solution of (1) of order n is a periodic solution of (1) of minimum period n.

Theorem. *Assume there exists a subharmonic solution of* (1) *of order* $n > 1$. *Then, for any* $k > 1$, *there exists a subharmonic solution of* (1) *of order* k. *In addition the set* $\mathcal{X}_k$ *of all subharmonic solutions of* (1) *of order* k *has dimension at least* k *as a subset of* $L^\infty(\mathbb{R})$.

As pointed out in [4, Remark 3] this result significantly improves [1, Theorem 5]

Date: March 6, 2006.

1991 *Mathematics Subject Classification.* 34A60.

Key words and phrases. First order scalar differential inclusion, periodic solution, subharmonic solution.

The first named author acknowledges the support of G.N.A.M.P.A., in the setting of the project "Soluzioni periodiche di equazioni differenziali ordinarie".

The second named author acknowledges the support of M.I.U.R, in the setting of the P.R.I.N. project "Equazioni differenziali ordinarie e applicazioni".

Proof. Let x be a subharmonic solution of (1) of order $n > 1$. Let $t_0 \in \mathbb{R}$ be such that $x(t_0) < x(t_0+1)$. Let $j \in \{1, \dots, n-1\}$ be such that

$$x(t_0) < x(t_0+1) \leq \cdots \leq x(t_0+j) \quad \text{and} \quad x(t_0+j) > x(t_0+j+1).$$

Let also $\ell \in \{0, \dots, j-1\}$ be such that

$$x(t_0+\ell) < x(t_0+\ell+1) = x(t_0+j).$$

Then we have $\max\{x(t_0+\ell), x(t_0+j+1)\} < x(t_0+j)$. Set

$$I = \,]\max\{x(t_0+\ell), x(t_0+j+1)\}, x(t_0+j)[.$$

Define $\alpha, \beta : [t_0, t_0+1] \to \mathbb{R}$ by

$$\alpha(t) = x(t+\ell), \qquad \beta(t) = x(t+j).$$

Then α and β are solutions of (1) such that

$$\alpha(t_0) < \beta(t_0), \qquad \beta(t_0+1) < \alpha(t_0+1).$$

Set

$$s_1 = \sup\{s \in \,]t_0, t_0+1[\; \mid \alpha(t) < \beta(t) \text{ on } [t_0, s[\},$$
$$s_2 = \inf\{s \in \,]t_0, t_0+1[\; \mid \beta(t) < \alpha(t) \text{ on }]s, t_0+1]\}.$$

Pick any $p \in I$. By Theorem 6 in [2, Chapter 2.7] there exists a solution v of (1) with $v(t_0) = p$, which can be continued to the right up to a point $r_1 \leq s_1$ where either $v(r_1) = \alpha(r_1)$ or $v(r_1) = \beta(r_1)$. In both cases we can extend v onto $[t_0, s_2]$ so that

$$\min\{\alpha(t), \beta(t)\} \leq v(t) \leq \max\{\alpha(t), \beta(t)\}$$

and $v(s_2) = \alpha(s_2) = \beta(s_2)$.

Similarly there exists a solution w of (1) with $w(t_0+1) = p$, which can be continued to the left up to a point $r_2 \geq s_2$ where either $w(r_2) = \alpha(r_2)$ or $w(r_2) = \beta(r_2)$. In both cases we can extend w onto $[s_2, t_0+1]$ so that

$$\min\{\alpha(t), \beta(t)\} \leq w(t) \leq \max\{\alpha(t), \beta(t)\}$$

and $w(s_2) = \alpha(s_2) = \beta(s_2)$. Set $u_p(t) = v(t)$ on $[t_0, s_2]$ and $u_p(t) = w(t)$ on $[s_2, t_0+1]$. Then u_p gives raise to a $1-$periodic solution of (1) satisfying $u_p(t_0) = p$.

We have just proved that for each $p \in I$ there exists a $1-$periodic solution u_p of (1) such that $u_p(t_0) = p$ and $u_p(s_2) = \alpha(s_2) = \beta(s_2)$. By the lattice structure of the set of solutions of (1), we can easily find an increasing sequence $(u_m)_m$ of $1-$periodic solutions of (1) such that $u_m(t_0) < u_{m+1}(t_0)$ and $u_m(s_2) = u_0(s_2)$ for every m.

A subharmonic solution v of (1) of order k can be constructed as follows: we define a $k-$periodic solution v of (1) by setting $v(t) = u_m(t)$ on $[s_2+m+ik, s_2+m+1+ik[$ for every $m \in \{0, \dots, k-1\}$ and $i \in \mathbb{Z}$. Let us prove that v has minimum period k. Since v is continuous and non-constant, v has a minimum period $T > 0$. Suppose, by contradiction, that $T < k$. Notice that, by definition of v, $T \neq 1$ and, as k is a multiple of T, $T \leq k/2$. This implies in particular that $T < k - 1$. If $T > 1$ (and hence $k > 2$), we get

$$\max v = \max v|_{[s_2, s_2+T]} \leq \max_{m=0,\dots,k-2} \max u_m = \max u_{k-2} < \max u_{k-1} = \max v.$$

Whereas if $T < 1$, we get

$$\max v = \max u_0|_{[s_2, s_2+T]} \leq \max u_0 < \max u_{k-1} = \max v.$$

In both cases a contradiction is achieved. Hence we conclude that $T = k$.

Finally, to prove that the dimension of $\mathcal{X}_k$ is at least k, we show that $[0, 1]^k$ is embedded into $\mathcal{X}_k$. Let $\mathcal{K}_m$ be the set of all solutions v of (1) on $[s_2, s_2 + 1]$ such that $u_m \leq v \leq u_{m+1}$. By Theorem 6 in [2, Chapter 2.7], $\mathcal{K}_m$ is a continuum in $C^0([s_2, s_2 + 1])$. Let $\mathcal{T}_m$ be a totally ordered subset of $\mathcal{K}_m$. By [3, Lemma 3.6] $\mathcal{T}_m$ is homeomorphic to a compact interval of $\mathbb{R}$. Extend all functions $v \in \mathcal{T}_m$ bt $1-$periodicity onto $\mathbb{R}$, so that each v is a $1-$periodic solutions of (1). Define

$$\Phi_k : \prod_{m=0}^{k-1} \mathcal{T}_m \to L^\infty(\mathbb{R})$$

by setting

$$\Phi_k(v_0, \dots, v_{k-1})(t) = v_m(t) \quad \text{ on } [s + m + ik, s + m + 1 + ik[$$

for every $m \in \{0, \dots, k-1\}$ and $i \in \mathbb{Z}$. Clearly Φ_k is one-to-one and continuous and hence it is a homeomorphism between $\prod_{m=0}^{k-1} \mathcal{T}_m$ and $\Phi\big(\prod_{m=0}^{k-1} \mathcal{T}_m\big) \subseteq \mathcal{X}_k$. $\square$

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