On threefolds covered by lines

by Emilia MEZZETTI and Dario PORTELLI*

Abstract

A classification theorem is given of projective threefolds that are covered by the lines of a two–dimensional family, but not by a higher dimensional family. Precisely, if X is such a threefold, let Σ denote the Fano scheme of lines on X and μ the number of lines contained in X and passing through a general point of X. Assume that Σ is generically reduced. Then $\mu \leq 6$. Moreover, X is birationally a scroll over a surface ($\mu = 1$), or X is a quadric bundle, or X belongs to a finite list of threefolds of degree at most 6. The smooth varieties of the third type are precisely the Fano threefolds with $-K_X = 2H_X$.

Introduction

Projective varieties containing "many" linear spaces appear naturally in several occasions. For instance, consider the following examples which, by the way, motivated our interest in this topic.

The first example concerns varieties of 4-secant lines of smooth threefolds in $\mathbf{P}^5$. The family of such lines has in general dimension four and the lines fill up the whole ambient space, but it can happen that they form a hypersurface.

A second example comes from the following recent theorem of Arrondo (see [1]), in some sense the analogous of the Severi theorem about the Veronese surface:

let Y be a subvariety of dimension n of the Grassmannian $\mathbf{G}(1, 2n+1)$ of lines of $\mathbf{P}^{2n+1}$ and assume that Y can be isomorphically projected into $\mathbf{G}(1, n+1)$. Then, if the lines parametrized by Y fill up a variety of dimension $n+1$, Y is isomorphic to the second Veronese image of $\mathbf{P}^n$.

If those lines generate a variety of lower dimension, nothing is known.

In both cases it would be very interesting to have a classification of such varieties. Moreover, these examples show that for such a classification it would be desirable to avoid any assumption concerning singularities.

The first general results about the classification of projective varieties containing a higher dimensional family of linear spaces were obtained by Beniamino Segre ([18]). In particular, in the case of lines, he proved:

*This work has been done in the framework of the activities of EAGER. Both authors have been supported by funds of MURST, Progetto Geometria Algebrica, Algebra Commutativa e Aspetti Computazionali. The paper has been widely improved while the second author was guest at the University of Oslo. The financial support from both the University of Oslo and the Norwegian Research Council is gratefully aknowledged.

Let $X \subset \mathbf{P}^N$ be an irreducible variety of dimension k, let $\Sigma \subset \mathbf{G}(1,N)$ be an irreducible component of maximal dimension of the variety of lines contained in X, such that the lines of Σ cover X. Then $\dim \Sigma \leq 2k-2$. *If equality holds, then $X = \mathbf{P}^k$. Moreover, if $k \geq 2$ and* $\dim \Sigma = 2k-3$, *then X is either a quadric or a scroll in $\mathbf{P}^{k-1}$'s over a curve.*

The case of a family Σ of dimension $2k-4$ is treated in some papers by Togliatti ([23]), Bompiani ([3]), M. Baldassarri ([2]), but their arguments are not easy to be followed. Recently, varieties of dimension $k \geq 3$ with a family of lines of dimension $2k-4$ have been classified by Lanteri–Palleschi ([14]), as particular case of a more general classification theorem. Their starting point is a pair (X, L) where L is an ample divisor on X, which is assumed to be smooth or, more in general, normal and $\mathbf{Q}$-Gorenstein. The assumptions on the singularities of X are removed by Rogora in his thesis ([17]), but he assumes $k \geq 4$ and codim $X > 2$.

The aim of this paper is the classification of the varieties of dimension k covered by the lines of a family of dimension $2k-4$, in the first non–trivial case: $k = 3$, i.e. threefolds covered by a family of lines of dimension 2. So, we classify threefolds covered by "few" lines.

A first remark is that among these varieties there are threefolds which are birationally scrolls over a surface or ruled by smooth quadrics over a curve. The first ones come from general surfaces contained in $\mathbf{G}(1,4)$, while the second ones come from general curves contained in the Hilbert scheme of quadric surfaces in $\mathbf{P}^n$. Note that these "quadric bundles" are built by varieties of lower dimension having a higher dimensional family of lines.

So we have focused our attention on threefolds not of these two types.

Observe that, if X is a threefold covered by the lines of a family of dimension two, then there is a fixed finite number μ of lines passing through any general point of X. In particular, having excluded scrolls, we have assumed $\mu > 1$.

It is interesting to remark that the surfaces Σ in $\mathbf{G}(1,4)$ corresponding to threefolds with $\mu > 1$, can be characterized by the property that the tangent space to $\mathbf{G}(1,4)$ at every point r of Σ intersects (improperly) Σ along a curve. This follows from the fact that the points of $\mathbf{G}(1,4) \cap T_r\mathbf{G}(1,4)$ represent the lines meeting r.

Our point of view, that we have borrowed from the quoted paper of Mario Baldassarri, is the following. Since we do not care about singularities, we are free to projected birationally into $\mathbf{P}^4$ our threefolds to hypersurfaces of the same degree and with the same μ. Hence, it is enough to classify hypersurfaces in $\mathbf{P}^4$ having a family of lines with the requested properties.

If $X \subset \mathbf{P}^4$ is a hypersurface of degree n, then the equation of X is a global section $G \in \Gamma(\mathbf{P}^4, \mathcal{O}_{\mathbf{P}^4}(n))$. The section G induces in a canonical way a global section $s \in \Gamma(\mathbf{G}(1,4), S^nQ)$, where Q is the universal quotient bundle on $\mathbf{G}(1,4)$. It is a standard fact that the points of the scheme of the zeros of the section s of Q correspond exactly to the lines on X. In this paper we will denote by Σ the Fano scheme of the lines on X, which is, by definition, the scheme of the zeros of the section s.

In this paper we will study threefolds X in $\mathbf{P}^4$ covered by lines such that Σ has dimension two.

The following theorem is the main result of the paper.

Theorem 0.1 *Let $X \subset \mathbf{P}^4$ be a projective, integral hypersurface over an algebraically closed field K, of characteristic zero, covered by lines. Let Σ denote the Fano scheme of the lines* *on X just introduced. Assume that Σ is generically reduced, that $\mu > 1$ and that X is not birationally ruled by quadrics over a curve. Then one of the following happens:*

1. *X is a cubic hypersurface with singular locus of dimension at most one; if X is smooth, then Σ is irreducible and $\mu = 6$;*
2. *X is a projection of a complete intersection of two hyperquadrics in $\mathbf{P}^5$; in general, Σ is irreducible and $\mu = 4$;*
3. $\deg X = 5$*: X is a projection of a section of $\mathbf{G}(1,4)$ with a $\mathbf{P}^6$, Σ is irreducible and $\mu = 3$;*
4. $\deg X = 6$*: X is a projection of a hyperplane section of $\mathbf{P}^2 \times \mathbf{P}^2$, Σ has two irreducible components and $\mu = 2$;*
5. $\deg X \leq 6$*: X is a projection of $\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$, Σ has at least three irreducible components and $\mu \geq 3$.*

Note that these five cases are precisely the projections of Fano varieties with $-K_X = L \otimes L$, L ample ([14]). This list is the same as in the article of Baldassarri.

It is interesting to remark that the bound $\mu = 6$ is attained only by cubic threefolds.

The assumption that Σ is generically reduced is necessary to make our method work. Note that this is a genericity assumption for X (however our threefolds are not general, if the degree is > 3; in fact, none of them is linearly normal in $\mathbf{P}^4$, so they have a big singular locus). This assumption is quite strong, because it implies in particular that the dual variety of X is a hypersurface and that a general line on X is never contained in a fixed tangent plane.

The paper is organized as follows.

In § 1 we prove that, under suitable conditions, on a general line of Σ there are $n - 3$ singular points of X, where n is the degree of X, and we derive from this many consequences we shall need in the paper. In particular, we will show that, if $n \geq 5$, then the singular locus of X is a surface and give an explicit lower bound for its degree (Theorem 1.11). Our main technical tool will be the family of planes containing a line of Σ. We prove that the assumption Σ generically reduced implies that there is no fixed tangent plane to X along a general line on X. From this it follows readily that the dual variety of X is a threefold (Theorem 1.6). In this section we also introduce the ruled surfaces $\sigma(r)$, generated by the lines on X meeting a fixed line r.

§ 2 contains the proof of the bound $\mu \leq 6$. Moreover, if $n > 3$ we prove that $\mu \leq 4$.

§ 3 is devoted to the classification of threefolds with an irreducible family of lines with $\mu > 1$. First of all, we check that, if $deg(X) > 3$ and X is not a quadric bundle, only two possibilities are allowed for μ, i.e. $\mu = 3, 4$. The threefolds with these invariants are then classified, respectively in Propositions 3.2 and 3.3.

§ 4 contains the classification of threefolds with a reducible 2-dimensional family of lines, such that all components of Σ have $\mu_i = 1$.

It is a pleasure to thank E. Arrondo, J.M. Landsberg, R. Piene and K. Ranestad for several useful conversations about the content of the paper, as well as for constant encouragement.

In the paper we will use the following:

Notations, general assumptions and conventions

1. We will always work over an algebraically closed field K of characteristic zero.
2. $X \subset \mathbf{P}^4$ will be a projective, integral hypersurface, of degree n, covered by lines.
3. We will denote by Σ the Fano scheme of the lines on X. (In particular, by the result of B. Segre quoted above, from $dim(\Sigma) = 2$ it follows $n \geq 3$.)
4. Let μ be the number of lines of Σ passing through a general point of X. If Σ is reducible, with $\Sigma_1, \ldots, \Sigma_s$ as irreducible components of dimension 2, then we will denote by μ_i the number of lines of Σ_i passing through a general point of X. Clearly $\mu = \mu_1 + \ldots + \mu_s$. We assume $\mu > 1$.
5. We will assume that X is not birationally ruled by quadric surfaces over a curve.
6. For a "general line in Σ" we mean any line which belongs to a subset $S \subset \Sigma$ (never given explicitly), such that S is Zariski dense in Σ. So "general line in Σ" is meaningful also in the case of a reducible Σ.
7. We will denote by the same letter both a line in $\mathbf{P}^4$ and the corresponding point of $\mathbf{G}(1,4)$. We hope that it will be always clear from the context which point of view is adopted.
8. For r general in Σ, the assumption $\mu > 1$ ensures that the union of all the lines of Σ meeting r is a surface $\sigma(r)$, which can also be seen as a curve inside $\mathbf{G}(1,4)$. As r varies in Σ, these curves describe an algebraic family in Σ of dimension ≤ 2. If Σ is reducible, with $\Sigma_1, \ldots, \Sigma_s$ as irreducible components of dimension 2, then the surfaces $\sigma(r)$ are unions $\sigma_1(r) \cup \ldots \cup \sigma_s(r)$, where $\sigma_i(r)$ is formed by the lines of Σ_i intersecting r.

1 Preliminary results

We consider the degree n of X. For $n = 3$, it is well known that all cubic hypersurfaces of $\mathbf{P}^4$ contain a family of lines of dimension at least 2, and of dimension exactly 2 if the singular locus of X has codimension at least 2. For $n \geq 4$, a *general* hypersurface of $\mathbf{P}^4$ of degree n is not covered by lines.

The following theorem is the main technical result of the paper. Here the assumption that the irreducible components of dimension two of Σ are reduced is essential.

Theorem 1.1 *If r is a general line of an irreducible component Σ_1 of Σ which is of dimension two and generically reduced, then $r \cap Sing(X)$ is a 0-dimensional scheme of lenght $n-3$ (we will express this briefly by saying that "on r there are exactly $n-3$ singular points of X"). In particular, if $n \geq 4$, then X is singular.*

Proof Let r be a general line of Σ_1 (in particular, Σ_1 is the only component of Σ containing r), and let π be a plane containing r. Then $\pi \cap X$ splits as a union $r \cup C$ where C is a plane curve of degree $n-1$. So $r \cap C$ has length $n-1$ and it is formed by points that are singular for $\pi \cap X$, hence either tangency points of π to X or singular points of X. We will prove that, if π is general among the planes containing r, then exactly $n-3$ of these points are singular for X. To this end, let us consider the family (possibily reducible) of planes $\mathcal{F} = \{\pi | \ \pi \supset r, r \in \Sigma\}$; its dimension is 4.

Claim. *The general plane through r cannot be tangent to X in more that two points.*

Proof of the Claim We have to prove that X does not possess a 4-dimensional family of k-tangent planes, with $k > 2$. Assume by contradiction that X possesses such a family $\mathcal{G}$. Let O be a general point of $\mathbf{P}^4$, $O \notin X$. The projection $p_O : X \to \mathbf{P}^3$, centered at O, is a covering of degree n, with branch locus a surface ρ contained in $\mathbf{P}^3$. There is a 2-dimensional subfamily $\mathcal{G}'$ of $\mathcal{G}$ formed by the planes passing through O: they project to lines k-tangent the surface ρ. Then ρ satisfies the assumptions of the following lemma:

Lemma 1.2 *Let $S \subset \mathbf{P}^3$ be a reduced surface and assume that there exists an irreducible subvariety $H \subset \mathbf{G}(1,3)$, with $dim(H) \geq 2$, whose general point represents a line in $\mathbf{P}^3$ which is tangent to S at $k > 2$ distinct points. Then $dim(H) = 2$ and H is a plane parametrizing the lines contained in a fixed plane $M \subset \mathbf{P}^3$, which is tangent to S along a curve.*

Therefore there exists a plane τ tangent to ρ along a curve of degree k. But τ is the projection of a 3-space α passing through O, which must contain the planes of $\mathcal{G}'$. So these planes are k-tangent also to $X \cap \alpha$, which is a surface of $\mathbf{P}^3$: this means that *all* planes tangent to $X \cap \alpha$ are k-tangent. Since $X \cap \alpha$ is not a plane, this is a contradiction.

Therefore, we have *at least* $n-3$ singular points of X on r. Assume there are $n-2$.

Let $H \subset \mathbf{P}^4$ be a hyperplane containing r. Let us denote by $\mathbf{G}(1,H) \simeq \mathbf{G}(1,3)$ the Schubert cycle in $\mathbf{G}(1,4)$ parametrizing lines contained in H. Then, for general H the intersection $\mathbf{G}(1,H) \cap \Sigma$ is proper, namely it is purely 0-dimensional. In fact, if infinitely many lines of Σ were contained in H, then $dim(\Sigma) \geq 3$, a contradiction. Moreover, since we assume that Σ is generically reduced, both Σ and $\mathbf{G}(1,H)$ are smooth at r. We will show, now, that if r contains $n-2$ singular points of X, then Σ and $\mathbf{G}(1,H)$ do not intersect transversally at r, and this will yield a contradiction. In fact, $PGL(4)$ acts transitively on $\mathbf{G}(1,4)$, and we can use [12] because we have assumed that our base field K has characteristic zero.

Before we start, let us recall briefly for the reader convenience some basic facts about $T_r\mathbf{G}(1,4)$. Let $\Lambda \subset K^5$ be the 2-dimensional linear subspace corresponding to r, i.e. $r = \mathbf{P}(\Lambda)$. Then $T_r\mathbf{G}(1,4)$ can be identified with $Hom_K(\Lambda, K^5/\Lambda)$, hence for a non zero $\varphi \in T_r\mathbf{G}(1,4)$ we have $rk\ \varphi = 1$ or 2. In both cases we can associate to φ in a canonical way a double structure on r. When $rk\ \varphi = 1$ this structure is obtained by doubling r on the plane $\mathbf{P}(\Lambda \oplus Im(\varphi))$, hence it has arithmetic genus zero ([11]). When $rk\ \varphi = 2$ the doubling of r is on a smooth quadric inside $\mathbf{P}(\Lambda \oplus Im(\varphi)) \simeq \mathbf{P}^3$, and the arithmetic genus is -1. In both cases we have $r \subset \mathbf{P}(\Lambda \oplus Im(\varphi))$ and $\varphi \in T_r\mathbf{G}(1, \mathbf{P}(\Lambda \oplus Im(\varphi)))$.

To prove the non transversality of Σ and $\mathbf{G}(1,H)$ at r, it is harmless to assume that H is not tangent to X at any smooth point of r. Therefore, the singularities of the surface $S := X \cap H$ on the line r are exactly those points which are already singular for X.

To fix ideas, let r be defined by the equations $x_2 = x_3 = x_4 = 0$, H defined by $x_4 = 0$, and S defined in H by $\overline{G} = 0$. Then, the restriction to r of the Gauss map of S is given analytically as follows:

$$\alpha : P \mapsto [\overline{G}_{x_0}(P), \overline{G}_{x_1}(P), \overline{G}_{x_2}(P), \overline{G}_{x_3}(P)].$$

We can regard the $\overline{G}_{x_i}(P)$'s as polynomials of degree $n-1$ in the coordinates of P on r. Since we assume X has $n-2$ singular points on r, the four polynomials $\overline{G}_{x_i}(P)$ have a common factor of degree $n-2$. Therefore, if we clean up this common factor, the above map can be represented analytically by polynomials of degree 1. Therefore, the double structure on r defined by α has arithmetic genus -1, and it arises from a *non zero* vector $\varphi \in T_r\mathbf{G}(1, H)$.

Now, for every $P \in r$ which is a smooth point for S we have $\alpha(P) = T_P S = T_P X \cap H$, and in particular we have $\alpha(P) \subset T_P X$. This means that φ is also a tangent vector to the Fano scheme Σ of the lines on X (see [10], pp. 209-210), i.e. $\varphi \in T_r\Sigma$. Since we assume that Σ_1 is the only component of Σ containing r and Σ_1 is reduced at r, by the usual criterion for multiplicity one, we conclude that $\mathbf{G}(1, H)$ and Σ are not transversal at r, and the proof is complete (for general facts about intersections multiplicities the reader is referred to [6]). $\square$

Proof *of Lemma 1.2*
The lines in $\mathbf{P}^3$ which are tangent to S are parametrized by a ruled threefold $K \subset \mathbf{G}(1, 3)$: any line on K corresponds to the pencil of lines in $\mathbf{P}^3$ which are tangent to S at a fixed smooth point. Then $H \subset K$.

H is a surface: otherwise, a general point $O \in \mathbf{P}^3$ would be contained in infinitely many lines of H, therefore, every tangent line to a general plane section C of S would be k-tangent to C, with $k > 2$, a contradiction.

Let $L \subset \mathbf{P}^3$ be a line corresponding to a smooth point of H; then L is tangent to S at least at points P, Q, R. Since a general point of K represents a line which is tangent to S at a unique point, K has three branches at L. We denote by U_P, U_Q, U_R the tangent spaces to these branches at L, i.e. $U_P \cup U_Q \cup U_R$ is contained in the tangent cone to K at L. We have $U_P \cap U_Q = T_L H$. The intersection of this plane with $\mathbf{G}(1, 3)$ is the union of two lines. Then, a direct, cumbersome computation proves that these lines inside $\mathbf{G}(1, 3)$ represent respectively *the pencil of lines in $T_P S$ through Q and the pencil of lines in $T_Q S$ through P.*

Claim. *For a general point $L \in H$ we have $T_L H \subset \mathbf{G}(1, 3)$.*

It is sufficient to show that $T_L H \cap \mathbf{G}(1, 3)$ contains three distinct lines.

From $U_P \cap U_Q \cap U_R = T_L H$ we get that the two lines of $T_L H \cap \mathbf{G}(1, 3)$ are contained also in U_R. If we translate all this into equations, an easy computation shows that $T_P S = T_R S$. By symmetry we get $T_P S = T_Q S = T_R S$. Therefore, the three distinct lines in $\mathbf{G}(1, 3)$ which correspond to the pencils in $T_P S$ of centres respectively P, Q, R are contained in $T_L H$, and the claim is proved.

By continuity, all the tangent planes $T_L H$ belong to one and the same system of planes on $\mathbf{G}(1, 3)$. Therefore, the tangent planes at two general points of H meet, and either H is a Veronese surface, or its linear span $\langle H \rangle$ is a $\mathbf{P}^4$.

The first case is impossible because the tangent planes to a Veronese surface fulfill a cubic hypersurface in $\mathbf{P}^5$, whereas $\mathbf{G}(1, 3)$ is a quadric. On the other hand, the quadric hypersurface $\mathbf{G}(1, 3) \cap \langle H \rangle$ in $\langle H \rangle = \mathbf{P}^4$ contains planes, hence it is singular. Therefore, the hyperplane $\langle H \rangle$ is tangent to $\mathbf{G}(1, 3)$ at some point r and all the lines of H meet the fixed line r in $\mathbf{P}^3$.

Were the lines of H not lying on a unique plane through r, then *any* plane N through r would contain infinitely many lines 3-tangent to the plane section $S \cap N$ of S, a contradiction. $\square$

In the statement of Theorem 1.1 we assume that an irreducible component of Σ is generically reduced. We will give now a criterion that leads to an easy way to check in practice if this hypothesis is satisfied.

We generalize a little and assume that an integral hypersurface $X \subset \mathbf{P}^N$ is covered by lines and that the dimension of the Fano scheme Σ of lines on X is $N-2$. Let the line r represent a general point of an irreducible component Σ_1 of Σ, of dimension $N-2$, and let p be a general point of r.

Let $[x_0, \ldots, x_N]$ be homogeneous coordinates in $\mathbf{P}^N$. Assume that the line $r \subset X$ is defined by $x_2 = \ldots = x_N = 0$, and that the point p is $[1, 0, \ldots, 0]$. We will work on the affine chart $p_{01} = 1$ of the Grassmannian $\mathbf{G}(1, N)$. Coordinates in this chart are $p_{02}, \ldots, p_{0N}, p_{12}, \ldots, p_{1N}$ and the line r is represented by the origin. It is easy to see that a line l in this affine chart contains the point p if and only if its coordinates satisfy the equations $p_{12} = \ldots = p_{1N} = 0$.

Moreover, we will work on the affine chart $x_0 = 1$ of $\mathbf{P}^N$, and we set $y_i := x_i/x_0$ for $i = 1, \ldots, N$. Then p is the origin.

Let $G = G_1 + G_2 + \ldots + G_n = 0$ be the equation of X in this chart, where the G_i are the homogeneous components of G. We can assume that the tangent space to X at p is defined by $y_N = 0$, and we can consider $y_1, \ldots, y_{N-1}$ as homogeneous coordinates in $\mathbf{P}(T_pX)$. Then, the line r is represented in $\mathbf{P}(T_pX)$ by the point $[1, 0, \ldots, 0]$. Finally, it is convenient to write $G_i = F_i + y_N H_i$, where the F_i's are polynomials in $y_1, \ldots, y_{N-1}$.

Proposition 1.3 *Assume that a hypersurface $X \subset \mathbf{P}^N$ is covered by lines and that the dimension of the Fano scheme Σ of lines on X is $N-2$. Let the line r represent a general point of an irreducible component Σ_1 of Σ, of dimension $N-2$, and let p be a general point of r. With the notations introduced above, Σ_1 is reduced at r if and only if the intersection of the hypersurfaces in $\mathbf{P}(T_pX)$ defined by $F_i = 0$ for $i = 2, \ldots, n$, is reduced at $[1, 0, \ldots, 0]$. Or, equivalently, if the $(y_2, \ldots, y_{N-1})$-primary component of the ideal $(F_2, \ldots, F_n) \subset K[y_2, \ldots, y_{n-1}]$ is $(y_2, \ldots, y_{N-1})$.*

Proof Let $s \subset \mathbf{P}^N$ be a line such that $s \not\subset X$ and $p \in s$. Let $A \subset \mathbf{G}(1, N)$ be the Schubert variety parametrizing the lines in $\mathbf{P}^N$ which intersect s. The only singular point of A is s. In fact, it is easily seen that A is the intersection of $\mathbf{G}(1, N)$ with the (projectivized) tangent space to $\mathbf{G}(1, N)$ at s. In particular, the points of A different from s are exactly the tangent vectors to $\mathbf{G}(1, N)$ at s which are of rank 1. Then, by using the facts on tangent vectors to Grassmannians briefly recalled in the proof of Thm. 1.1, it is easily seen that A is the affine cone inside $T_s\mathbf{G}(1, N)$, over a $\mathbf{P}^1 \times \mathbf{P}^{N-2} \subset \mathbf{P}(T_s\mathbf{G}(1, N))$. It is clear that Σ_1 and A intersect properly at r.

We claim that Σ_1 is reduced at r if and only if $\Sigma_1 \cap A$ is reduced at r. Assume that $\Sigma_1 \cap A$ is reduced at r. Let $\mathcal{O}$ be the local ring of $\mathbf{G}(1, N)$ at r and let I and J denote respectively the ideals of Σ_1 and A in $\mathcal{O}$. Then the Artinian ring $\mathcal{O}/I + J$ is reduced, i.e. it is a field, and we want prove that $\mathcal{O}/I$ is reduced. The Cohen-Macaulay locus of Σ_1 is certainly open and non empty. So, by genericity, we can assume that $\mathcal{O}/I$ is Cohen-Macaulay. We have $dim(\mathcal{O}/I) = N - 2 = ht(J)$. But J is generated by a regular sequence of length $N-2$ since

A is smooth at r. Therefore, the same is true for $J + I/I$, being $\mathcal{O}/I$ a Cohen-Macaulay ring. But $\mathcal{O}/I + J$ is a field, hence $J + I/I$ is the maximal ideal of $\mathcal{O}/I$. It follows that this last ring is a regular local ring.

Assume, conversely, that Σ_1 is reduced at r. Then $\Sigma_1 \cap A$ is reduced at r bevause r is general and because of Kleiman's criterion of transversality of the generic translate, already used in the proof of Thm. 1.1.

Denote by B the Schubert cycle in $\mathbf{G}(1, N)$ parametrizing the lines in $\mathbf{P}^N$ through p. A moment's thought shows that the local rings at r of $\Sigma_1 \cap A$ and $\Sigma_1 \cap B$ are the same. Then we are reduced to compute the ideal of $\Sigma_1 \cap B$ inside $\mathcal{O} = \mathcal{O}_{\mathbf{G}(1,N),r}$.

To do this, we replace the parametric representation of a general line l containing p, namely $y_1 = t$, and $y_i = p_{0i}t$ $(i \geq 2$, where t varies in the base field $K)$ in all the equations $G_i = 0$, for $i \geq 1$. From $G_1(t, p_{02}t, \ldots, p_{0N}t) = 0$ we get simply $p_{0N} = 0$. Then, since the F_i are homogeneous polynomials, the other generators for the ideal of $\Sigma_1 \cap A$ at r are the $F_i(1, p_{02}, \ldots, p_{0,N-1})$ $i = 2, \ldots, n$. An obvious change of variables completes the proof. $\square$

Example 1.4: Let X be the variety of the secant lines of a rational normal quartic curve $\Gamma \subset \mathbf{P}^4$. It is well known that the degree of X is 3. On X we have two families of lines of dimension two, each covering X. We denote by Σ_1 the family of the secant lines of Γ. By Terracini's Lemma, these lines are also the fibres of the Gauss map. Hence $dim(\check{X}) = 2$ and for the family Σ_1 we have $\mu_1 = 1$.

Since $deg(X) = 3$, the intersection of X with its tangent space along r is a cubic surface which is singular along r, hence ruled. These new lines form the second family Σ_2.

With a suitable choice of coordinates, a concrete case of such an X is given by the equation:

$$y_4 + y_1y_4 - y_2^2 - y_3^2 - y_1y_2^2 - 2y_2y_3y_4 - y_4^3 = 0 \ ,$$

and the line r defined by $y_2 = y_3 = y_4 = 0$ is one of the secant lines of Γ. Now $F_2 = y_2^2 + y_3^2$ and $F_3 = y_1y_2^2$. Then, the curves $F_2 = 0$ and $F_3 = 0$ do not intersect transversally at $[1, 0, 0]$, and Σ_1 is not reduced at r. In fact, on any line of Σ_1 there are two points of $Sing(X)$. This shows that *the hypothesis "Σ_1 is generically reduced" in Theorem 1.1 is essential.*

Note also that the curves $F_2 = 0$ and $F_3 = 0$ intersect outside $[1, 0, 0]$ transversally at two points. These points represent two lines on X through p, which belong to Σ_2. Therefore $\mu_2 = 2$.

The following proposition deals with a delicate point, namely the possibility for a general line r of Σ to be contained in a plane which is tangent to X at any point of r.

Proposition 1.5 *Let $X \subset \mathbf{P}^4$ be an irreducible hypersurface covered by the lines of a family of dimension 2 such that Σ is generically reduced. Let $r \in \Sigma$ be general. Then there is no plane containing r which is tangent to X at any general point of r.*

Proof Assume by contradiction that there exists a plane M such that $M \subset T_qX$ for every $q \in r \cap X_{sm}$. We perform some local computations and we use the same notations as in Proposition 1.3. So, let $\mathbf{A}^4$ be an affine chart in $\mathbf{P}^4$, with coordinates $y_1, \ldots, y_4$. Assume that the origin is a general point p of X, and that T_pX is defined by $y_4 = 0$. Let r and M be defined respectively by $y_2 = y_3 = y_4 = 0$ and $y_3 = y_4 = 0$. Let $G = G_1 + G_2 + \ldots + G_n = 0$ be the equation of X in this chart. We write also $G_i = F_i + y_4 H_i$, where the F_i's are homogeneous polynomials in y_1, y_2, y_3. Since the line r is represented in $\mathbf{P}(T_pX)$ by the point $[1,0,0]$, we have

$$F_i = y_1^{i-1} A_{i,1}(y_2, y_3) + y_1^{i-2} A_{i,2}(y_2, y_3) + \ldots + A_{i,i}(y_2, y_3) \ ,$$

where the $A_{i,j}$ are homogeneous polynomials of degree j, or zero.

Now, if we move the origin of our system of coordinates to the point $q \in r$ by a change of coordinates of type $Y_1 = y_1 - t$ and $Y_i = y_i$ for $i = 2, 3, 4$ and $t \in K$ (hence $q = (t, 0, 0, 0)$), then in the new system of coordinates X is defined by the equation

$$\widetilde{G}_t(Y_1, \ldots, Y_4) = G(Y_1 + t, Y_2, Y_3, Y_4) = Y_4 + \sum_{i=2}^{n} \{ F_i(Y_1 + t, Y_2, Y_3) + Y_4 H_i(Y_1 + t, Y_2, Y_3, Y_4) \}$$

$$= (1 + f(t))Y_4 + \sum_{i=2}^{n} t^{i-1} A_{i,1}(Y_2, Y_3) + \ H.O.T. \ ,$$

where $f(t) \in K$. Now, since $M \subset T_qX$ for every $q \in r \cap X_{sm}$, the above equation shows that, necessarily the linear term of $\widetilde{G}_t$ belongs to the ideal (Y_3, Y_4) for every $t \in K$. Therefore, the linear forms $A_{i,1}(Y_2, Y_3)$ are in the ideal (Y_3) for every $i \geq 2$. But in this case the curves in $\mathbf{P}(T_pX)$ defined by $F_i = 0$ are either singular at $[1,0,0]$, or with tangent line $y_3 = 0$ at $[1,0,0]$. This contradicts Prop. 1.3, and the proof is complete. □

From Proposition 1.5 we will deduce the following very useful corollaries.

Let $\gamma : X \cdots \to \check{\mathbf{P}}^4$ be the Gauss map, which is defined on the smooth locus X_{sm} of X. The closure of the image is $\check{X}$, the dual variety of X. If $\dim \check{X} < 3$, then the fibres of γ are linear subvarieties of X, and the tangent space to X is constant along each fibre.

Corollary 1.6 *Let $X \subset \mathbf{P}^4$ be an irreducible hypersurface covered by the lines of a family of dimension 2 such that Σ is generically reduced. Then the dual variety $\check{X}$ of X is a hypersurface of $\check{\mathbf{P}}^4$.*

Proof First of all, the dimension of $\check{X}$ must be at least 2 : otherwise X would contain a 1-dimensional family of planes, hence a 3-dimensional family of lines, a contradiction. So assume by contradiction that $dim(\check{X}) = 2$. But then along each fibre of the Gauss map there is even a fixed tangent hyperplane, contradicting Proposition 1.5. □

Corollary 1.7 *Let $X \subset \mathbf{P}^4$ be an irreducible hypersurface covered by the lines of a family of dimension 2 such that Σ is generically reduced. Let Σ_1 be an irreducible component of Σ of dimension two, such that $\mu_1 > 1$. Let $r \in \Sigma_1$ be general, and set $\sigma_1(r) = \{r' \in \Sigma_1 \mid r \cap r' \neq \emptyset\}$. Then $r \notin \sigma_1(r)$.*

Proof Assume the contrary. Then, when $r' \in \sigma_1(r)$ moves on $\sigma_1(r)$ to r, the plane $\langle r' \cup r\rangle$ moves to a limit plane M. The intersection $X \cap M$ is a curve which has the line r as a "double component" ; in particular, this curve is singular along r.

Then $M \subset T_q X$ for every $q \in r \cap X_{sm}$. In fact, if $M \not\subset T_q X$, then $X \cap M$ would be smooth at q, contradiction. □

Let $\mathcal{F}$ be the 4-dimensional family of planes introduced in the proof of Theorem 1.1. We will consider now its subfamily $\mathcal{F}'$ of dimension 3, formed by the planes generated by pairs of coplanar lines of Σ.

Proposition 1.8 *Let π be a general plane of $\mathcal{F}'$ generated by the lines r and r' of Σ. Then π is tangent to X at exactly 3 points of $r \cup r'$ (but maybe π is tangent to X elsewhere, outside $r \cup r'$).*

Proof By Theorem 1.1 there are two tangency points of π to X on r and two on r'. The point $r \cap r'$ is singular for $X \cap \pi$, but it cannot be singular for X, because, otherwise, letting r and r' vary, every point of X would be singular. So $r \cap r'$ is a tangency point of π to X. Hence, π is tangent to X at exactly three points *lying on r or r'*. □

To prove the next proposition, and also in the sequel, we will need the following refined form of the connectedness principle of Zariski, due to A.Nobile ([15]):

Lemma 1.9 *Let $f : X \to T$ be a flat family of projective curves, parametrized by a quasi–projective smooth curve, such that the fibres X_t are all reduced and X_t is irreducible for $t \neq 0$. Assume that, for $t \neq 0$, X_t has a fixed number d of singular points $P_1^t, \ldots, P_d^t$ and that there exist d sections $s_j : T \to X$ such that $s_j(t) = P_j^t$ if $t \neq 0$, that $s_i(t) \neq s_j(t)$ if $i \neq j$ and that $\delta(X_t, P_j^t)$ is constant (where $\delta(X_t, P_j^t)$ denotes the length of the quotient $\overline{A}/A$, A being the local ring of X_t at P_j^t and $\overline{A}$ its normalization). If the singularities of X_0 are $s_1(0), \ldots, s_d(0), Q_1, \ldots, Q_r$, then $X_0 \setminus \{s_1(0), \ldots, s_d(0)\}$ is connected.*

Proposition 1.10 *Let $n \geq 4$ and let π be a general plane of an arbitrary irreducible component of $\mathcal{F}'$. Then π does not contain three lines of Σ.*

Proof Assume by contradiction that π contains the lines r, r', r''. Then the residual curve of r in $\pi \cap X$ splits as $r' \cup r'' \cup C$. Hence, by Lemma 1.9, there is a new tangency point on $r' \cup r''$, against Proposition 1.8. □

Since our hypersurfaces $X \subset \mathbf{P}^4$ contain "too many" lines if $n \geq 4$, it is quite natural that they are far from general in the linear system of all hypersurfaces of $\mathbf{P}^4$ of a fixed degree n. In fact, it will turn out that, if $n \geq 4$ none of them is linearly normal. Hence their singular loci have always dimension 2. We will prove, now, directly this last property, under the more restrictive assumption that $n \geq 5$, which is sufficient for our application of the theorem.

Theorem 1.11 *Let $X \subset \mathbf{P}^4$ be a hypersurface of degree $n \geq 5$, covered by a family of lines Σ of dimension 2, with $\mu > 1$ Let Δ denote the singular locus of X. Then Δ is a surface. If X is not birationally ruled by quadrics, then* $\deg(\Delta) \geq 2(n-3)$*.*

Proof We assume by contradiction that Δ is a curve. Then every point of Δ belongs to infinitely many lines of Σ. The curve Δ is not a line because every line of Σ meets Δ in $n-3$ points, and $n \geq 5$. If $x \in X$ is general, from $\mu > 1$ it follows that through x there are two secant lines of Δ, say r and s. By Terracini's lemma the tangent space to X must be constant along r and also along s. Therefore, the plane spanned by r and s is (contained in) a fibre of the Gauss map, hence it is contained in X. So, through a general point of X there is a plane on X, contradiction. This proves that Δ is a surface.

To prove the assertion on the degree, we consider a general plane of $\mathcal{F}'$. If it intersects properly Δ, then this intersection contains at least $2(n-3)$ points, and the claim follows. If the intersection is not proper, then Δ contains a family of plane curves of dimension 3, hence it is a plane. Let H be a hyperplane containing Δ; then $X \cap H$ splits as the union of Δ with a surface S. If $P \in S$ is general, there are at least two lines on X passing through P. Each of them meets Δ, hence is contained in H, and therefore in S. This shows that S is a union of smooth quadrics. $\square$

We will give in the next proposition some generalities on the surfaces $\sigma(r)$.

Proposition 1.12 *Let $X \subset \mathbf{P}^4$ be a hypersurface of degree n covered by the lines of the family Σ of dimension 2, with $\mu \geq 2$. Let r be a general line of Σ and $\sigma(r)$ be the union of the lines of Σ intersecting r. Then:*

(i) $\sigma(r)$ is a ruled surface, having r as line of multiplicity $\mu - 1$;

(ii) if the surfaces $\sigma(r)$ describe, as r varies in Σ, an algebraic family in X of dimension < 2, then X is covered by a 1-dimensional family of quadrics such that there is one and only one quadric of the family passing through any general point of X.

Proof The first assertion of (i) is clear. To prove the second, it is enough to observe that exactly $\mu - 1$ lines of Σ, different from r, pass through a general point of r, and that these lines are separated by the blow-up of X along r.

The assumption of (ii) means that, for every r, the lines of Σ intersecting r intersect also infinitely many other lines of the family, so $\sigma(r)$ is doubly ruled, hence it is a smooth quadric, or a finite union of smooth quadrics. In the second case, the algebraic family described by the surfaces $\sigma(r)$ has dimension two, so this case is excluded. $\square$

We will refer to threefolds X as in (ii) as "quadric bundles".

In the following we will analyze the self–intersection of the curves $\sigma(r)$ on Σ assuming it positive. If the family of these curves is one–dimensional, then the self–intersection is zero and X is a quadric bundle. This is the reason why we exclude quadric bundles in our classification.

Our final task concerning the surfaces $\sigma(r)$ will be the determination of their degree. For this we need another proposition.

Let r and r' denote two general lines in the same irreducible component Σ_i of Σ. We will call $\overline{\mu}_i$ the number of lines of *all* Σ intersecting both r and r'.

Recall that, for every $r \in \Sigma$, the curve $\sigma(r) \subset \Sigma$ (we switch our point of view, now) parametrizes the lines of Σ intersecting r. If Σ is reducible, with $\Sigma_1, \ldots, \Sigma_s$ as irreducible components of dimension 2, then the curves $\sigma(r)$ are unions $\sigma_1(r) \cup \ldots \cup \sigma_s(r)$, where $\sigma_i(r)$

is formed by the lines of Σ_i intersecting r. Note that, if $\mu_i = 1$ for some index i and $r \in \Sigma_i$, then $\sigma_i(r)$ is empty.

Then $\overline{\mu}_i$ is the intersection number $\sigma(r) \cdot \sigma(r')$ on (a normalization of) Σ.

Proposition 1.13 *Let X be a threefold such that $deg(X) > 3$. Let r and r' be two general lines in the same irreducible component Σ_i of Σ. Then $\overline{\mu}_i = \mu - 2$ (independent of i !)*

Proof To evaluate $\overline{\mu} = \sigma(r) \cdot \sigma(r')$ we choose the lines r and r' so that they intersects at a point p, smooth for X. Since $deg(X) > 3$, by Proposition 1.10 we can also assume that r and r' are the only lines of Σ contained in the plane $\langle r \cup r' \rangle$, so that the lines intersecting both r and r' are those passing through p. The conclusion follows from Corollary 1.7 $\square$

Proposition 1.14 *Assume* $\deg X \geq 4$ *and let r be a general line on X. Then* $\deg \sigma(r) = 3\mu - 4$.

Proof Note first that $\deg \sigma(r)$ is equal to the degree of the curve, intersection of $\sigma(r)$ with a hyperplane H. We can assume $r \subset H$; then $H \cap \sigma(r)$ splits in the union of r with m other lines meeting r. Indeed, if $P \in H \cap \sigma(r)$ and $P \notin r$, there exists a line passing through P and meeting r, which is necessarily contained in H. Moreover, $\sigma(r)$ and H meet along r with intersection multiplicity $\mu - 1$ (Proposition 1.12). Therefore $\deg \sigma(r) = \mu - 1 + m$.

To compute m, the number of lines meeting r and contained in a 3-space H, we can assume that H is tangent to X at a point P of r. In this case H contains the $\mu - 1$ lines through P different from r. To control the other $m - (\mu - 1)$ lines, we use the following degeneration argument.

Since H is tangent to X at $p \in r$, the intersection multiplicity of Σ and $\mathbf{G}(1, H)$ at r is 2 (this will be proved in §2, Lemma 2.3). According to the so called"dynamical interpretation of the multiplicity of intersection", in any hyperplane H' "close" to H (if we are working over $\mathbf{C}$ this means: in a suitable neighbourhood of H for the Euclidean topology of $\check{\mathbf{P}}^4$) there are two distinct lines $g, g' \in \Sigma$ which both have r as limit position when H' specializes to H. Note that the lines g and g' are skew, because otherwise $g \in \sigma(g')$, which becomes $r \in \sigma(r)$ when H' specializes to H, a contradiction with Prop. 1.7.

Therefore, we can choose a family of 3-spaces H_t, parametrized by a smooth curve T, such that $H_0 = H$ and, for general t, H_t is generated by two skew lines r_t and r'_t, having both r as limit position for $t = 0$. The lines in H meeting r come from lines in H_t meeting either r_t or r'_t. In other words, the intersections $\sigma(r_t) \cap H_t$ and $\sigma(r'_t) \cap H_t$ both move to $\sigma(r) \cap H$. Therefore to preserve the degree of these intersections, the remaining lines intersecting r have to come from the $\overline{\mu}$ lines of H_t meeting both r_t and r'_t. Note that, if l is one of these "remaining" lines, then the multiplicity of l in $\Sigma \cap \mathbf{G}(1, H)$ is 1. In fact, otherwise, H would be tangent to X at some point of l; but H is already tangent to X at p, and $p \notin l$. We can conclude by the previous proposition that $m = \overline{\mu} + \mu - 1 = 2\mu - 3$. $\square$

2 Bounds for μ

It is well known that for a surface covered by the lines of a 1-dimensional family, there are at most two lines through any general point. The following theorem is the analogous for threefolds.

Theorem 2.1 *Let $X \subset \mathbf{P}^4$ be a 3-fold covered by lines. Assume that the Fano scheme Σ is generically reduced and of dimension 2. Then $\mu \leq 6$.*

Proof It was already remarked in the Introduction that for the degree n of X we have $n \geq 3$. Let p be a general point of X and fix a system of affine coordinates $y_1, \ldots, y_4$ such that $p = (0, \ldots, 0)$. Let $G = G_1 + \ldots + G_n$ be the equation of X. As usual, we assume that T_pX is defined by $y_4 = 0$, and, moreover, we write $G_i = F_i + y_4 H_i$, for $i \geq 2$.

The polynomials $F_2, \ldots, F_n$ define (if not zero) curves in the plane $\mathbf{P}(T_pX)$. In particular, $F_2 = 0$ is a conic C_2, whose points represent tangent lines to X having at p a contact of order > 2, and $F_3 = 0$ is a cubic C_3; the points of $C_2 \cap C_3$ represent the tangent lines to X having at p a contact of order > 3, and so on. Clearly the points of $\mathbf{P}(T_pX)$ corresponding to lines contained in X are exactly those of $C_2 \cap C_3 \cap \ldots \cap C_n$.

We have $F_2 \neq 0$ at any general point of X because, otherwise X would be a hyperplane of $\mathbf{P}^4$. On the other hand, since $deg(X) \geq 3$, at any general point of X we have also that F_3 is not a multiple of F_2 ([9], Lemma (B.16)). In particular, we have $F_3 \neq 0$, and C_2 is not contained in C_3.

Now, $dim(\check{X}) = 3$, so C_2 is an irreducible conic (see [19] or [7]), and we are done. $\square$

Remark 2.2 Actually, it is possible to give a proof of Theorem 2.1 which is independent of Theorem 1.6, hence of the assumption that Σ is generically reduced.

Lemma 2.3 *For general $H \in \check{\mathbf{P}}^4$ the intersection $\Sigma \cap \mathbf{G}(1, H)$ is proper. Moreover, if $r \in \Sigma$ is general and $r \subset H$, then the intersection multiplicity of Σ and $\mathbf{G}(1, H)$ at r is always ≤ 2 and it is 1 if and only if H is not tangent to X at any point of $r \cap X_{sm}$.*

Proof The first part of the statement was already shown in the proof of Thm. 1.1.

Moreover, in the same proof we saw that, if H is not tangent to X at some point of r, then $T_r\Sigma$ and $T_r\mathbf{G}(1, H)$ are transversal inside $T_r\mathbf{G}(1, 4)$. In fact, the GCD of the polynomials $\overline{G}_{x_1}$ in (1) has degree exactly $n - 3$. Hence the double structure on r they define has arithmetic genus -2 and does not represent any vector in $T_r\mathbf{G}(1, H)$. Therefore $T_r\Sigma \cap T_r\mathbf{G}(1, H) = (0)$ and the intersection is transversal.

So we have proved that $i(r) = 1$ if and only if H is not tangent to X at any point of r. Hence, we assume now that H is tangent to X at some point of r. To show that $i(r) \leq 2$ we perform some local computations. Let $[x_0, \ldots, x_4]$ be a system of homogeneous coordinates in $\mathbf{P}^4$ such that the line r is defined by the equations $x_2 = x_3 = x_4 = 0$. Let $H = T_PX$, where $P = [0, 1, 0, 0, 0]$ and H is defined by $x_4 = 0$. Let $[p_{01}, \ldots, p_{34}]$ be the related Plücker coordinates. So r has coordinates $[1, 0, \ldots, 0]$, hence $p_{01} \neq 0$. We will restrict, from now on, to work in the affine chart U_{01} of $\mathbf{G}(1, 4)$ given by $p_{01} \neq 0$; coordinates in this chart are $p_{02}, p_{03}, p_{04}, p_{12}, p_{13}$ and p_{14}. The equations of $\mathbf{G}(1, H)$ inside U_{01} are $p_{04} = p_{14} = 0$. Then the general point of a line $r \in U_{01} \cap \mathbf{G}(1, H)$ is $[s, 1, p_{02} - sp_{12}, p_{03} - sp_{13}, 0]$.

In a suitable system of coordinates, the equation of X is of the form:

$$F = x_2 \Psi x_0^2 + x_3 \Psi x_0 x_1 + x_4 \Psi x_1^2 + a x_2^2 + b x_2 x_3 + c x_2 x_4 + \ldots + f x_4^2 + \\ + \text{terms of degree} > 2 \text{ in } x_2, x_3, x_4 \qquad (1)$$

where $\Psi, a, \ldots, f \in K[x_0, x_1]$ are forms of degree $n-3$ and $n-2$ respectively. Here we have used the condition $r \subset X$. Moreover the homogeneous part of degree 1 in x_2, x_3, x_4 of F can be normalized in this way because there is no fixed tangent plane to X along r ([4]).

From $P \notin Sing(X)$ it follows that the coefficient of x_1^{n-3} in the polynomial Ψ is not zero, and we can set $\Psi = x_1^{n-3} + \rho_1 x_0 x_1^{n-4} + \rho_2 x_0^2 x_1^{n-5} + \ldots$. The point P is $(0,0,0,0)$ in the affine chart $x_1 \neq 0$, and if we dehomogeneize F w.r.t. x_1 we get

$$ {}^aF = ({}^aF)_1 + ({}^aF)_2 + \ldots = x_4 + x_0x_3 + \rho_1 x_0 x_4 + V(0, 1, x_2, x_3, x_4) + \ldots \qquad (2) $$

where $V := ax_2^2 + bx_2x_3 + cx_2x_4 + \ldots + fx_4^2$.

The condition $r \subset X$ implies that $F(s, 1, p_{02} - sp_{12}, p_{03} - sp_{13}, 0)$ is identically zero as a polynomial in s. If we set $F(s, 1, p_{02} - sp_{12}, p_{03} - sp_{13}, 0) = \alpha + \beta s + \gamma s^2 + \delta s^3 + \ldots$, then we can compute $\alpha, \beta, \gamma, \delta$ from (2), and we get:

$$ \alpha = \overline{a} p_{02}^2 + \overline{b} p_{02} p_{03} + \overline{d} p_{03}^2 $$

$$ \beta = p_{03} + \text{ terms of degree} > 1 \text{ in } p_{12}, p_{13}, p_{02}, p_{03} $$

$$ \gamma = -p_{13} + p_{02} + \rho_1 p_{03} + \text{ terms of degree} > 1 \text{ in } p_{12}, p_{13}, p_{02}, p_{03} $$

$$ \delta = -p_{12} + \rho_1(-p_{13} + p_{03}) + \rho_2 p_{03} + \text{ terms of degree} > 1 \text{ in } p_{12}, p_{13}, p_{02}, p_{03} $$

where $\overline{a}, \overline{b}, \overline{d}$ are the constant terms of the polynomials $a(x_0, 1), b(x_0, 1)$ and $d(x_0, 1)$ respectively. Note that $\alpha, \beta, \gamma, \delta$ are some of the equations of $\mathbf{G}(1, H) \cap \Sigma$.

By setting $\beta = \gamma = \delta = 0$ we define inside the four dimensional affine space H a curve which is smooth at $(0,0,0,0)$, the point in $\mathbf{G}(1, H)$ which represents r. The direction of the tangent line to this curve at r is given by the vector $(\rho_1, -1, 1, 0)$.

Assume by contradiction that $i(r) > 2$. Then this vector annihilates α, and $\overline{a} = 0$. It follows that x_0 divides $a(x_0, x_1)$, hence $x_2^2 a(x_0, 1)$ does not give any contribution to $({}^aF)_2$. Therefore, the reduction modulo x_4 (the equation of T_PX in $\mathbf{P}^4$) of the polynomial $({}^aF)_2$ is $x_3(x_0 + \overline{b}x_2 + \overline{d}x_3)$. This is the equation of the conic C_2 embedded in $\mathbf{P}(T_PX)$. But, since the dual variety of X has dimension 3, by Theorem 1.6 this conic should be smooth ([19], [7]). $\square$

Theorem 2.4 *Let X be a threefold such that $deg(X) > 3$. Then $\mu \leq 4$.*

Proof We analyze in detail the case $\mu = 5$. A similar proof can be given if $\mu = 6$. For a different proof of this last case, see [21].

Let us recall that $\overline{\mu} = 3$, so given $r, r' \in \Sigma$ general and skew, there are three lines $a, b, c \in \Sigma$ meeting both r and r'.

The lines a, b, c are pairwise skew, otherwise r, r' would fail to be skew. Since $\overline{\mu} = 3$, there exists a third line in Σ, besides r and r', meeting both a and b. The same conclusion holds for the pairs (a, c) and (b, c).

Claim: If r and r' are general lines of Σ, then the three lines of Σ constructed above starting from the pairs (a, b), (a, c) and (b, c) are distinct.

Assume the contrary. Then there exists a unique line $s \in \Sigma$, different from both r and r', which meets a, b, c. Note that all the six lines a, b, c, r, r', s are contained in the linear span of r and r'.

We consider now a family of pairs of lines $\{(r_t, r_t')\}$ on X, parametrized by a smooth quasi–projective curve T, such that r_t and r_t' are disjoint for a general $t \in T$,, while for $t = 0$ the lines r_0 and r_0' meet at a point P, general on X. Therefore for t general $\alpha_t := \langle r_t, r_t' \rangle$ is a $\mathbf{P}^3$: we get a family of 3-spaces whose limit position α_0 is the tangent space $T_P X$.

We can assume that the plane of r_0 and r_0' does not contain other lines of Σ (because $n > 3$). For general t, we have three lines a_t, b_t, c_t, meeting r_t and r_t', and a third line s_t, meeting a_t, b_t and c_t, which exists by assumption. For $t = 0$, the lines a_0, b_0, c_0 still meet r_0 and r_0', and s_0 meets a_0, b_0 and c_0. Hence a_0, b_0, c_0 pass through P. By Corollary 1.7, s_0 cannot coincide with a_0, b_0 or c_0, therefore by the assumption $\overline{\mu} = 3$ and $\mu = 5$, either $s_0 = r_0$ or $s_0 = r_0'$.

Assume $s_0 = r_0$.

By Lemma 2.3, the intersection multiplicity of $\mathbf{G}(1, \alpha_0)$ and Σ is two at each of the five points corresponding to the lines r_0, r_0', a_0, b_0, c_0, therefore, by the dynamical interpretation of the intersection multiplicity, there exist four more lines in α_t moving to r_0', a_0, b_0, c_0 respectively. Let u_t be a line of α_t, having r_0' as limit position: by Corollary 1.7 $r_t' \cap u_t = \emptyset$.

Let us assume that $u_t \cap (a_t \cup b_t \cup c_t \cup r_t \cup s_t) = \emptyset$. In this case, from $\overline{\mu} = 3$, it follows that there exist six lines in α_t, three of them meeting both s_t and u_t, three meeting both r_t and u_t.

The limit position of each of these six lines passes through P: but in this way we get too many lines passing through P in $T_P X$, contradicting the "multiplicity two " statement of Lemma 2.3.

Therefore u_t meets either r_t (or, symmetrically, s_t) or a_t (or, symmetrically, b_t or c_t).

Case (i): $u_t \cap r_t \neq \emptyset$.

In this case $u_t \cap s_t = \emptyset$, otherwise we would have four lines meeting both r_t and s_t. Also $u_t \cap a_t = \emptyset$ (and analogously $u_t \cap b_t$ and $u_t \cap c_t$), otherwise the three lines r_t, a_t and u_t would be coplanar. Therefore there exist three lines meeting u_t and s_t, two more lines meeting u_t and a_t, two meeting u_t and b_t, two meeting u_t and c_t: summing up, we get nine new lines.

We get again a contradiction with Lemma 2.3, because we have found 16 lines tending to lines of $T_P X$ passing through P. We conclude that $u_t \cap r_t = \emptyset$.

Case (ii): $u_t \cap a_t \neq \emptyset$.

So, being $\overline{\mu} = 3$, $u_t \cap b_t = u_t \cap c_t = \emptyset$. In this case, we can construct four new lines, two meeting s_t and u_t and two meeting r_t and u_t. Summing up we have 11 lines moving to lines of $T_P X$ passing through P: this contradiction proves the Claim.

Hence, given r and r', general lines on X, there exist lines a, b and c meeting both of them, and two by two distinct lines s_1, s_2, s_3 meeting a and b, a and c, b and c respectively. Moreover: $s_i \cap s_j = \emptyset$ for $i \neq j$; $r \cap s_i = r' \cap s_i = \emptyset$, $\forall i$.

Using the assumption $\overline{\mu} = 3$, we get the existence of six more lines: l meeting r and s_1, l' meeting r' and s_1; m meeting r and s_2, m' meeting r' and s_2; n meeting r and s_3, n' meeting r' and s_3. Altogether there is a configuration of 14 lines obtained from r and r'.

The first observation is that the s_i's tend to lines through P, but s_1 tends neither to a_0 nor to b_0, because s_1 meets a and b. Therefore there are three possibilities, that we examine separately:

(i) $s_1 \to r_0$; in this case the lines tending to r_0 are only r and s_1. Now we consider s_2: there are two subcases:

- $s_2 \to r'_0$: hence $s_3 \to a_0$. Since l' meets both r' and s_1, then it moves either to b_0 or to c_0; similarly m', which meets both r and s_2, moves either to b_0 or to c_0, and also n does the same. This contradicts Lemma 2.3 and Corollary 1.7.
- $s_2 \to b_0$: then we consider l', which moves either to a_0 or to c_0. If $l' \to a_0$: then s_3, which meets b and c, goes to r'_0; m', which meets r and s_2, goes to c_0; n which meets r and s_3 could go to a_0 or to b_0 or to c_0: but all three cases are excluded by Lemma 2.3 and Corollary 1.7 again. If $l' \to c_0$, the conclusion is similar.

(ii) $s_1 \to r'_0$; this case is analogous to case (i).

(iii) $s_1 \to c_0$. We consider s_2: since it meets a and c, it goes to b_0, or to r_0, or to r'_0. The last two possibilities are excluded as in (i) and (ii) for s_1, so $s_2 \to b_0$ and finally $s_3 \to a_0$. By considering the limit positions of l, l', m, we find that also in this case the "multiplicity two" statement of Lemma 2.3 is violated.

This concludes the proof. $\square$

The statement of Theorem 0.1 shows that the families of lines in $\mathbf{P}^4$ we want to classify are characterized by the number s of irreducible components $\Sigma_1, \ldots, \Sigma_s$ of Σ and by the relative μ_i's. Therefore the proof can be organized according to the following two possibilities:

- there exists an irreducible component Σ_i of Σ with $\mu_i > 1$;
- for every irreducible component Σ_i of Σ, $\mu_i = 1$.

By Theorem 2.1, there are only finitely many values of s and μ_i to analyze. A posteriori, it will turn out that, actually, in the first case there do not exist other irreducible components of Σ.

3 There exists an irreducible component Σ_i of Σ with $\mu_i > 1$

Let Σ_i be an irreducible component of Σ of dimension 2, such that $\mu_i > 1$. In this section we will consider and use only the lines of Σ_i, e.g. for constructing the surfaces $\sigma(r)$ and so on. So, for simplicity, we will denote Σ_i by Σ and μ_i by μ. Note that Proposition 1.13 is still true (with the same proof) even if we use in the statement our "μ" and "$\overline{\mu}$" defined by using only the lines of Σ_i.

Proposition 3.1 *Assume that X is not a quadric bundle and that $deg(X) > 3$. Then $\mu > 2$.*

Proof Since we assume that X is not a quadric bundle we have that the dimension of $\{\sigma(r)\}_{r\in\Sigma}$ is 2 by Prop. 1.12. Then, through a general point of Σ there are infinitely many curves $\sigma(r)$, and, by Proposition 1.13 we conclude

$$\mu - 2 = \sigma(r)^2 > 0 \ .$$

□

Then, if we assume that $deg(X) > 3$ and that X is not a quadric bundle, by the above proposition and by Theorem 2.4, the only possibilities for μ are $\mu = 3, 4$.

The case $\mu = 3$.

Proposition 3.2 *Let $X \subset \mathbf{P}^4$ be a hypersurface of degree > 3, containing an irreducible family of lines Σ with $\mu = 3$. Then X has degree 5, sectional genus $\pi = 1$ and it is a projection of a Fano threefold of $\mathbf{P}^6$ of the form $\mathbf{G}(1,4) \cap \mathbf{P}^6$.*

Proof The algebraic system of dimension two $\{\sigma(g)\}_{g\in\Sigma}$ on the surface Σ is linear because there is exactly one curve of the system passing through two general points ($\overline{\mu} = 1$). Also the self–intersection is equal to $\overline{\mu} = 1$, therefore $\{\sigma(g)\}$ is a homaloidal net of rational curves, which defines a birational map f from Σ to the plane, such that the curves of the net correspond to the lines of $\mathbf{P}^2$. The degree of the curves $\sigma(g)$ is 5 by Proposition 1.14. So the birational inverse of f is given by a linear system of plane curves of degree 5. Hence we get immediately the weak bound $\deg \Sigma \leq 25$. Let ν denote the number of lines of Σ contained in a 3-plane: by Schubert calculus, $\deg \Sigma = \mu n + \nu$. To evaluate ν, we consider two general skew lines r, r' on X, generating a 3-space H. The lines r and r' have a common secant line l. The set–theoretical intersection $\sigma(r) \cap H$ is the union of r, l and two more lines l_1, l_2 by Proposition 1.14. Similarly we get two new lines m_1, m_2 in $\sigma(r') \cap H$. The line l_1 (resp. l_2) cannot meet both m_1 and m_2 because $\overline{\mu} = 1$, so there are two new lines in H.

So we have found at least 9 lines in H, hence $\nu \geq 9$. The assumption $\mu = 3$ together with $\nu \geq 9$ gives at once $n \leq 5$.

Let S be a general hyperplane section of X. If $n = 4$, then it is well known (see for example the classical book of Conforto [5]) that under our assumptions one of the following happens: S is a ruled surface (in particular a cone) or a Steiner surface or a Del Pezzo surface with a double irreducible conic. None of these surfaces is section of a threefold X with the required properties. In the first case X would have a family of lines of dimension 3, in the second case X would be a cone, in the third case $\mu = 4$ (see [8] and [22]). Therefore the degree of X is exactly 5.

We can apply, now, Theorem 1.11 which gives $\deg \Delta \geq 4$ since $n = 5$. If π denotes the sectional genus of X (i.e. the geometric genus of a general plane section of X) we deduce $\pi \leq 2$.

To exclude $\pi = 2$, we show that there exist planes containing three lines of Σ. Indeed let r be a general line of Σ. We fix in $\mathbf{P}^4$ a 3-plane H not containing r, intersecting r at a point O. Let $\gamma := \sigma(r) \cap H$ be a hyperplane section of $\sigma(r)$. By Proposition 1.14, $\sigma(r)$ has degree 5, hence there exists a trisecant line t passing through O and meeting γ again at two points P and Q. Let M be the plane generated by r and t: it contains also the lines of $\sigma(r)$

passing through P and Q, so M contains three lines contained in X. Now we consider $M \cap \Delta$. By Lemma 1.9 in $M \cap X = r \cup r' \cup r'' \cup C$ there must be a "new" tangency point, hence $\Delta \cap M$ contains at least five points. Therefore $\deg \Delta \geq 5$ and $\pi \leq 1$. If $\pi = 0$, the curves intersection of S with its tangent planes have a new singular point, so they split. Then by the Kronecker–Castelnuovo theorem, S is ruled, a contradiction. So we have $\pi = 1$ and S is a projection of a linearly normal Del Pezzo surface S' of $\mathbf{P}^5$ of the same degree 5 (see [5]), which is necessarily a linear section of $\mathbf{G}(1,4)$. This proves the theorem. $\square$

The case $\mu = 4$.

Proposition 3.3 *Let $X \subset \mathbf{P}^4$ be a hypersurface of degree > 3, containing an irreducible family of lines Σ with $\mu = 4$. Then X has degree 4 and sectional genus $\pi = 1$, hence it is a projection of a Del Pezzo threefold of $\mathbf{P}^5$, complete intersection of two quadric hypersurfaces of $\mathbf{P}^5$.*

Proof Let $\overline{g} \in \Sigma$ be general and set $\sigma := \sigma(\overline{g})$, for simplicity. Let γ denote a normalization of σ. The proof of the proposition is based on the following two lemmas.

Lemma 3.4 *The curve γ is irreducible, hyperelliptic of genus 2. Hence γ can be embedded into $\mathbf{P}^3$ as a smooth quintic.*

Let $S \subset \mathbf{G}(1,3)$ be the surface parametrizing the secant lines of γ. Let $r \subset \mathbf{P}^3$ be a fixed general secant line of γ; we will denote by A and B the points of $r \cap \gamma$. The family of all secant lines of γ that intersect r has three irreducible components: the secant lines through A, those through B and "the other ones". This last component is represented on S by an irreducible curve that we will denote by I_r.

Lemma 3.5 *There exists a birational map $\tau: \Sigma \cdots \to S$ such that the image via τ of every curve $\sigma(g) \subset \Sigma$ is the curve $I_{\tau(g)}$ on S just introduced. If $g, g' \in \Sigma$ are general, then $g \cap g' \neq \emptyset$ if and only if $\tau(g) \cap \tau(g') \neq \emptyset$.*

We will prove now Proposition 3.3 assuming Lemmas 3.4 and 3.5 .

Let p be a general point of $\mathbf{P}^3$, $p \notin \gamma$. There are four secant lines $l_1, \ldots, l_4$ of γ through p and we can assume that $l_i = \tau(g_i)$, with $g_i \in \Sigma$, $i = 1, ..., 4$. By Lemma 3.5 we have $g_i \cap g_j \neq \emptyset$ for every $i \neq j$.

The first possibility is that, for a general $p \in \mathbf{P}^3$, the four lines $g_1, \ldots, g_4$ all lie in a plane $M_p \subset \mathbf{P}^4$. By Prop. 1.10 the family of such planes has dimension at most 2 and, therefore, the same plane M_p corresponds to infinitely many points of $\mathbf{P}^3$. This implies that every plane M_p contains infinitely many lines of Σ, hence $M_p \subset X$. Then X contains at least a 1-dimensional family of planes: a contradiction.

Therefore, for a general $p \in \mathbf{P}^3$, the four lines $g_1, \ldots, g_4$ all contain one fixed point $P \in X$, and we get a rational map $\alpha: \mathbf{P}^3 \cdots \to X$ by setting $\alpha(p) := P$. This map is dominant because $\tau: \Sigma \cdots \to S$ is birational, and it has degree 1, because $\mu = 4$. Hence X is birational to $\mathbf{P}^3$ via α.

Note that α is not regular at the points of γ, so α is defined by a linear system of surfaces $F \subset \mathbf{P}^3$ of degree m, all containing γ. Let s be the maximum integer such that these surfaces contain the s^{th} infinitesimal neighbourhood of γ. So $F \in |mH - (s+1)\gamma|$, where H is a plane divisor in $\mathbf{P}^3$. We claim that $s = 0$ and $m = 3$.

The second part of the statement of Lemma 3.5 makes clear that *any secant line of* γ *is transformed by* α *into a line of* Σ. Therefore we must have $m = 2(s+1)+1$; if we intersect one of the surfaces F with the unique quadric surface Q containing γ, by Bezout and $deg(\gamma) = 5$ we get

$$2m = 2[2(s+1)+1] \geq 5(s+1),$$

hence $s \leq 1$.

If $s = 1$ we get $m = 5$ and the surfaces F contain the first infinitesimal neighbourhood of γ. Let $I \subset K[x_0, \ldots, x_3]$ denote the saturated ideal of γ. Since $\gamma \subset \mathbf{P}^3$ is arithmetically Cohen-Macaulay, the saturated ideal of the first infinitesimal neighbourhood of γ is I^2 ([16], 2.3.7). Now, I can be minimally generated by one polynomial q of degree 2 (the equation of Q) and two polynomials of degree 3; therefore, *every homogeneous polynomial of degree* 5 *in* I^2 *must contain* q *as a factor.* So the case $s = 1$ is excluded.

Hence, the linear system defining α is a system of *cubic* surfaces of $\mathbf{P}^3$, containing γ with multiplicity 1. The linear system of all such surfaces defines a rational map $\mathbf{P}^3 \cdots \rightarrow \mathbf{P}^5$, whose image is a Del Pezzo threefold, complete intersection of two quadric hypersurfaces of $\mathbf{P}^5$. This completes the proof of Proposition 3.3. $\square$

Proof of Lemma 3.4 The proof is divided into several steps.

Step 1. *There is a birational map* $\psi: \Sigma \cdots \rightarrow \sigma^{(2)}$, *where* $\sigma^{(2)}$ *denotes the symmetric product of the curve* σ *by itself.*

On Σ there is the algebraic system of curves $\{\sigma(g)\}_{g \in \Sigma}$, of dimension 2. Since $\overline{\mu} = 2$, there are exactly 2 curves of the system containing two fixed general points on Σ; moreover $\sigma(g)^2 = 2$.

The map ψ is defined as follows: let r be a general line of Σ; let a, b be the two lines of Σ intersecting both r and $\overline{g}$. The corresponding points on Σ actually lie on σ. We set $\psi: r \mapsto (a, b)$; it is easily seen that ψ is birational. Note that the map ψ depends on the choice of $\overline{g} \in \Sigma$.

In particular, from Σ irreducible it follows that σ *is also irreducible.*

Step 2. *The characteristic series of the algebraic system* $\{\sigma(g)\}_g$ *on the curve* σ *is a complete* g^1_2. *Therefore also the algebraic system* $\{\sigma(g)\}_g$ *is complete.*

From the fact that the dimension and the degree of the algebraic system $\{\sigma(g)\}_g$ are both 2, it follows at once that the characteristic series has degree 2 and dimension 1, i.e. it is a g^1_2.

Assume it is not complete; then σ is necessarily a rational curve and the characteristic series generates a complete g^2_2. In this case Σ is a rational surface and we can embed $\{\sigma(g)\}_g$ into the complete linear system $|\sigma(g)|$ of dimension 3. Let L be the linear span of $\{\sigma(g)\}_g$ inside $|\sigma(g)|$. Let $\mathcal{L}$ be the linear system of those ruled surfaces on X which correspond to the curves of L. Fix a general point P of X and denote by $\mathcal{M}$ the subsystem of surfaces of $\mathcal{L}$ containing P: $\mathcal{M}$ contains 4 linearly independent surfaces, hence its dimension is at least 3: a contradiction.

Step 3. *Let π denote the geometric genus of σ. Then $\pi \geq 2$.*

By the previous step we already know that $\pi \geq 1$; assume $\pi = 1$. Then, by the well known fact that the irregularity of $\sigma^{(2)}$ equals the (geometric) genus of σ, the irregularity of Σ is 1. But the surface Σ, which parametrizes the curves of $\{\sigma(g)\}_{g\in\Sigma}$, is therefore fibered by a 1-dimensional family of lines, each line representing a linear pencil of curves $\sigma(g)$; from $\sigma(g)^2 = 2$ it follows that every such pencil has 2 base points. This also means that on X we have a 1-dimensional family of linear pencils of elliptic ruled surfaces $\sigma(g)$, each pencil having exactly two base lines.

We fix one of these pencils $\{\sigma(g_t)\}_{t\in\mathbf{P}^1}$, and we let r and r' denote the two base lines. Every surface of the pencil is of the type $\sigma(g)$, with g intersecting both r and r'. Set

$$R := \bigcup_{t\in\mathbf{P}^1} g_t \subset X$$

We claim that, for general $t, t' \in \mathbf{P}^1$, the lines g_t and $g_{t'}$ don't meet on r. Indeed, if $g_t \cap g_{t'} = P \in r$, then also the fourth line of Σ through P would be contained in $\sigma(g_t) \cap \sigma(g_{t'})$, the base locus of the pencil: a contradiction.

So r is a simple unisecant for R. Since $\sigma(r)$ is irreducible, from $R \subseteq \sigma(r)$ it follows that $R = \sigma(r)$. Then we have a contradiction because r has multiplicity 3 on $\sigma(r)$ by Proposition 1.12. Therefore, σ is hyperelliptic of geometric genus $\pi \geq 2$.

To complete the proof of Lemma 3.4 it remains to show:

Step 4. *The genus of γ is 2. In particular, γ is embedded in $\mathbf{P}^3$ with degree 5.*

Let $p \in \overline{g}$ be a general point, and let $a, b, c \in \Sigma$ denote the lines through p, different from $\overline{g}$. Moreover, let $d, e \in \sigma$ be such that $d + e \in g^1_2$ on γ. Then $H := a + b + c + d + e$ is a positive divisor on γ, of degree 5. When p varies on $\overline{g}$, the divisors on γ of type $a + b + c$ are all linearly equivalent because they are parametrized by the rational variety $\overline{g}$. We denote by $\mathcal{D}$ the pencil of such divisors. Since the two rational maps $\gamma \to \mathbf{P}^1$ defined respectively by $\mathcal{D}$ and g^1_2 are clearly different, it is easily seen that $dim\ |H| \geq 3$. Hence, by Clifford's theorem H is non special. Since $\pi \geq 2$, it follows then by Riemann–Roch that $dim\ |H| = 3$, and that $\pi = 2$. Then H is also very ample on γ. □

To prove Lemma 3.5 we need

Lemma 3.6 *$\{I_r\}_{r\in S}$ is an algebraic system of curves on S of dimension 2, degree 2 and index 2.*

Proof Since $deg(\gamma) = 5$ and $\pi = 2$, there are 4 secant lines of γ through a general point of $\mathbf{P}^3$, and 10 secant lines of γ contained in a general plane of $\mathbf{P}^3$. Therefore, the class of S in the Chow group $CH_2(\mathbf{G}(1,3))$ is $4\alpha + 10\beta$, with traditional notations. It follows that the degree of $S \subset \mathbf{P}^5$ is 14; this means that there are 14 secant lines of γ intersecting two general lines r and r' in $\mathbf{P}^3$.

Assume, now, that r and r' are chords of γ, and set $r \cap \gamma = \{A, B\}$, $r' \cap \gamma = \{C, D\}$. To compute $I_r \cdot I_{r'}$ we have just to compute the number of the spurious solutions among these 14 secant lines. Let M be the plane generated by r and C; besides A, B, C the plane M intersects γ at the points P, Q. Therefore, we have the 4 secant lines AC, BC, PC, QC on

M. By repeating this argument for the planes $\langle r \cup D\rangle$, $\langle r' \cup A\rangle$, $\langle r' \cup B\rangle$, we get 16 spurious secant lines, 4 of them have been counted twice. Hence, $I_r \cdot I_{r'} = 2$.

It follows easily that the index of $\{ I_r \}_r$ is also 2. $\square$

Proof of Lemma 3.5 Let us remark first of all that the curves γ and I_r are birational. Indeed let $r \cap \gamma = \{A, B\}$. If $P \in \gamma$, and $P \notin r$, then the plane $\langle r \cup P\rangle$ intersects γ at the points A, B, P, C, D. We get a birational map $f\colon \gamma \to I_r$ by setting $f\colon P \mapsto \overline{CD}$.

We fix now a general secant line r of γ. Starting from the just constructed map f, we can also construct, in a canonical way, a map $f^{(2)}\colon \gamma^{(2)} \to I_r^{(2)}$, which is again birational.

In the first step of the proof of Lemma 3.4 we have constructed a birational map $\psi\colon \Sigma \cdots\to \sigma^{(2)}$. Since γ and σ are birational, we get also a map $\varphi\colon \Sigma \cdots\to \gamma^{(2)}$.

Finally, the algebraic system $\{ I_r \}_{r\in S}$ allows us to construct a birational map $\chi\colon I_r^{(2)} \cdots\to S$ as follows. Let a, b be a general pair of secant lines of γ, and assume that each of them intersects r. By Lemma 3.6 we have $I_a \cdot I_b = 2$; one of these intersections is r, the other one is, by definition, $\chi(a, b)$.

If we compose φ, f and χ we get the desired map $\tau\colon \Sigma \cdots\to S$.

It remains to show that $\tau(\sigma(g)) = I_{\tau(g)}$. Consider a curve $\sigma(g)$ such that g intersects $\overline{g}$. It is mapped by φ to the curve on $\gamma^{(2)}$ formed by all the pairs of elements of γ containing g. Therefore, $f^{(2)} \circ \varphi$ sends $\sigma(g)$ to the curve on $I_r^{(2)}$ formed by all the pairs of elements of I_r containing $f(g)$, and clearly χ maps this last curve to $I_{\tau(g)}$. $\square$

Remark 3.7 *Note that, if X is one of the threefolds found in this section with $\mu = 3, 4$, then the Fano scheme Σ of X is actually irreducible.*

4 Every irreducible component Σ_i of Σ has $\mu_i = 1$

In this section we assume that the family of lines Σ on X is reducible and that for every irreducible component Σ_i of Σ we have $\mu_i = 1$.

Note that, from $\mu_i = 1$ for all i and from Theorem 2.1, it follows that $s = \mu \leq 6$.

The case $s = 2$.

Proposition 4.1 *Let $X \subset \mathbf{P}^4$ be a threefold containing two irreducible families of lines Σ_i ($i = 1, 2$) both with $\mu_i = 1$. Assume that X is not a quadric bundle. Then X is a threefold of degree 6 with sectional genus $\pi = 1$, projection of a Fano threefold of $\mathbf{P}^7$, hyperplane section of $\mathbf{P}^2 \times \mathbf{P}^2$ (see [20]).*

Proof If g_1 is a fixed line of Σ_1, then the lines of Σ_2 meeting it generate the *rational* ruled surface $\sigma_2(g_1)$ having g_1 as simple unisecant. Hence Σ_2 results to be a rational surface. Similarly for Σ_1.

There are two possibilities regarding the algebraic system $\{\sigma_2(g_1)\}_{g_1\in\Sigma_1}$, whose dimension is two (because X is not a quadric bundle): either it is already linear, or it can be embedded in a larger linear system of curves in Σ_2, which corresponds to a linear system of rational ruled surfaces on X. We will prove now that the second case can be excluded.

To this end, we reformulate the problem in a slightly different way. We consider the rational map $\phi : X \to \mathbf{P}^r := \mathbf{P}^{H^0(\sigma_2(g_1))^*}$ associated to the complete linear system $\mid \sigma_2(g_1) \mid$. The map ϕ sends a point p to the subsystem formed by the ruled surfaces passing through p. From $\mu_2 = 1$, it follows that ϕ contracts the lines of Σ_2, which are therefore the fibres of ϕ. Hence $\phi(X)$ is a surface S of degree $d = \sigma_2(g_1)^2$. By an argument similar to that of Proposition 1.14, we have that $\deg \sigma_2(g_1) = d + 2$.

The inverse images of the hyperplane sections of S are the surfaces of $\mid \sigma_2(g_1) \mid$, so S is a surface with rational hyperplane sections. We replace now S with a general projection in $\mathbf{P}^3$, so we can apply the theorem of Kronecker–Castelnuovo and we get only three possibilities:

1. $S = \mathbf{P}^2$: in this case the considered algebraic system is already linear and $d = 1$;
2. S is a scroll and $d > 1$;
3. S is a Steiner surface, projection of a Veronese surface, with $d = 4$.

We have to prove that only the first case happens. Assume by contradiction that S is like in 2. or 3. Note that any section of S with a tangent plane is reducible. If S is a scroll, such a section is the union of a line l with a plane curve C of degree $d - 1$. Let π be the arithmetic genus of C. The following relation expresses the arithmetic genus of a reducible plane section of S: $\pi + d - 2 = 0$, so $d = 2$, $\pi = 0$ and S is a quadric. Moreover $\deg(\sigma_2(g_1)) = 4$, so a general ruled surface in the linear system $\mid \sigma_2(g_1) \mid$ is a scroll of type $(1, 3)$ or $(2, 2)$. The case $(1, 3)$ is excluded because every surface of the system should have a unisecant line and our threefold X contains a family of lines of dimension exactly 2. So a general scroll of the system should be of type $(2, 2)$, hence contain a 1-dimensional family of conics. In this case X contains a 4-dimensional family of conics, and a general hyperplane section $X \cap H$ of its contains a 2-dimensional family of conics. By the usual argument, $X \cap H$ is a quadric or a cubic scroll or a Steiner surface: all three possibilities are easily excluded.

We assume now that S is a projection of a Veronese surface. In this case $\deg \sigma_2(g_1) = 6$, so a general ruled surface in the linear system $\mid \sigma_2(g_1) \mid$ is a scroll of type $(2, 4)$ or $(3, 3)$. The reducible plane sections of S are unions of conics and correspond to reducible ruled surfaces on X, unions of two scrolls of degree three. Necessarily they are both of type $(1, 2)$ so each of them contains a family of conics of dimension 2: we conclude as in the previous case.

So we have proved that for both systems of lines $d = 1$, hence $\deg \sigma_2(g_1) = \deg \sigma_1(g_2) = 3$. Also the curves in the Grassmannian $\mathbf{G}(1, 4)$ corresponding to these ruled surfaces have degree 3. So the surface Σ_i (for $i = 1, 2$) contains a linear system of dimension two of rational cubics, with self–intersection one: it defines a birational map from Σ_i to $\mathbf{P}^2$, whose inverse map is defined by a linear system of plane cubic curves. Hence $\deg \Sigma_i \leq 9$ and Σ_i has rational or elliptic hyperplane sections.

Moreover there is a natural birational map between plane sections of X and some hyperplane sections of Σ_i. Precisely, let H be the singular hyperplane section of $\mathbf{G}(1, 4)$, given by lines meeting a plane π: then $\Sigma_i \cap H$ represents lines of Σ_i passing through the points of $X \cap \pi$. Since there is only one line of Σ_i through a general point of X, we get the required birational map between $\Sigma_i \cap H$ and $X \cap \pi$.

We conclude that also the plane sections of X are rational or elliptic curves. In particular a general hyperplane section of X is a surface of $\mathbf{P}^3$ with the same property. The case of rational sections can be excluded using the Kronecker–Castelnuovo theorem as in Proposition 3.2. So a hyperplane section of X is a Del Pezzo surface and X is a (projection of) a Fano threefold. Looking at the list of Fano threefolds we get the proposition. $\square$

The case $s > 2$.

If Σ has three or more components, a new situation can appear, precisely X could be a quadric bundle in more than one way.

For example, if $X = \mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$ (or one of its projections), Σ has three components with $\mu_i = 1$, so that there are three lines passing through any point P of X, one for each of the three systems. The lines of a system Σ_i meeting a fixed line of another system Σ_j fill up a smooth quadric, so the surfaces $\sigma_i(g_j)$ are all quadrics. Moreover the 1-dimensional families $\{\sigma_i(g_j)\}_{g_j \in \Sigma_j}$ and $\{\sigma_j(g_i)\}_{g_i \in \Sigma_i}$ coincide. Hence there are three different structures of quadric bundle on X giving raise to six families of conics in $\mathbf{G}(1,4)$.

Let X be a threefold of $\mathbf{P}^4$ covered by $s \geq 3$ two-dimensional families of lines Σ_i, $i = 1, \ldots, s$. We distinguish the following two cases:

- there exists a pair of indices $(\bar{\imath}, \bar{\jmath})$ such that the family $\{\sigma_{\bar{\imath}}(g_{\bar{\jmath}})\}_{g_{\bar{\jmath}} \in \Sigma_{\bar{\jmath}}}$ has dimension two;
- for all (i,j), $\dim\{\sigma_i(g_j)\}_{g_j \in \Sigma_j} = 1$.

In the first case, we consider only the two components $\Sigma_{\bar{\jmath}}$ and $\Sigma_{\bar{\imath}}$: we can argue on these components as we did in the case $s = 2$, obtaining that X has to be a projection of a Fano threefold. Since there are no Fano threefolds satisfying our assumption, we can exclude the first case.

Therefore, if $s \geq 3$, necessarily the surfaces $\sigma_i(g_j)$ are smooth quadrics for all pair (i,j). To get the classification, our strategy will be the usual one: to fix three of the families of lines and argue with them. Our result is:

Proposition 4.2 *Let X be a threefold of $\mathbf{P}^4$ containing three or more irreducible families of lines Σ_i all with $\mu_i = 1$. Then X is a threefold of degree ≤ 6 with sectional genus $\pi = 1$, projection of $\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$.*

Proof For every pair of indices (i,j) and general $g_j \in \Sigma_j$, the surface $\sigma_i(g_j)$ is a smooth quadric and it is clear that the linear systems $\{\sigma_i(g_j)\}_{g_j \in \Sigma_j}$ and $\{\sigma_j(g_i)\}_{g_i \in \Sigma_i}$ coincide: we call it Σ_{ij}. We want to study the intersection of two quadrics belonging to two families of the form Σ_{ik} and Σ_{jk}, $i \neq j$.

Let us remark first that, if g_j, g_k are two general coplanar lines in Σ_j, Σ_k respectively, then two cases are possible: either the plane $\langle g_j, g_k \rangle$ does contain a line of Σ_i, or it does not. In the first case X is a cubic (Prop. 1.10). So if $\deg X > 3$ and $p \in \sigma_j(g_k)$, $p \notin g_k$, then $p \notin \sigma_i(g_k)$. This immediately implies that $\sigma_j(g_k) \cap \sigma_i(g_k) = g_k$. Let us consider now $\sigma_j(g_k) \cap \sigma_i(g'_k)$: it can be written also as $\sigma_k(g_j) \cap \sigma_k(g_i^t)$ for a fixed $g_j \in \Sigma_j$ and g_i^t varying in a ruling of the second quadric. Now g_j certainly meets all the quadrics of Σ_{ik} and is not contained in any of them, so there exists a $\bar{t}$ such that g_j and $g_i^{\bar{t}}$ meet at a point q. Let $\overline{g_k}$ be the line of Σ_k through q. Then:

$$\sigma_j(g_k) \cap \sigma_i(g'_k) = \sigma_k(g_j) \cap \sigma_k(g_i^{\bar{t}}) = \sigma_j(\overline{g_k}) \cap \sigma_i(\overline{g_k}),$$

so we fall in the previous case. We conclude that two general quadrics of these families meet along a line of the family having the common index.

As a consequence, we have that through a general point p of X there pass one quadric of the family Σ_{ij} and one line of Σ_k.

Now, we embed the $\mathbf{P}^4$ containing X as a subspace of a $\mathbf{P}^7$, and call $Y \subset \mathbf{P}^7$ the image of the Segre embedding $\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1 \to \mathbf{P}^7$. If $Q \subset X$ is a fixed general quadric of the family Σ_{12}, by acting on Y with an element of the projective linear group, we can assume that $Q \subset Y$ as well. Let $L \subset \mathbf{P}^7$ be a linear subspace of dimension 5, in "general position" with respect to X, i.e. $L \cap X$ is a curve. Let Σ_1', Σ_2' and Σ_3' denote the three families of lines on Y; to fix ideas, assume that Q contains lines of the families Σ_1', Σ_2' on Y.

We define a rational map $\alpha: X \backslash L \cdots \to Y$ as follows. Let $p \in X$ be general; then, the line $r \in \Sigma_3$, such that $p \in r$, intersects Q at a single point p'. Let $r' \in \Sigma_3'$ be the line (on Y) containing p'. Set $\alpha(p) := \langle L \cup p \rangle \cap r'$. It is clear that α is birational. Moreover, by considering the case of a hyperplane through L, we see that α takes hyperplane sections of X to hyperplane sections of Y.

There are suitable $\mathbf{P}^3$'s in $\mathbf{P}^7$, let us call M one of them, such that the restriction $\beta: Y \setminus M \cdots \to \mathbf{P}^3$ of the projection $\mathbf{P}^7 \setminus M \cdots \to \mathbf{P}^3$ is birational. The inverse map $\beta^{-1}: \mathbf{P}^3 \cdots \to Y$ is defined by a linear system $|3H_{\mathbf{P}^3} - l_1 - l_2 - l_3|$, where the l_i's are three lines, pairwise skew.

Since α takes hyperplane sections of X to hyperplane sections of Y, the birational map $(\beta \circ \alpha)^{-1}: \mathbf{P}^3 \cdots \to X$ is defined by a linear subsystem of $|3H_{\mathbf{P}^3} - l_1 - l_2 - l_3|$, i.e. X is a projection of $Y = \mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$, and the proof is complete. □

References

[1] E. Arrondo: *Projections of Grassmannians of lines and characterization of Veronese varieties* , J. Algebraic Geom. **8** (1999), no. 1, 85–101.

[2] M. Baldassarri: *Le varietà pluririgate a tre dimensioni*, Rend. Sem. Mat. Univ. Padova, **19** (1950), 172-200.

[3] E. Bompiani: *Sulle varietà a k dimensioni con ∞^k rette*, Rend. Acc. Lincei (8) **1** (1946), 1001-1004

[4] H. Clemens – Ph. Griffiths: *The intermediate Jacobian of the cubic threefold*, Annals of Math., **95** (1972), 281-356.

[5] F. Conforto: Le superficie razionali, Zanichelli, Bologna, 1939

[6] W. Fulton: Intersection theory, Springer – Verlag, 1984.

[7] I.M. Gelfand – M.M. Kapranov –A.V. Zelevinsky: Discriminants, resultants and multidimensional determinants, Birkhäuser, 1994

[8] Ph. Griffiths – J. Harris: Principles of Algebraic Geometry, John Wiley & Sons, 1978.

[9] Ph. Griffiths – J. Harris: *Algebraic geometry and local differential geometry*, Ann. scient. Éc. Norm. Sup., **12** (1979), 355-432.

[10] J. Harris: Algebraic Geometry, Springer – Verlag 1992.

[11] J. Harris: Curves in projective space, Les Presses de L'Université de Montréal, 1982.

[12] S.L. Kleiman: *The transversality of the general translate*, Comp. Math., **28** (1974), 287-297.

[13] J.M. Landsberg: *Is a linear space contained in a submanifold? On the number of derivatives needed to tell*, J. reine angew. Math., **508** (1999), 53-60.

[14] A. Lanteri – M. Palleschi: *Projective manifolds containing many rational curves*, Indiana Univ. Math. J., **36** (1987), 857-865.

[15] A. Nobile:*Variation du genre des fibres d'une famille de courbes*, Semin. Theor. Nombres, Univ. Bordeaux I 1986/1987, Exp. No.10, 8 p. (1987).

[16] C.S. Peterson: *Applications of liaison theory to schemes supported on lines, grow of the deficiency module and low rank vector bundles*, Ph. D. Thesis, Duke University (1994).

[17] E. Rogora:*Metodi proiettivi per lo studio di alcune questioni relative alle varietà immerse*, Ph. D. Thesis, Università di Roma I (1993)

[18] B. Segre: *Sulle V_n contenenti più di $\infty^{n-k} S_k$*, I, Lincei - Rend. Sc. fis. mat. e nat. **5** (1948), 193–197, II, Lincei - Rend. Sc. fis. mat. e nat. **5** (1948), 275–280

[19] B. Segre: *Bertini forms and Hessian matrices*, J. Lond. Math. Soc. **26** (1951), 164–176

[20] M. E. Serpico: *Le varietà di Fano sezioni iperpiane di $\mathbf{P}^2 \times \mathbf{P}^2$*, Le Matematiche, **35** (1980), 113-121.

[21] C. H. Sisam: *On Varieties of Three Dimensions with Six Right Lines through Each Point*, Amer. J. Math., **52** (1930), 607-610.

[22] A. N. Tjurin: *On the intersection of quadrics*, Russian Math. Surveys, **30** (1975), 51-105.

[23] E. Togliatti: *Sulle varietà a k dimensioni contenenti almeno ∞^k rette*, Atti Accad. Torino **57** (1921)

Dipartimento di Scienze Matematiche
Università di Trieste
34127 – Trieste
Italia