arXiv:q-alg/9607016 v2 17 Jul 1996

April 22, 2004

NONCOMMUTATIVE LATTICES AND THE ALGEBRAS OF THEIR CONTINUOUS FUNCTIONS

Elisa Ercolessi[1], Giovanni Landi[2,3], Paulo Teotonio-Sobrinho[2,4]

[1] *Dipartimento di Fisica, Università di Bologna and INFM, Via Irnerio 46, I-40126, Bologna, Italy.*

[2] *The E. Schrödinger International Institute for Mathematical Physics, Pasteurgasse 6/7, A-1090 Wien, Austria.*

[3] *Dipartimento di Scienze Matematiche, Università di Trieste, P.le Europa 1, I-34127, Trieste, Italy and INFN, Sezione di Napoli, Napoli, Italy.*

[4] *Department of Physics, University of Illinois at Chicago, 60607-7059 Chicago, IL, USA and Universidade de Sao Paulo, Instituto de Fisica - DFMA, Caixa Postal 66318, 05389-970, Sao Paulo, SP, Brasil*.*

Abstract

Recently a new kind of approximation to continuum topological spaces has been introduced, the approximating spaces being partially ordered sets (posets) with a finite or at most a countable number of points. The partial order endows a poset with a nontrivial non-Hausdorff topology. Their ability to reproduce important topological information of the continuum has been the main motivation for their use in quantum physics. Posets are truly noncommutative spaces, or *noncommutative lattices*, since they can be realized as structure spaces of noncommutative C^*-algebras. These noncommutative algebras play the same role of the algebra of continuous functions $\mathcal{C}(M)$ on a Hausdorff topological space M and can be thought of as algebras of operator valued functions on posets. In this article, we will review some mathematical results that establish a duality between finite posets and a certain class of C*-algebras. We will see that the algebras in question are all postliminal approximately finite dimensional (AF) algebras.

*Permanent address

1 Introduction

It is well known that the standard discretization methods used in quantum physics (where a manifold is replaced by a lattice of points with the discrete topology) are not able to describe any significant topological attribute of the continuum, this being equally the case for both the local and global properties. For example, there is no nontrivial concept of winding number and hence no way to formulate theories with topological solitons or instantons on these lattices.

A new kind of finite approximation to continuum topological spaces has been first introduced in [1], with the name of posets or partially ordered sets. As we will see in section 3, posets are also T_0 topological spaces and can reproduce important topological properties of the continuum, such as the homology and the homotopy groups, with remarkable fidelity [1, 2]. This ability to capture topological information has been the main motivation for their use in quantum physics in substitution of the ordinary discrete lattices. In [3], quantum mechanics has been formulated on posets and it has been proved that it is possible to study nontrivial topological configurations, such as θ-states for particles on the poset approximations to a circle. Some promising results have also been obtained in the formulation of solitonic field theories [3] as well as of gauge field theories [4].

In [5], the poset approximation scheme has been developed in a novel direction. Indeed, it has been observed that posets are truly noncommutative spaces, or *noncommutative lattices*, since they can be realized as structure spaces (spaces of irreducible representations) of noncommutative C^*-algebras. These noncommutative algebras play the same rôle of the algebra of continuous functions $\mathcal{C}(M)$ on a manifold M and can be thought of as algebras of operator valued functions on posets. This naturally leads to the use of noncommutative geometry [6] (see also [7]) as the tool to rewrite quantum theories on posets and gives a remarkable connection between topologically meaningful finite approximations to quantum physics and noncommutative geometry.

The duality relation between Hausdorff topological spaces and commutative C*-algebras is provided by the Gel'fand-Naimark theorem. There is no analogue of this theorem in the noncommutative setting. In this article, we will review how it is possible to establish a relation between finite posets and a particular class of noncommutative C*-algebras. For such class of algebras the situation is very similar to the commutative case. We will see that the algebras in question are all approximately finite dimensional (AF) postliminal algebras [8, 9], i.e. C^*-algebras that can be approximated in norm by a sequence of finite dimensional matrix algebras and whose irreducible representations are completely characterized by the kernels. This is exactly what makes them of some interest in mathematics: they present virtually all the attributes and complications of other infinite dimensional algebras, but many techniques and results valid in the finite dimensional case can be used in their study. Thus, for example, a complete classification of AF C^*-algebras is available [10].

These algebras have been first extensively studied by Bratteli [8], who also introduced a diagrammatic representation which is very useful for the study of their algebraic properties. In particular we will see how to use the Bratteli diagram of an AF algebra to construct its structure space. Then we will see how, given any finite poset P, it is possible to construct the Bratteli diagram of an AF algebra whose structure space is the given poset P. Being noncommutative, this AF C^*-algebras is far from being unique. Indeed there is a whole family of AF algebras that have P as structure space and that can be classified by means of results due to Behncke and Leptin [11].

In this article, we will not present the classification of AF C^*-algebras that can be formulated in terms of their algebraic K-theory [10]. In view of their relation with posets, this would represent also a first step in the construction of bundles and characteristic classes over noncommutative lattices. This is indeed the content of [12], to which we refer the interested reader for a detailed analysis of the K-theory of AF algebras.

This article is organized as follows. In section 2 we review some elementary algebraic concepts as well as the Gel'fand Naimark theorem in order to clarify the notation and the terminology used in the sequel. In section 3 we briefly describe the topological approximation of continuous spaces that leads to partially ordered sets (posets). In section 4, AF C^*-algebras are introduced and the connection between Bratteli diagrams and posets is discussed in detail. Finally, in section 5 we present the classification theorem of the AF C^*-algebras that have a poset as structure space due to Behncke and Leptin. Several interesting examples will be examined throughout the article.

2 C^*-algebras and Structure Spaces

Let us start with some elementary algebraic preliminaries [13, 14] that will be also useful to set the notation.

In the sequel, $\mathcal{A}$ will always denote a C^*-algebra over the field of complex numbers $\mathbb{C}$. We remind that this means that $\mathcal{A}$ is equipped with a norm of algebra $||\cdot|| : \mathcal{A} \to \mathbb{C}$ (with respect to which $\mathcal{A}$ is complete) and an involution $^* : \mathcal{A} \to \mathcal{A}$, satisfying the identity:

$$\| a^* a \| = \| a \|^2 , \quad \forall\, a \in \mathcal{A} . \tag{2.1}$$

The following are examples of commutative and noncommutative C^*-algebras which will be used in the article:

1) the (noncommutative) algebra $\mathbb{M}(n, \mathbb{C})$ of $n \times n$ complex matrices T, with T^* given by the hermitian conjugate of T and the squared norm $\| T \|^2$ being equal to the largest eigenvalue of T^*T;

2) the (noncommutative) algebra $\mathcal{B}(\mathcal{H})$ of bounded operators B on a separable (infinite-dimensional) Hilbert space $\mathcal{H}$ as well as its subalgebra $\mathcal{K}(\mathcal{H})$ of compact op-

erators. Now * is given by the adjoint and the norm is the operator norm: $||B|| = \sup_{||\xi||\leq 1} ||B\xi||$ $(\xi \in \mathcal{H})$.

3) the (commutative) algebra $\mathcal{C}(M)$ of continuous functions on a compact Hausdorff topological space M, with * denoting complex conjugation and the norm given by the supremum norm, $\| f \|_\infty = \sup_{x\in M} |f(x)|$. If M is not compact but only locally compact, then one can consider the algebra $\mathcal{C}_0(M)$ of functions vanishing at infinity.

Notice that $\mathcal{K}(\mathcal{H})$ and $\mathcal{C}_0(M)$ (with M only locally compact) are examples of C^*-algebras without unit $\mathbb{I}$, contrary to $\mathbb{M}(n, \mathbb{C})$ and $\mathcal{B}(\mathcal{H})$.

2.1 Commutative C^*-algebras : The Gel'fand-Naimark Theorem

In the third example above we have seen how it is possible to associate a commutative C^*-algebra with (without) unit, namely $\mathcal{C}(M)$ $(\mathcal{C}_0(M))$, to any Hausdorff compact (locally compact) topological space M. Viceversa, given *any* commutative C^*-algebra $\mathcal{C}$ with (without) unit, it is possible to construct a Hausdorff compact (locally compact) topological space M such that $\mathcal{C}$ is isometrically $*$-isomorphic to the the algebra of continuous functions $\mathcal{C}(M)$ $(\mathcal{C}_0(M))$. This is precisely the content of the Gel'fand-Naimark theorem [14] that will be discussed in this paragraph. For simplicity we will consider only the case when $\mathcal{C}$ is a commutative C^*-algebra with unit.

Given such a $\mathcal{C}$, we let $\widehat{\mathcal{C}}$ denote the *structure space* of $\mathcal{C}$, namely the space of equivalence classes of irreducible *-representations (IRR's) of $\mathcal{C}$. The trivial IRR given by $\mathcal{C} \to \{0\}$ is not included in M and will therefore be ignored here and hereafter. Since the C^*-algebra $\mathcal{C}$ is commutative, every IRR is one-dimensional, i.e. is a (non-zero) linear functional $\phi : \mathcal{C} \to \mathbb{C}$ satisfying $\phi(ab) = \phi(a)\phi(b)$ and $\phi(a^*) = \overline{\phi(a)}$, $\forall a, b \in \mathcal{C}$. It follows that $\phi(\mathbb{I}) = 1, \forall \phi \in \widehat{\mathcal{C}}$. The space $\widehat{\mathcal{C}}$ is made into a topological space by endowing it with the *Gel'fand topology*, namely with the topology of pointwise convergence on $\mathcal{C}$. Then $\mathcal{C}$ can be proved to be a Hausdorff compact topological space.

For a commutative C^*-algebra, two-irreducible representations are unitarily equivalent if and only if they have the same kernel. Thus one can consider also the space of kernels of IRR's, the so called *primitive spectrum* $Prim\mathcal{C}$. Now, these kernels are maximal ideals of $\mathcal{C}$ and, viceversa, any maximal ideal is the kernel of an irreducible representation [14]. Indeed, suppose that $\phi \in \widehat{\mathcal{C}}$, then $Ker(\phi)$ is of codimension 1 and so is a maximal ideal of $\mathcal{C}$. Conversely, suppose that I is a maximal ideal of $\mathcal{C}$, then the natural representation of $\mathcal{C}$ on $\mathcal{C}/I$ is irreducible, hence one-dimensional. It follows that $\mathcal{C}/I \cong \mathbb{C}$, so that the quotient homomorphism $\mathcal{C} \to \mathcal{C}/I$ can be identified with an element $\phi \in \widehat{\mathcal{C}}$. Clearly, $I = Ker(\phi)$. Thus $Prim\mathcal{C}$ can be thought of as the space of maximal ideals. As such, $Prim\mathcal{C}$ is equipped with the *Jacobson* or *hull kernel topology*, that will be described in the next paragraph.

The map that to each class of unitary representations associates its kernel gives a map $\widehat{\mathcal{C}} \to Prim\mathcal{C}$, which turns out to be a homeomorphism of the two topological spaces so that we can equivalently talk of the structure space or of the primitive spectrum of a commutative C^*-algebra.

If $c \in \mathcal{C}$, the *Gel'fand transform* $\hat{c}$ of c is the complex-valued function on $\widehat{\mathcal{C}}$ given by

$$\hat{c}(\phi) = \phi(c) \ , \quad \forall \phi \in \widehat{\mathcal{C}} \ . \tag{2.2}$$

It is clear that $\hat{c}$ is continuous for each c. We thus get the interpretation of elements in $\mathcal{C}$ as $\mathbf{C}$-valued continuous functions on $\widehat{\mathcal{C}}$. The Gel'fand-Naimark theorem states that all continuous functions on $\widehat{\mathcal{C}}$ are indeed of the form (2.2) for some $c \in \mathcal{C}$ [14]:

Proposition 2.1 *Let $\mathcal{C}$ be a commutative C^*-algebra. Then the Gel'fand transform map $c \mapsto \hat{c}$ is an isometric $*$-isomorphism of the C^*-algebra $\mathcal{C}$ onto the C^*-algebra $\mathcal{C}(\widehat{\mathcal{C}})$ (equipped with the supremum norm $\| \cdot \|_\infty$).*

Suppose now that M is a compact topological space. We have a natural C^*-algebra, $\mathcal{C}(M)$, associated to it. It is then natural to ask what is the relation between the Gel'fand space $\widehat{\mathcal{C}(M)}$ and M itself. It turns out that this two spaces can be identified both setwise and topologically. We notice first that each $m \in M$ gives a complex homomorphism $\phi_m \in \widehat{\mathcal{C}(M)}$ through the evaluation map:

$$\phi_m : \mathcal{C}(M) \to \mathbb{C} \ , \quad \phi_m(f) = f(m) \ . \tag{2.3}$$

Let I_m denote the kernel of ϕ_m, namely the maximal ideal of $\mathcal{C}(M)$ consisting of all functions vanishing at m. We have the following theorem [14]:

Proposition 2.2 *The map $\Phi : m \mapsto \phi_m$ given by (2.3) is a homeomorphism of M onto $\widehat{\mathcal{C}(M)}$, namely $M \cong \widehat{\mathcal{C}(M)}$. Moreover, every maximal ideal of $\mathcal{C}(M)$ is of the form I_m for some $m \in M$.*

In conclusion, the previous two theorems set up a one-to-one correspondence between the $*$-isomorphism classes of commutative C^*-algebras and the homeomorphism classes of locally compact Hausdorff spaces. If $\mathcal{C}$ has a unit, then $\hat{\mathcal{C}}$ and $Prim\mathcal{C}$ are compact.

2.2 Noncommutative Algebras and Associated Spaces

The scheme described in the previous section cannot be directly generalized to noncommutative C^*-algebras. There is more than one candidate for the analogue of the topological space M. In particular, since non-equivalent unitary transformations may now have the same kernel, we have to distinguish even setwise between:

1) the *structure space* $\widehat{\mathcal{A}}$ of $\mathcal{A}$ or the space of all unitary equivalence classes of irreducible *-representations;

2) the *primitive spectrum* $Prim\mathcal{A}$ of $\mathcal{A}$ or the space of kernels of irreducible *-representations. Any element of $Prim\mathcal{A}$ is automatically a two-sided *-ideal of $\mathcal{A}$.

One can define a natural topology on both $\widehat{\mathcal{A}}$ and $Prim\mathcal{A}$. While for a commutative C^*-algebra the resulting topological spaces are homeomorphic, this is no longer true in the noncommutative case. For instance, in section (4.1) we will describe a C^*-algebra $\mathcal{A}$ associated to the Penrose tiling of the plane [6], whose structure space $\widehat{\mathcal{A}}$ consists of an infinite countable set of points, whereas $Prim\mathcal{A}$ consists of a single point.

The topology one puts on $\widehat{\mathcal{A}}$ is called *regional topology* [14] and is a generalization of the pointwise convergence we have used in the previous paragraph, to which it reduces in the commutative case. This topology is constructed by defining a basis of neighborhoods for the points (classes of representations) of $\widehat{\mathcal{A}}$ as follows. Given a $T \in \widehat{\mathcal{A}}$, let us denote with $\mathcal{H}_T$ the Hilbert space of the representation T. Then an open neighborhood of T is identified by a finite sequence $\xi_1, \xi_2, \cdots, \xi_n$ of vectors in $\mathcal{H}_T$, a positive number ϵ and a finite not void set $\mathcal{F} \subset \mathcal{A}$ by means of:

$$\begin{aligned} &U(T; \epsilon; \xi_1, \xi_2, \cdots, \xi_n; \mathcal{F}) =: \{T' \in \widehat{\mathcal{A}} \;:\; \exists\, \xi'_1, \xi'_2, \cdots, \xi'_n \in \mathcal{H}_{T'} \text{ with} \\ &|(\xi'_i, \xi'_j)_{\mathcal{H}_{T'}} - (\xi_i, \xi_j)_{\mathcal{H}_T}| < \epsilon \;,\; |(T'(a)\xi'_i, \xi'_j)_{\mathcal{H}_{T'}} - (T(a)\xi_i, \xi_j)_{\mathcal{H}_T}| < \epsilon \\ &\text{for } i, j = 1, 2, \cdots, n \text{ and } \forall a \in \mathcal{F}\} \;. \end{aligned} \tag{2.4}$$

On $Prim\mathcal{A}$ we instead define a closure operation as follows [13, 14]. Given any subset W of $Prim\mathcal{A}$, the closure $\overline{W}$ of W is by definition the set of all elements in $Prim\mathcal{A}$ containing the intersection of the elements of W, namely

$$\overline{W} = \{I \in Prim\mathcal{A} : \bigcap W \subset I\} \;. \tag{2.5}$$

This 'closure operation' satisfies the Kuratowski axioms [15] and thus defines a topology on $Prim\mathcal{A}$, which is called *Jacobson* or *hull-kernel topology*. With respect to this topology we have:

Proposition 2.3 *Let W be a subset of $Prim\mathcal{A}$. Then W is closed if and only if W is exactly the set of primitive ideals containing some subset of $\mathcal{A}$.*

Proof. If W is closed, by 2.5, W is the set of primitive ideals containing $\bigcap W \subset I$. Conversely, let $V \subseteq \mathcal{A}$. If W is the set of primitive ideals of $\mathcal{A}$ containing V, then $V \subseteq \bigcap W \subset I$, for all $I \in W$, so that $\overline{W} \subset W$, and $\overline{W} = W$.

In general $\widehat{\mathcal{A}}$ and $Prim\mathcal{A}$ fail to be Hausdorff (or T_2). Recall [15] that a topological space is called T_0 if for any two distinct points of the space there is a neighborhood of one of the two points which does not contain the other. It is called T_1 if any point of the space is closed. It is called T_2 if there exist disjoint neighborhoods of any two points. Whereas nothing can be said on the separability properties of $\widehat{\mathcal{A}}$, it turns out that $Prim\mathcal{A}$ is always a T_0 space and that it is T_1 if and only if all primitive ideals in $\mathcal{A}$ are maximal, as it is established by the following propositions [13, 14].

Proposition 2.4 *The space $Prim\mathcal{A}$ is a T_0 space.*

Proof. *Suppose I_1 and I_2 are two distinct points of $Prim\mathcal{A}$ so that say $I_1 \not\subset I_2$. Then the set of those $I \in Prim\mathcal{A}$ which contain I_1 is a closed subset W (by proposition 2.3) such that $I_1 \in W$ and $I_2 \notin W$. Then its complement W^c is an open set containing I_2 and not I_1.*

Proposition 2.5 *Let $I \in Prim\mathcal{A}$. Then the point $\{I\}$ is closed in $Prim\mathcal{A}$ if and only if I is maximal among primitive ideals.*

Proof. *Indeed $\overline{\{I\}}$ is just the set of primitive ideals of $\mathcal{A}$ containing I.*

One should be aware of the fact that, for a general algebra $\mathcal{A}$, a maximal ideal needs not be primitive. This is however always the case if $\mathcal{A}$ admits a unit [13].

As in the commutative case, both $\widehat{\mathcal{A}}$ and $Prim\mathcal{A}$ are locally compact topological spaces. In addition, if $\mathcal{A}$ has a unit, then they are compact. Notice that, in general, $\widehat{\mathcal{A}}$ being compact does not implies that $\mathcal{A}$ has a unit. For instance, the algebra $\mathcal{K}(\mathcal{H})$ of compact operators on an infinite dimensional Hilbert space $\mathcal{H}$ has no unit but its structure space consists of a single point.

Let us now come to a comparison between the space $\widehat{\mathcal{A}}$ and $Prim\mathcal{A}$. There is a canonical surjection of $\widehat{\mathcal{A}}$ onto $Prim\mathcal{A}$, given by the map that to each IRR π associates its kernel $ker\pi$. The pull-back of the Jacobson topology from $Prim\mathcal{A}$ to $\widehat{\mathcal{A}}$ defines another topology on the latter that turns out to be equivalent to the regional topology defined above [14]. But $\widehat{\mathcal{A}}$ and $Prim\mathcal{A}$ are homeomorphic only under the hypotheses stated below [14].

Proposition 2.6 *Let $\mathcal{A}$ be a C^*-algebra , then the following conditions are equivalent:*

(i) *$\widehat{\mathcal{A}}$ is a T_0 space.*

(ii) *Two irreducible representations of $\widehat{\mathcal{A}}$ with the same kernel are equivalent.*

(iii) The canonical map $\widehat{\mathcal{A}} \longrightarrow Prim\mathcal{A}$ is a homeomorphism.

Proof. *By construction, a subset $\mathcal{S} \in \widehat{\mathcal{A}}$ will be closed if and only if it is of the form $\{\pi \in \widehat{\mathcal{A}} : ker\pi \in W\}$ for some W closed in $Prim\mathcal{A}$. As a consequence, given any two (classes of) representations $\pi_1, \pi_2 \in \widehat{\mathcal{A}}$, the representation π_1 will be in the closure of π_2 if and only if $ker\pi_1$ is in the closure of $ker\pi_2$, or, by proposition 2.5 if and only if $ker\pi_2 \subset ker\pi_1$. In turn, π_1 and π_2 are one in the closure of the other if and only if $ker\pi_2 = ker\pi_1$. Therefore, π_1 and π_2 will not be distinguished by the topology of $\widehat{\mathcal{A}}$ if and only if they have the same kernel. On the other side, if $\widehat{\mathcal{A}}$ is T_0 one is able to distinguish points. It follows that (i) implies (ii), namely that if $\hat{\mathcal{A}}$ is a T_0 space, two representations with the same kernel must be equivalent. The other implications are obvious.*

3 Noncommutative Lattices

For convenience, we will review in this section the content of [1, 2, 3], where it is shown how it is possible to approximate a continuum topological space by means of a finite or countable set of points P [1] which, being equipped with a partial order relation, is a partially ordered set or a *poset* . As explained there, these approximating spaces are able to reproduce important topological properties of the continuum. Moreover, in section 4.1 we will see that any of these spaces can be identified with the space $\widehat{\mathcal{A}} = Prim\mathcal{A}$ of primitive ideals of some (noncommutative) AF algebra $\mathcal{A}$, which thus plays the rôle of the algebra of continuous functions on P [5]. This fact will make any poset a truly noncommutative space [6], hence also the name *noncommutative lattice.*

This is the reason why, in this article, we will consider only a special class of algebras, namely postliminal approximately finite (AF) algebras. In section 4 we will see in detail that AF algebras are approximated in norm by direct sums of finite dimensional matrices [8, 9]. As for postliminal we refer to [13] for the exact definition. For what concerns this article, we need just to know that, as a consequence of general theorems, this implies that $\widehat{\mathcal{A}}$ and $Prim\mathcal{A}$ are homeomorphic. In other words, in the following we will have to deal only with structure spaces (or primitive spectrum spaces) which are T_0 locally compact topological spaces.

3.1 The Finite Topological Approximation

Let M be a continuum topological space. Experiments are never so accurate that they can detect events associated with points of M, rather they only detect events as occurring in certain sets O_λ. It is therefore natural to identify any two points x, y of M if they can never be separated or distinguished by the sets O_λ.

Let us assume that each O_λ is open and that the family $\{O_\lambda\}$ covers M:

$$M = \bigcup_\lambda O_\lambda \ . \tag{3.1}$$

We also assume that $\{O_\lambda\}$ is a topology for M [15]. This implies that both $O_\lambda \cup O_\mu$ and $O_\lambda \cap O_\mu$ are in $\mathcal{U}$ if $O_{\lambda,\mu} \in \mathcal{U}$. This hypothesis is physically consistent because experiments can isolate events in $O_\lambda \cup O_\mu$ and $O_\lambda \cap O_\mu$ if they can do so in O_λ and O_μ separately, the former by detecting an event in either O_λ or O_μ, and the latter by detecting it in both O_λ and O_μ.

Given x and y in M, we write $x \sim y$ if every set O_λ containing either point x or y contains the other too:

$$x \sim y \text{ means } x \in O_\lambda \Leftrightarrow y \in O_\lambda \quad \text{for every} \ \ O_\lambda \ . \tag{3.2}$$

Then $\sim$ is an equivalence relation, and it is reasonable to replace M by $M/\sim \equiv P(M)$ to reflect the coarseness of observations.

We assume that the number of sets O_λ is finite when M is compact so that $P(M)$ is an approximation to M by a finite set in this case. When M is not compact, we assume instead that each point has a neighborhood intersected by only finitely many O_λ, so that $P(M)$ is a "finitary" approximation to M [1]. In the notation we employ, if $P(M)$ has N points, we sometimes denote it by $P_N(M)$.

The space $P(M)$ inherits the quotient topology from M [15], i.e. a set in $P(M)$ is declared to be open if its inverse image for Φ is open in M, Φ being the map from M to $P(M)$ obtained by identifying equivalent points. The topology generated by these open sets is the finest one compatible with the continuity of Φ.

Let us illustrate these considerations for a cover of $M = S^1$ by four open sets as in Figure 1(a). In that figure, $O_1, O_3 \subset O_2 \cap O_4$. Figure 1(b) shows the corresponding discrete space $P_4(S^1)$, the points x_i being images of sets in S^1. The map $\Phi : S^1 \to P_4(S^1)$ is given by

$$\begin{aligned} &O_1 \to x_1, \qquad O_2 \setminus [O_2 \cap O_4] \to x_2 \ , \\ &O_3 \to x_3, \qquad O_4 \setminus [O_2 \cap O_4] \to x_4 \ . \end{aligned} \tag{3.3}$$

The quotient topology for $P_4(S^1)$ can be read off from Figure 1, the open sets being

$$\{x_1\} \ , \ \{x_3\} \ , \ \{x_1, x_2, x_3\} \ , \ \{x_1, x_4, x_3\} \ , \tag{3.4}$$

and their unions and intersections (an arbitrary number of the latter being allowed as $P_4(S^1)$ is finite).

Notice that our assumptions allow us to isolate events in certain sets of the form $O_\lambda \setminus [O_\lambda \cap O_\mu]$ which may not be open. This means that there are in general points in

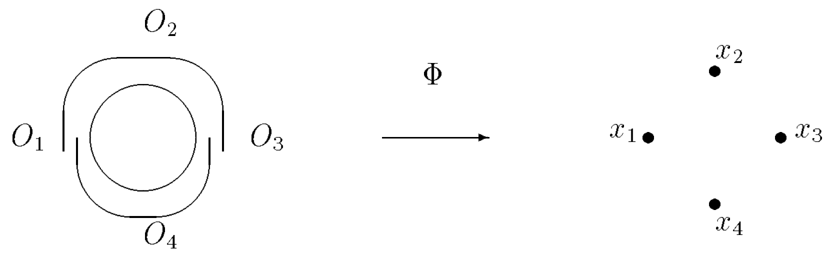

Figure 1:
The covering of S^1 that gives rise to the poset $P_4(S^1)$.

$P(M)$ coming from sets which are not open in M and therefore are not open in the quotient topology. This implies that in general $P(M)$ is neither Hausdorff nor T_1. However, it can be shown [1] that it is always a T_0 space. For example, given the points x_1 and x_2 of $P_4(S^1)$, the open set $\{x_1\}$ contains x_1 and not x_2, but there is no open set containing x_2 and not x_1.

We will see now how the topological properties of $P(M)$ can also be encoded in a combinatorial structure, namely a partial order relation, that can be defined on it.

Since $P(M)$ is finite (finitary), its topology is generated by the smallest open neighborhoods O_x of its points x. It is possible to introduce a partial order relation $\preceq$ [2, 16] by declaring that:

$$x \preceq y \;\Leftrightarrow O_x \;\subset O_y \; . \tag{3.5}$$

In this way $P(M)$ becomes a <u>p</u>artially <u>o</u>rdered <u>set</u> or a *poset*.

Later, we will write $x \prec y$ to indicate that $x \preceq y$ and $x \neq y$. A point $x \in P$ such that there exists no $y \in P$ with $x \prec y$ ($x \succ y$) is said to be maximal (minimal). In addition, a set $\{x_1, x_2, \cdots, x_k\}$ of points in P is said to be a chain if x_{j+1} covers x_j ($j = 1, \cdots, k-1$). A chain is maximal if x_1 and x_k are respectively a minimal and a maximal point.

It is easy to read the topology of $P(M)$ once the partial order is given. It is not difficult to check that

$$O_x = \{y \in P(M) \; : \; y \preceq x\} \; .$$

Indeed, one can even prove a stronger result [1, 2], namely that any finite set P on which a partial order $\preceq$ is defined can be made into a finite T_0 topological space by declaring that the smallest open neighborhood O_x containing x is given exactly by the above set.

Throughout this article, we will use 'finite poset' and 'finite T_0 space' interchangeably.

It is convenient to graphically represent a poset by a diagram, the Hasse diagram,

constructed by arranging its points at different levels and connecting them using the following rules [1, 16]:
1. if $x \prec y$, then x is at a lower level than y;
2. if $x \prec y$ and there is no z such that $x \prec z \prec y$, then x is at a level immediately below y and these two points are connected by a line called a link.

Let us consider $P_4(S^1)$ again. The partial order reads

$$x_1 \preceq x_2 \ , \quad x_1 \preceq x_4 \ , \quad x_3 \preceq x_2 \ , \quad x_3 \preceq x_4 \ , \tag{3.6}$$

where we have omitted writing the relations $x_j \preceq x_j$. The corresponding Hasse diagram is shown in Figure 2.

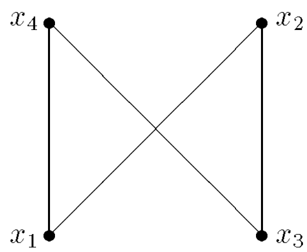

Figure 2:
The Hasse diagram for the circle poset $P_4(S^1)$.

In the language of partially ordered sets, the smallest open set O_x containing a point $x \in P(M)$ consists of all y preceding x: $O_x = \{y \in P(M) \ : \ y \preceq x\}$. In the Hasse diagram, it consists of x and all points we encounter as we travel along links from x to the bottom. In Figure 2, this rule gives $\{x_1, x_2, x_3\}$ as the smallest open set containing x_2, just as in (3.4).

As one example of a three-level poset, consider the Hasse diagram of Figure 3 for a finite approximation $P_6(S^2)$ of the two-dimensional sphere S^2 derived in [1]. Its open sets are generated by

$$\begin{aligned} &\{x_1\} \ , \quad \{x_3\} \ , \quad \{x_1, x_2, x_3\} \ , \quad \{x_1, x_4, x_3\} \ , \\ &\{x_1, x_2, x_3, x_4, x_5\} \ , \quad \{x_1, x_2, x_3, x_4, x_6\} \ , \end{aligned} \tag{3.7}$$

by taking unions and intersections.

One of the most remarkable properties of a poset is its ability to accurately reproduce the homology and the homotopy groups of the Hausdorff topological space it approximates. For example, as for S^1, the fundamental group of $P_N(S^1)$ is $\mathbf{Z}$ whenever $N \geq 4$

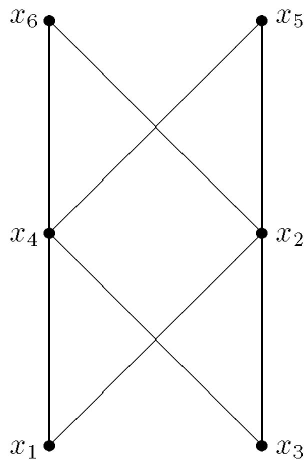

Figure 3:
The Hasse diagram for the sphere poset $P_6(S^2)$.

[1]. Similarly, as for S^2, $\pi_1(P_6(S^2)) = \{0\}$ and $\pi_2(P_6(S^2)) = \mathbf{Z}$. This has been widely discussed in our previous work [3, 5], where we argued that global topological information relevant for quantum physics can be captured by such discrete approximations. Furthermore, the topological space being approximated can be recovered by considering a sequence of finer and finer coverings, the appropriated framework being that of projective systems of topological spaces. We refer to [1, 17] for details.

In this article we are however mostly concerned with the algebraic properties of a poset, i.e. with the fact that any finite poset can be regarded as the structure space of a C^*-algebra. This will be extensively discussed and proved in the following sections, but let us first illustrate a simple example. Consider the C^*-algebra :

$$\mathcal{A} = \{\lambda_1 \mathbb{I}_1 + \lambda_2 \mathbb{I}_2 + k_{12} \ : \ \lambda_j \in \mathbf{C}, k_{12} \in \mathcal{K}_{12}\} \tag{3.8}$$

acting on the direct sum of two Hilbert spaces $\mathcal{H} = \mathcal{H}_1 \bigoplus \mathcal{H}_2$ and generated by multiples of the identity $\mathbb{I}_1$ on $\mathcal{H}_1$, multiples of the identity $\mathbb{I}_2$ on $\mathcal{H}_2$ and compact operators $\mathcal{K}_{12}$ on the whole Hilbert space $\mathcal{H}$. This algebra admits only three classes of irreducible representations, two finite dimensional ones and an infinite dimensional one:
1. $\pi_1 : \lambda_1 \mathbb{I}_1 + \lambda_2 \mathbb{I}_2 + k_{12} \mapsto \lambda_1$,
2. $\pi_2 : \lambda_1 \mathbb{I}_1 + \lambda_2 \mathbb{I}_2 + k_{12} \mapsto \lambda_2$,

3. $\rho : \lambda_1 I\!I_1 + \lambda_2 I\!I_2 + k_{12} \mapsto \lambda_1 I\!I_1 + \lambda_2 I\!I_2 + k_{12}$.

Hence the corresponding structure space consists of only three points $p_1 = ker\pi_1, p_2 = ker\pi_2, q = ker\rho$, corresponding respectively to the three representations given above. This space has to be given the Jacobson topology as explained in the previous section. This is easily done if one notices that, the space being finite, this amounts to give a partial order relation on the set $\{p_1, p_2, q\}$ [14]. Indeed one can show that, on any finite structure space $Prim\mathcal{A}$ of a C^*-algebra $\mathcal{A}$, the Jacobson topology is equivalent to the following partial order relation:

$$p_j \prec p_k \;\Leftrightarrow\; ker\pi_j \subset ker\pi_k \;, \tag{3.9}$$

where p_j is the point in $Prim\mathcal{A}$ corresponding to the IRR π_j of $\mathcal{A}$. Thus in our example, since $ker\rho \subset ker\pi_1$ and $ker\rho \subset ker\pi_2$, the set $\{p_1, p_2, q\}$ is equipped with the order relations $q \prec p_1, q \prec p_2$ and therefore corresponds to the poset of Figure 4, which will be referred to as the $\bigvee$ poset from now on.

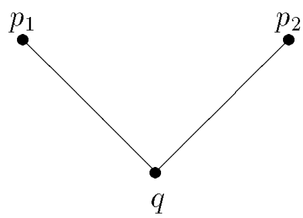

Figure 4:
The $\bigvee$ poset, structure space of $\mathcal{A} = \mathbf{C}I\!I_1 + \mathbf{C}I\!I_2 + \mathcal{K}_{12}$.

4 AF algebras

4.1 Bratteli diagrams

A C^*-algebra $\mathcal{A}$ is said to be *approximately finite dimensional* (AF) [8, 9] if there exists an increasing sequence

$$\mathcal{A}_0 \overset{I_0}{\hookrightarrow} \mathcal{A}_1 \overset{I_1}{\hookrightarrow} \mathcal{A}_2 \overset{I_2}{\hookrightarrow} \mathcal{A}_3 \overset{I_2}{\hookrightarrow} \cdots \overset{I_{n-1}}{\hookrightarrow} \mathcal{A}_n \overset{I_n}{\hookrightarrow} \cdots \tag{4.10}$$

of finite dimensional subalgebras of $\mathcal{A}$, such that $\mathcal{A}$ is the norm closure of $\cup_n \mathcal{A}_n$. Here the maps I_n are injective *-homomorphisms. In other words, $\mathcal{A}$ is the direct limit in the category of C^*-algebras with morphisms given by *-algebras maps (not isometries) of the sequence $(\mathcal{A}_n)_{n\in\mathbb{N}}$. As a set, $\bigcup_n \mathcal{A}_n$ is made of coherent sequences,

$$\bigcup_n \mathcal{A}_n = \{a = (a_n)_{n\in\mathbb{N}} \ , a_n \in \mathcal{A}_n \mid \exists N_0 \ : \ a_{n+1} = I_n(a_n) \ , \forall \ n > N_0\}. \tag{4.11}$$

Now the sequence $(||a_n||_{\mathcal{A}_n})_{n\in\mathbb{N}}$ is eventually decreasing, since $||a_{n+1}|| \leq ||a_n||$ (the maps I_n are norm decreasing) and therefore convergent. One writes for the norm

$$||(a_n)_{\mathbb{N}}|| = \lim_{n\to\infty} ||a_n||_{\mathcal{A}_n} \ . \tag{4.12}$$

Since the maps I_n are injective, the expression (4.12) gives directly a true norm and not simply a seminorm and there is no need to quotient out the zero norm elements.

Each subalgebra $\mathcal{A}_n$, being a finite dimensional C^*-algebra, is a matrix algebra and therefore can be written as $\mathcal{A}_n = \bigoplus_{k=1}^{N_n} \mathbb{M}^{(n)}(d_k, \mathbb{C})$ where $\mathbb{M}^{(n)}(d_k, \mathbb{C})$ is the algebra of $d_k \times d_k$ matrices with complex coefficients. Given any two such matrix algebras $\mathcal{A}_1 = \bigoplus_{j=1}^{N_1} \mathbb{M}^{(1)}(d_j, \mathbb{C})$ and $\mathcal{A}_2 = \bigoplus_{k=1}^{N_2} \mathbb{M}^{(2)}(d_k, \mathbb{C})$ with $\mathcal{A}_1 \hookrightarrow \mathcal{A}_2$, one can always choose suitable bases in $\mathcal{A}_1$ and $\mathcal{A}_2$ such that $\mathcal{A}_1$ is identified with a subalgebra of $\mathcal{A}_2$ of the following form [8]:

$$\mathcal{A}_1 \simeq \bigoplus_{k=1}^{N_2} \left(\bigoplus_{j=1}^{N_1} N_{kj} \mathbb{M}^{(1)}(d_j, \mathbb{C}) \right) \ . \tag{4.13}$$

Here, for any nonnegative integers p and q, the symbol $p\,\mathbb{M}(q, \mathbb{C})$ stands for $\mathbb{M}(q, \mathbb{C}) \otimes \mathbb{I}_p$. In (4.13), the coefficients N_{kj} represent the multiplicity of the partial embedding of $\mathbb{M}^{(1)}(d_j, \mathbb{C})$ in $\mathbb{M}^{(2)}(d_k, \mathbb{C})$ and satisfy the condition

$$\sum_{j=1}^{N_1} N_{kj} d_j = d_k \ . \tag{4.14}$$

A useful way to represent the algebras $\mathcal{A}_1$, $\mathcal{A}_2$ and the embedding $\mathcal{A}_1 \hookrightarrow \mathcal{A}_2$ is by means of a diagram, the *Bratteli diagram* [8], which can be constructed out of the dimensions, d_j $(j = 1, \ldots, N_1)$ and d_k $(k = 1, \ldots, N_2)$, of the diagonal blocks of the two algebras and the numbers N_{kj} that describe the partial embeddings. To construct the diagram, we draw two horizontal rows of vertices, the top (bottom) one representing $\mathcal{A}_1$ ($\mathcal{A}_2$) and consisting of N_1 (N_2) vertices, labeled by the corresponding dimensions $d_1, \ldots, d_{N_1}$ $(d_1, \ldots, d_{N_2})$. Then for each $j = 1, \ldots, N_1$ and $k = 1, \ldots, N_2$, we draw N_{kj} edges between d_j and d_k. We will also write $d_j^{(1)} \searrow^{N_{kj}} d_k^{(2)}$ to denote the fact that $\mathbb{M}(d_j^{(1)}, \mathbb{C})$ is embedded in $\mathbb{M}(d_k^{(2)}, \mathbb{C})$ with multiplicity N_{kj}. By repeating the procedure at each level, we obtain a semi-infinite diagram denoted by $\mathcal{D}(\mathcal{A})$ which completely defines $\mathcal{A}$ up to isomorphisms. Notice that the diagram $\mathcal{D}(\mathcal{A})$ depends not only on $\mathcal{A}$ but also on the particular sequence $\{\mathcal{A}_n\}_{n\in\mathbb{N}}$ which generates $\mathcal{A}$. However, it is possible to show [8] that all diagrams corresponding to

AF algebras which are isomorphic to $\mathcal{A}$ can be obtain from the chosen $\mathcal{D}(\mathcal{A})$ by means of an algorithm.

As an example of an AF algebra, let us consider the subalgebra $\mathcal{A}$ of the algebra $\mathcal{B}(\mathcal{H})$ of bounded operators on $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ given in (3.8). This C^*-algebra algebra can be obtained as the direct limit of the following sequence of finite dimensional algebras:

$$
\begin{aligned}
&\mathcal{A}_0 = \mathbb{M}(1, \mathbb{C}) \\
&\mathcal{A}_1 = \mathbb{M}(1, \mathbb{C}) \oplus \mathbb{M}(1, \mathbb{C}) \\
&\mathcal{A}_2 = \mathbb{M}(1, \mathbb{C}) \oplus \mathbb{M}(2, \mathbb{C}) \oplus \mathbb{M}(1, \mathbb{C}) \\
&\vdots \\
&\mathcal{A}_n = \mathbb{M}(1, \mathbb{C}) \oplus \mathbb{M}(2n-2, \mathbb{C}) \oplus \mathbb{M}(1, \mathbb{C}) \\
&\vdots
\end{aligned}
$$

where, for $n \geq 1$, $\mathcal{A}_n$ is embedded in $\mathcal{A}_{n+1}$ as the subalgebra $\mathbb{M}(1, \mathbb{C}) \oplus [\mathbb{M}(1, \mathbb{C}) \oplus \mathbb{M}(2n-2, \mathbb{C}) \oplus \mathbb{M}(1, \mathbb{C})] \oplus \mathbb{M}(1, \mathbb{C})$:

$$
a_n = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & m_{2n-2 \times 2n-2} & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix} \hookrightarrow \begin{bmatrix} \lambda_1 & 0 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 & 0 \\ 0 & 0 & m_{2n-2 \times 2n-2} & 0 & 0 \\ 0 & 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & 0 & \lambda_2 \end{bmatrix} . \tag{4.15}
$$

It is therefore described by the diagram of Figure 5.

As a second example, consider the C^*-algebra of the Penrose tiling. This is an example of an AF algebra which is not postliminal, since this algebra admits an infinite number of nonequivalent representations all with the same kernel. At each level, the finite dimensional algebra is given by [6]

$$
\mathcal{A}_n = \mathbb{M}(d_n, \mathbb{C}) \oplus \mathbb{M}(d'_n, \mathbb{C}) \ , \quad n \geq 1 \ , \tag{4.16}
$$

with inclusion $\mathcal{A}_n \hookrightarrow \mathcal{A}_{n+1}$:

$$
\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \hookrightarrow \begin{bmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & A \end{bmatrix} \ ; \ A \in \mathbb{M}(d_n, \mathbb{C}) \ , \quad B \in \mathbb{M}(d'_n, \mathbb{C}) \ , \tag{4.17}
$$

so that $d_{n+1} = d_n + d'_n$ and $d'_{n+1} = d_n$. The corresponding Bratteli diagram is shown in Figure 6.

To conclude this section we remark that an AF algebra is commutative if and only if all its factors $\mathbb{M}^{(n)}(d_k, \mathbb{C})$ are one dimensional, i.e. they are just $\mathbb{C}$. Thus the corresponding diagram has the property that for each $\mathbb{M}^{(n)}(d_k, \mathbb{C})$ with $n \geq 1$ there is exactly one $\mathbb{M}^{(n-1)}(d_j, \mathbb{C})$ and $\mathbb{M}^{(n-1)}(d_j, \mathbb{C}) \searrow^{p_{kj}} \mathbb{M}^{(n)}(d_k, \mathbb{C})$ with $p_{kj} = 1$. An interesting example is given in Figure 7, which corresponds to the AF C^*-algebra of continuous functions on the Cantor set [10].

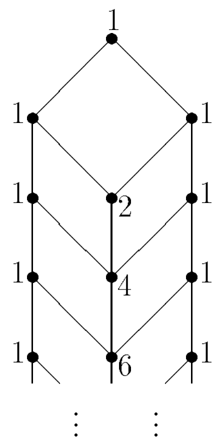

Figure 5:
The Bratteli diagram corresponding to $\mathcal{A} = \mathbb{C}I\!I_1 + \mathbb{C}I\!I_2 + \mathcal{K}_{12}$.

4.2 From Bratteli Diagrams to Posets

The Bratteli diagram $\mathcal{D}(\mathcal{A})$ of an AF algebra $\mathcal{A}$ is useful not only because it gives the finite approximations of the algebra explicitly, but also because it is possible to read the ideals and the primitive ideals of the algebra (hence the topological properties of $Prim\mathcal{A}$) out of it very easily. Indeed one can show that the following proposition holds [8]:

Proposition 4.1

1. *There is a one-to-one correspondence between the proper ideals $\mathcal{I}$ of $\mathcal{A}$ and the subsets $\Lambda = \Lambda_{\mathcal{I}}$ of the Bratteli diagram satisfying the following two properties:*
   i) if $\mathbb{M}^{(n)}(d_k, \mathbb{C}) \in \Lambda$ and $\mathbb{M}^{(n)}(d_k, \mathbb{C}) \searrow \mathbb{M}^{(n+1)}(d_j, \mathbb{C})$ then necessarily $\mathbb{M}^{(n+1)}(d_j, \mathbb{C})$ belongs to Λ;
   ii) if all factors $\mathbb{M}^{(n+1)}(d_j, \mathbb{C})$ ($j = \{1, 2, \cdots, N_{n+1}\}$), for which $\mathbb{M}^{(n)}(d_k, \mathbb{C}) \searrow \mathbb{M}^{(n+1)}(d_j, \mathbb{C})$, belong to Λ, then $\mathbb{M}^{(n)}(d_k, \mathbb{C}) \in \Lambda$.

2. *A proper ideal $\mathcal{I}$ of $\mathcal{A}$ is primitive if and only if the associated subdiagram $\Lambda_{\mathcal{I}}$ satisfies:*
   iii) $\forall n$ there exists an $\mathbb{M}^{(m)}(d_j, \mathbb{C})$, with $m > n$, not belonging to $\Lambda_{\mathcal{I}}$ such that, for all $k \in \{1, 2, \cdots, N_n\}$ with $\mathbb{M}^{(n)}(d_k, \mathbb{C})$ not in $\Lambda_{\mathcal{I}}$, one can find a sequence $\mathbb{M}^{(n)}(d_k, \mathbb{C}) \searrow \mathbb{M}^{(n+1)}(d_\alpha, \mathbb{C}) \searrow \mathbb{M}^{(n+2)}(d_\beta, \mathbb{C}) \searrow \cdots \searrow \mathbb{M}^{(m)}(d_j, \mathbb{C})$.

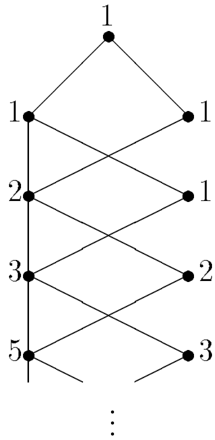

Figure 6:
The Bratteli diagram of the Penrose tiling.

For example, consider the diagram of Figure 5, representing the AF C^*-algebra $\mathcal{A} = \mathbb{C}\mathbb{I}_1 + \mathbb{C}\mathbb{I}_2 + \mathcal{K}_{12}$ already discussed in section 3.1. This algebra contains only three nontrivial ideals, whose diagrams are represented in Figure 8(a,b,c). In this pictures the points belonging to the ideals are marked with a "$\clubsuit$". It is not difficult to check that only $\mathcal{I}_1$ and $\mathcal{I}_2$ are primitive ideals, since $\mathcal{I}_3$ does not satisfy property (iii) above.

We remark the following here:
1) The whole $\mathcal{A}$ is an ideal, which by definition is not primitive since the trivial representation $\mathcal{A} \to 0$ is not irreducible.
2) The set $\{0\} \subset \mathcal{A}$ is an ideal, which is primitive if and only if $\mathcal{A}$ has one irreducible faithful representation. This can also be understood from the Bratteli diagram in the following way. The set $\{0\}$ is not a subdiagram of $\mathcal{D}(\mathcal{A})$, being represented by the element 0 of the matrix algebra of each finite level, so that there is at least one element $a \in \mathcal{A}$ not belonging to the ideal $\{0\}$ at any level. Thus to check if $\{0\}$ is primitive, i.e. to check property (iii) above, we have to examine whether *all* the points at a given level, say n, can be connected to a *single* point at a level $m > n$. For example this is the case for the diagram of Figure 5 and not for that of Figure 7.

Proposition 4.1 above allows us to understand the topological properties of $Prim\mathcal{A}$ at once. This becomes particularly simple if the algebra admits only a finite number of nonequivalent irreducible representations. In this case $Prim\mathcal{A}$ is a T_0 topological space

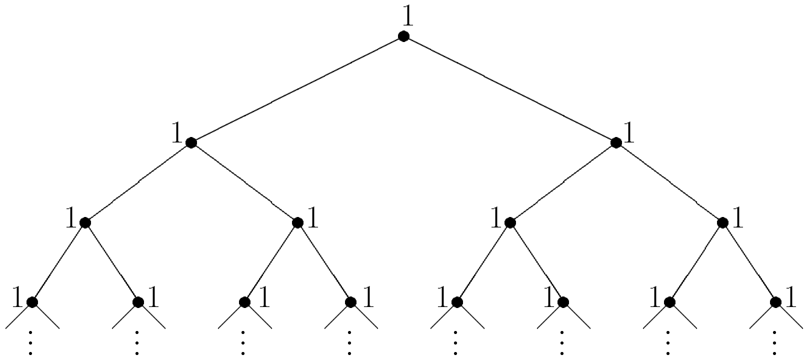

Figure 7:
The Bratteli diagram corresponding to the AF C^*-algebra
of continuous functions on the Cantor set.

with only a finite number of points, hence a finite poset P. To reconstruct the latter we just need to draw the Bratteli diagram $\mathcal{D}(\mathcal{A})$ and find the subdiagrams that, according to properties (i, ii, iii), correspond to primitive ideals. Then P has so many points as the number of primitive ideals and the partial order relation in P that determines the T_0 topology is simply given by the inclusion relations that exist among the primitive ideals.

As an example consider again Figure 5. We have seen that the corresponding AF algebra has only three primitive ideals: the $\{0\}$ ideal and the ideals $\mathcal{I}_1, \mathcal{I}_2$ represented in Figure 8(a),(b). Clearly $\{0\} \subset \mathcal{I}_1, \mathcal{I}_2$ so that $Prim\mathcal{A}$ is the $\bigvee$ poset of Figure 4.

Figure 7 leads to another interesting topological space. As we have mentioned, such a diagram corresponds to a commutative AF algebra $\mathcal{C}$ and hence to a Hausdorff $Prim\mathcal{C}$, which is homeomorphic to the Cantor set.

4.3 From Posets to Bratteli Diagrams

In the preceding subsection we have described the properties of the Bratteli diagram $\mathcal{D}(\mathcal{A})$ of an AF algebra $\mathcal{A}$ and in particular we have seen how, out of it, it is possible to read the primitive ideal space of $\mathcal{A}$, in particular when the latter is a finite poset. In the following we will see that, under some rather mild hypotheses which are always verified in the cases of interest to us, it is possible to reverse the construction and thus build the AF algebra

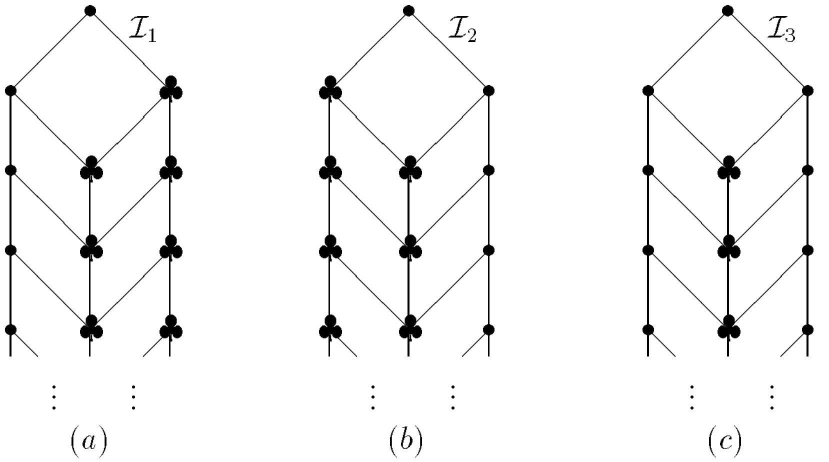

Figure 8:
The representation of the ideals of $\mathcal{A} = \mathbb{C}I\!I_1 + \mathbb{C}I\!I_2 + \mathcal{K}_{12}$
in the corresponding Bratteli diagram.

that corresponds to a given (finite) T_0 topological space.

Such a reconstruction rests on the following theorem of Bratteli [8], which specifies a class of topological spaces which are the primitive ideal spaces of AF algebras:

Proposition 4.2 *A topological space Y is the primitive ideal space $Prim\mathcal{A}$ of an AF algebra $\mathcal{A}$ if it has the following properties:*
i) Y is T_0;
ii) Y contains at most a countable number of closed sets;
iii) if $\{F_n\}_{n\in\Lambda}$, Λ being any direct set, is a decreasing sequence of closed subsets of Y, then $\cap_n F_n$ is an element in $\{F_n\}_{n\in\Lambda}$;
iv) if $F \subset Y$ is a closed set which is not the union of two proper closed subsets, then F is the closure of a one-point set.

It is not difficult to check that all the above conditions hold true if Y is a T_0 topological space with a finite number of points, so that we have the corollary:

Corollary.
A finite poset P is the primitive ideal space $Prim\mathcal{A}$ for some AF algebra $\mathcal{A}$.

Here we will not report the proof of proposition 4.2, which can be found in [8]. However, starting from the techniques used in such a proof, we want to show how one can explicitly find an AF algebra $\mathcal{A}$ whose primitive ideal space is a given finite poset P. First we will give the general construction and then discuss an example.

Let $\{K_1, K_2, K_3, \ldots\}$ be the collection of all closed sets in P, where $K_1 = P$. To construct the n-th level of the Bratteli diagram $\mathcal{D}(\mathcal{A})$, we consider the subcollection of closed sets $\mathcal{K}_n \equiv \{K_1, K_2, \ldots, K_n\}$ and denote with $\mathcal{K}'_n$ the smallest collection of closed sets in P that contains $\mathcal{K}_n$ and is closed under union and intersection. The collection $\mathcal{K}_n$ determines a partition of the topological space P, by taking intersections and complements of the sets $K_j \in \mathcal{K}_n$ $(j = 1, \ldots, n)$. We denote with $Y(n,1)$, $Y(n,2)$, $\ldots$, $Y(n,k_n)$ the sets of such partition. Also, we write $F(n,j)$ for the smallest closed set which contains $Y(n,j)$ and belongs to the subcollection $\mathcal{K}'_n$. Then we can construct a Bratteli diagram following the rules:

1. the n-th level of $\mathcal{D}(\mathcal{A})$ has k_n points, one for each set $Y(n,j)$;

2. the point at the level n of the diagram corresponding to $Y(n,\alpha)$ is linked to the point at the level $n+1$ corresponding to $Y(n+1,\beta)$ if and only if $Y(n,\alpha) \cap F(n+1,\beta) \neq \emptyset$. In this case, the multiplicity of the embedding is always 1.

To illustrate this construction, let us consider the $\bigvee$ poset of Figure 4: $P = \{p_1, p_2, q\}$. Now there are four closed sets:

$$K_1 = \{p_1, p_2, q\},\ K_2 = \{p_1\},\ K_3 = \{p_2\},\ K_4 = \{p_1, p_2\}\ .$$

Thus it is not difficult to check that:

$$
\begin{array}{llll}
\mathcal{K}_1 = \{K_1\} & \mathcal{K}'_1 = \{K_1\} & Y(2,1) = \{p_1, p_2, q\} & \subset\ F(1,1) = K_1 \\
 & & & \\
\mathcal{K}_2 = \{K_1, K_2\} & \mathcal{K}'_2 = \{K_1, K_2\} & Y(2,1) = \{p_1\} & \subset\ F(2,1) = K_2 \\
 & & Y(2,2) = \{p_2, q\} & \subset\ F(2,2) = K_1 \\
 & & & \\
\mathcal{K}_3 = \{K_1, K_2, K_3\} & \mathcal{K}'_3 = \{K_1, K_2, K_3, K_4\} & Y(3,1) = \{p_1\} & \subset\ F(3,1) = K_2 \\
 & & Y(3,2) = \{q\} & \subset\ F(3,2) = K_1 \\
 & & Y(3,3) = \{p_2\} & \subset\ F(3,3) = K_3 \\
 & & & \\
\mathcal{K}_4 = \{K_1, K_2, K_3, K_4\} & \mathcal{K}'_4 = \{K_1, K_2, K_3, K_4\} & Y(4,1) = \{p_1\} & \subset\ F(4,1) = K_2 \\
 & & Y(4,2) = \{q\} & \subset\ F(4,2) = K_1 \\
 & & Y(4,3) = \{p_2\} & \subset\ F(4,3) = K_3 \\
 & & \vdots &
\end{array}
$$

Notice that, since P has only a finite number of points and hence a finite number of closed sets, the partition of P we have to consider at each level n repeats itself after a certain point ($n = 3$ in this case). Figure 9 shows the corresponding diagram, obtained through rules (1) and (2) above. Recalling then that the first matrix algebra that gives origin to an AF algebra is $\mathbb{C}$ and using the fact that all the embeddings have multiplicity one, we eventually obtain the sequence of finite dimensional algebras shown by the Bratteli diagram of Figure 5. As we have said previously, such a diagram corresponds to the AF algebra $\mathcal{A} = \mathbb{C}\mathbb{I}_1 + \mathbb{C}\mathbb{I}_1 + \mathcal{K}_{12}$.

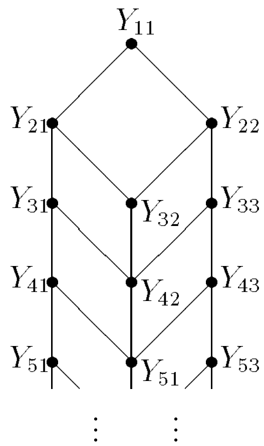

Figure 9:
The construction of the Bratteli diagram of the AF algebra corresponding to the $\bigvee$ poset of Figure 4.

It is a general fact that the Bratteli diagram describing any finite poset "stabilizes", i.e. repeats itself, after a certain level n_0, when the family $\mathcal{K}_{n_0}$ of closed sets we choose is such that it determines a partition of the poset which distinguishes each point of the poset itself. In particular, this is the case if we choose n_0 in such a manner that $\mathcal{K}_{n_0}$ contains all closed sets. Then, each $Y(n_0, j)$ will contain a single point of the poset and $F(n_0 + 1, j)$ will be the smallest closed set containing $Y(n_0, j)$. It is only this stable part of the diagram which is relevant for the inductive limit and hence for the determination of the AF algebra it represents. Indeed, diagrams (or sequences of finite dimensional algebras) that differ only for a finite numbers of initial levels give different finite approximations to the same AF algebra [8, 9].

To conclude this section, we want to describe the AF algebras whose structure spaces are the poset approximations of the circle, $P_4(S^1)$, and of the sphere, $P_6(S^2)$.

As for $P_4(S^1)$, given in Figure 2, the Bratteli diagram repeats itself for $n > n_0 = 4$ and the stable partition is given by

$$\begin{array}{ll} Y(n_0,1) = \{x_2\} & F(n_0+1,1) = \{x_2\} \\ Y(n_0,2) = \{x_1\} & F(n_0+1,2) = \{x_1,x_2,x_4\} \\ Y(n_0,3) = \{x_3\} & F(n_0+1,3) = \{x_2,x_3,x_4\} \\ Y(n_0,4) = \{x_4\} & F(n_0+1,4) = \{x_4\}\ . \end{array} \tag{4.18}$$

The corresponding Bratteli diagram is in Figure 10. The set $\{0\}$ is not an ideal. The

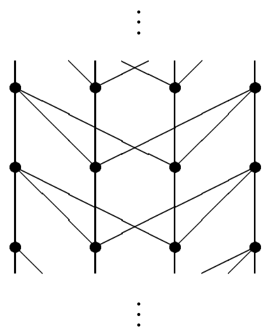

Figure 10:
The stable part of the Bratteli diagram for the circle poset $P_4(S^1)$.

limit algebra $\mathcal{A}$ turns out to be a subalgebra of bounded operators on the Hilbert space $\mathcal{H} = \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_4$, with $\mathcal{H}_i,\ i = 1, \ldots, 4$ infinite dimensional Hilbert spaces:

$$\mathcal{A} = \mathbb{C}\mathbb{I}_{13} \oplus \mathbb{C}\mathbb{I}_{24} \oplus \mathcal{K}_{12} \oplus \mathcal{K}_{34}\ . \tag{4.19}$$

Here $\mathbb{I}_{ij}$ and $\mathcal{K}_{ij}$ denote the identity operator and the algebra of compact operators on $\mathcal{H}_i \oplus \mathcal{H}_j$ respectively.

For the poset $P_6(S^2)$ for the two-dimensional sphere, given in Figure 3, $n_0 = 6$ and the stable partition is given by

$$\begin{array}{ll} Y(n_0,1) = \{x_5\} & F(n_0+1,1) = \{x_5\} \\ Y(n_0,2) = \{x_2\} & F(n_0+1,2) = \{x_2,x_5,x_6\} \\ Y(n_0,3) = \{x_1\} & F(n_0+1,3) = \{x_1,x_2,x_4,x_5,x_6,\} \\ Y(n_0,4) = \{x_3\} & F(n_0+1,4) = \{x_2,x_3,x_4,x_5,x_6\} \\ Y(n_0,5) = \{x_4\} & F(n_0+1,5) = \{x_4,x_5,x_6\} \\ Y(n_0,6) = \{x_6\} & F(n_0+1,6) = \{x_6\}\ . \end{array} \tag{4.20}$$

The corresponding Bratteli diagram is in Figure 11. The set $\{0\}$ is not an ideal. The

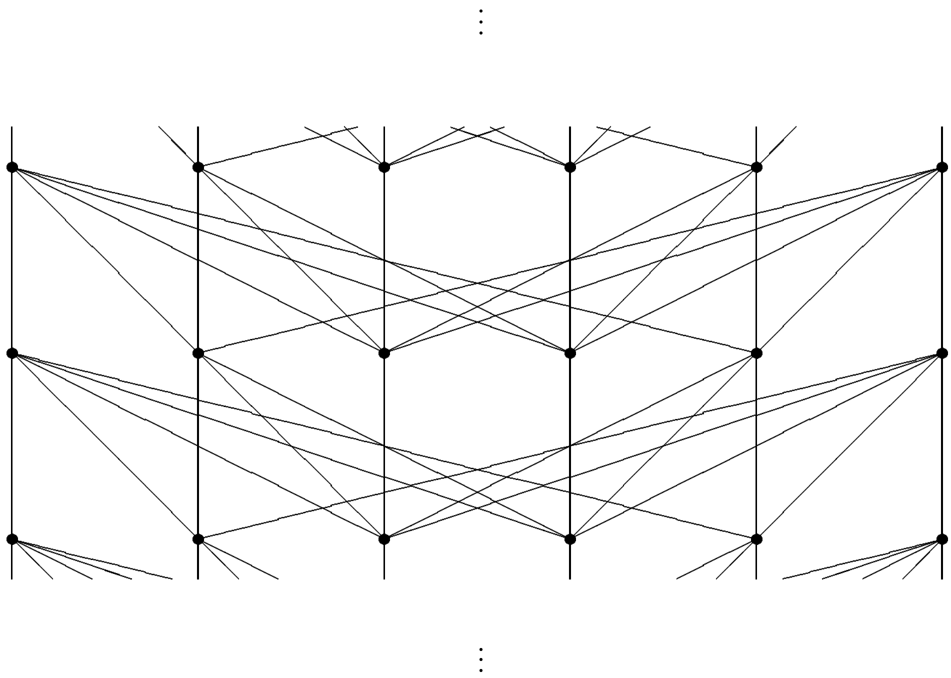

Figure 11:
The stable part of the Bratteli diagram for the sphere poset $P_6(S^2)$.

inductive limit is a subalgebra of bounded operators on the Hilbert space $\mathcal{H} = \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_8$ with $\mathcal{H}_i,\ i = 1, \ldots, 8$ infinite dimensional Hilbert spaces, given by:

$$\begin{aligned}\mathcal{A} &= \mathbb{C}\mathbb{I}_{\mathcal{H}_1 \otimes (\mathcal{H}_5 \oplus \mathcal{H}_6) \oplus \mathcal{H}_2 \otimes (\mathcal{H}_7 \oplus \mathcal{H}_8)} \oplus \mathbb{C}\mathbb{I}_{\mathcal{H}_3 \otimes (\mathcal{H}_5 \oplus \mathcal{H}_6) \oplus \mathcal{H}_4 \otimes (\mathcal{H}_7 \oplus \mathcal{H}_8)} \\ &\oplus \left(\mathcal{K}_{\mathcal{H}_1 \oplus \mathcal{H}_3} \otimes \mathbb{C}\mathbb{I}_{\mathcal{H}_5 \oplus \mathcal{H}_6}\right) \oplus \left(\mathcal{K}_{\mathcal{H}_2 \oplus \mathcal{H}_4} \otimes \mathbb{C}\mathbb{I}_{\mathcal{H}_7 \oplus \mathcal{H}_8}\right) \\ &\oplus \mathcal{K}_{\mathcal{H}_5 \otimes (\mathcal{H}_1 \oplus \mathcal{H}_3) \oplus \mathcal{H}_7 \otimes (\mathcal{H}_2 \oplus \mathcal{H}_4)} \oplus \mathcal{K}_{\mathcal{H}_6 \otimes (\mathcal{H}_1 \oplus \mathcal{H}_3) \oplus \mathcal{H}_8 \otimes (\mathcal{H}_2 \oplus \mathcal{H}_4)}\end{aligned} \tag{4.21}$$

5 The Behncke-Leptin Construction

Given a poset P, there is always an AF algebra $\mathcal{A}$ such that $\hat{\mathcal{A}} = P$. A particular procedure to find such an algebra has been described in the previous section, but it is known that there exists more than one C^*-algebra $\mathcal{A}$ whose structure space is P. For example, if P consists of a single point, we can take for $\mathcal{A}$ any of the C^*-algebras $\mathbb{M}(n, \mathbb{C})$ of all $n \times n$ matrices valued in $\mathbb{C}$.

It is natural then to ask what are all the algebras associated to a given finite T_0 topological space P. This problem was solved by Behncke and Leptin in 1973. In [11] they give a complete classification of all separable C^*-algebras $\mathcal{A}$ with finite structure spaces. Such classification requires the definition of a function d on P, called defector, valued in $\overline{\mathbb{N}} = \{\infty, 0, 1, 2, ...\}$. Given P and d, the Behncke-Leptin construction gives a separable C^*-algebra $\mathcal{A}(P, d)$ such that $\hat{\mathcal{A}}(P, d) = P$. Furthermore, any separable C^*-algebra $\mathcal{A}$ satisfying $\hat{\mathcal{A}} = P$ is isomorphic to $\mathcal{A}(P, d)$ for some d.

A *defector* d on the poset P is a $\overline{\mathbb{N}}$-valued function on P such that

$$d(x) > 0 \quad \text{if } x \text{ is maximal.} \tag{5.22}$$

Two defectors d and d' are declared to be equal if there exists an automorphism φ of P such that $d' = d \circ \varphi$. They are called immediately equivalent if $d(x) = d(x')$ for all $x \in P$ with the exception of at most one nonmaximal $y \in P$, such that

$$d(y) = d'(y) + d'(z) \quad \text{or} \quad d'(y) = d(y) + d(z)$$

for some z covering y *, if $d(z) = d'(z) < \infty$. In the case $d(z) = \infty$, $d(y)$ and $d'(y)$ may be arbitrary. Then two defectors are defined to be equivalent ($d \sim d'$) if there exists a finite sequence of immediately equivalent defector connecting them.

We will start by describing the Behncke-Leptin construction for a special class of posets called forests. Then we will give the generalization for an arbitrary finite poset.

5.1 The Behncke-Leptin Construction for a Forest

A forest is a poset F such that

$$\{x, y, z \in F, x \preceq z, y \preceq z\} \Rightarrow \{x \preceq y \text{ or } y \preceq x\}. \tag{5.23}$$

Given a forest F and a defector d on F, the Behncke-Leptin construction consists of the following steps. First we introduce a Hilbert space $\mathcal{H}(F, d)$ associated to the whole forest F. Second, for each point $x \in F$, we introduce a subspace $\mathcal{H}(x) \subseteq \mathcal{H}(F, d)$ and a set of operators $\mathcal{R}_x$ acting on $\mathcal{H}(x)$. Actually, $\mathcal{R}_x$ can be thought of as acting on the whole $\mathcal{H}(F, d)$ by defining its action on the complement of $\mathcal{H}(x)$ to be zero. Then the C^*-algebra $\mathcal{A}$ associated to the forest is the one generated by the $\mathcal{R}_x$'s as x varies in F.

Now we explain how to determine $\mathcal{H}(F, d), \mathcal{H}(x)$ and $\mathcal{R}_x$. The Hilbert space $\mathcal{H}(F, d)$ can be obtained using an auxiliary forest F' constructed from F in the following way. The forest F' contains a point $x^{(1)}$ for each maximal point $x \in F$ and a pair of points $x_i^{(1)}$

*We say that y covers x if $x \prec y$ and there is no z such that $x \prec z \prec y$.

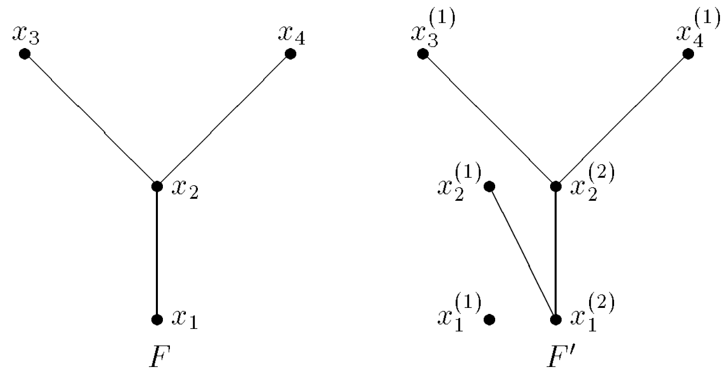

Figure 12:
An example of a forest F and the auxiliary forest F'.

and $x_i^{(2)}$ for each non maximal point $x_i \in F$. Then on F' we introduce a partial order by declaring that $x_i^{(2)}$ is covered by both $x_j^{(1)}$ and $x_j^{(2)}$ if and only if x_i is covered by x_j. Figure 12 shows an example of F and the corresponding F'.

In F' we consider the maximal chains $C_\alpha = \{x_1^{(p_1)}, x_2^{(p_2)}, ..., x_k^{(p_k)}\}$, which can be seen to be necessarily of the form

$$C_\alpha = \{x_1^{(2)}, x_2^{(2)}, ..., x_{k-1}^{(2)}, x_k^{(1)}\} \ . \tag{5.24}$$

For example, in F' of Figure 12, the maximal chains are $\{x_1^{(1)}\}, \{x_1^{(2)}, x_2^{(1)}\}, \{x_1^{(2)}, x_2^{(2)}, x_3^{(1)}\}, \{x_1^{(2)}, x_2^{(2)}, x_4^{(1)}\}$.

To each maximal chain C_α we associate the Hilbert space

$$h(C_\alpha) = l_{x_1} \otimes l_{x_2} \otimes ... \otimes l_{x_{k-1}} \otimes \mathbf{C}^{d(x_k)} \ , \tag{5.25}$$

where $d(x_k)$ is the value of the defector d at the point $x_k \in F$ and l_{x_i} can be realized as the Hilbert space ℓ^2 of sequences $(f_1, f_2, ...)$ of complex numbers with $\sum_n |f_n|^2 < \infty$. We then define the total Hilbert space $\mathcal{H}(F, d)$ associated to F to be

$$\mathcal{H}(F, d) = \bigoplus_\alpha h(C_\alpha) \ , \tag{5.26}$$

where we sum over all maximal chains C_α in F'.

In a similar way, we introduce the subspaces $\mathcal{H}(x_i)$ associated to a single point $x_i \in F$ by

$$\mathcal{H}(x_i) = \bigoplus_\beta h(C_\beta) \ \text{ for all } C_\beta \text{ such that } x_i^{(p)} \in C_\beta \ . \tag{5.27}$$

Notice that if we consider the subforest F_x of F given by

$$F_x = \{y \in F | x \preceq y\} , \tag{5.28}$$

we can construct the Hilbert space $\mathcal{H}(F_x, d_x)$, where d_x is the restriction of d to F_x. An important property of $\mathcal{H}(x)$ defined in (5.27) is that it satisfies

$$\mathcal{H}(x) = \mathcal{H}_x \otimes \mathcal{H}(F_x, d_x) \tag{5.29}$$

where

$$\mathcal{H}_x = \bigotimes_i l_{x_i} \quad \text{for all } x_i \prec x \tag{5.30}$$

and $\mathcal{H}_x = \mathbb{C}$ if x is a minimal point.

Now we are ready to define the C^*-algebra $\mathcal{A}(F, d)$. First, let us introduce the algebra of operators $\mathcal{R}_x$, acting on $\mathcal{H}(x)$, given by

$$\mathcal{R}_x = \mathbb{C}\mathbb{I}_{\mathcal{H}_x} \otimes \mathcal{K}(\mathcal{H}(F_x, d_x)) , \tag{5.31}$$

$\mathbb{I}_{\mathcal{H}_x}$ being the identity operator on $\mathcal{H}_x$ and $\mathcal{K}(\mathcal{H}(F_x, d_x))$ being the algebra of compact operators on $\mathcal{H}(F_x, d_x)$. In other words, $\mathcal{R}_x$ acts as multiples of the identity on the Hilbert space $\mathcal{H}_x$ determined by the points $x_i \prec x$ which precede x, as in (5.30), and as compact operators on the Hilbert space $\mathcal{H}(F_x, d_x)$ determined by the points $x_j \succeq x$ which follow x.

Then $\mathcal{A}(F, d)$ is the algebra of operators on $\mathcal{H}(F, d)$ generated by all $\mathcal{R}_x$ as x varies in F.

The algebras $\mathcal{R}_x$, with $x \in F$, satisfy:

$$\mathcal{R}_x \mathcal{R}_y \subset \mathcal{R}_x \quad \text{if } x \preceq y$$

and

$$\mathcal{R}_x \mathcal{R}_y = 0 \quad \text{if } x \text{ and } y \text{ are incomparable.} \tag{5.32}$$

One of the major results of [11] is the following theorem, which establishes that the structure space of the C^*-algebra $\mathcal{A}(F, d)$ constructed according to the rules given above is homeomorphic to the forest F:

Proposition 5.1 *Let F be a finite forest with defector d and $\mathcal{A}(F, d)$ the algebra of operators on $\mathcal{H}(F, d)$ defined as above. Then we have:*

(i) if E is a closed subset of F with complement U, then $I_E = \bigotimes_{x \in U} \mathcal{R}_x$ is a closed two-sided ideal of $\mathcal{A}(F, d)$, and $A_E = \bigotimes_{x \in E} \mathcal{R}_x$ is a closed subalgebra of $\mathcal{A}(F, d)$;

(ii) every two-sided ideal of $\mathcal{A}(F, d)$ is of the form I_E for some closed $E \subset F$ and I_E is primitive iff $E = \overline{\{x\}}$. In particular, $\hat{\mathcal{A}}(F, d) = F$.

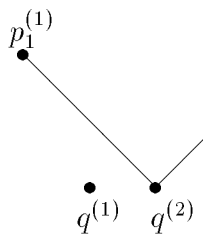

Figure 13:
The forest associated to the $\bigvee$ poset.

Let us illustrate the Behncke-Leptin construction for a very simple forest, namely the $\bigvee$ poset of Figure 4. The correspondent associated forest P' is illustrated in Figure 13.

We consider a generic defector d. From the diagram of P' in Figure 13 we can write down all its maximal chains:

$$\{q^{(2)}, p_1^{(1)}\}\ ,\ \{q^{(2)}, p_2^{(1)}\}\ ,\ \{q^{(1)}\}$$

and following (5.25) and (5.26) we see that $\mathcal{H}(F, d)$ is given by

$$\mathcal{H}(F, d) = \left(l_q \otimes \mathbb{C}^{d(p_1)}\right) \oplus \left(l_q \otimes \mathbb{C}^{d(p_2)}\right) \oplus \mathbb{C}^{d(q)}\ . \tag{5.33}$$

The subspaces $\mathcal{H}(x_i)$ can also be determined from the diagram of P':

$$\begin{aligned} \mathcal{H}(p_1) &= l_q \otimes \mathbb{C}^{d(p_1)} \\ \mathcal{H}(p_2) &= l_q \otimes \mathbb{C}^{d(p_2)} \\ \mathcal{H}(q) &= \mathcal{H}(F, d)\ . \end{aligned} \tag{5.34}$$

Notice that the factorization expressed in (5.29) is satisfied, where now

$$\begin{aligned} \mathcal{H}_{p_1} = l_q \quad &, \quad \mathcal{H}(F_{p_1}, d_{p_1}) = \mathbb{C}^{d(p_1)} \\ \mathcal{H}_{p_2} = l_q \quad &, \quad \mathcal{H}(F_{p_2}, d_{p_2}) = \mathbb{C}^{d(p_2)} \\ \mathcal{H}_q = \mathbb{C} \quad &, \quad \mathcal{H}(F_q, d_q) = \mathcal{H}(F, d)\,. \end{aligned} \tag{5.35}$$

The C^*-algebra $\mathcal{A}(F, d)$ is generated by all $\mathcal{R}_x$, $x \in F$. The latter reads

$$\begin{aligned} \mathcal{R}_{p_1} &= \mathbb{C}\mathbb{I}_{\mathcal{H}_{p_1}} \otimes \mathcal{K}(\mathbb{C}^{d(p_1)}) \\ \mathcal{R}_{p_2} &= \mathbb{C}\mathbb{I}_{\mathcal{H}_{p_2}} \otimes \mathcal{K}(\mathbb{C}^{d(p_2)}) \\ \mathcal{R}_q &= \mathcal{K}(\mathcal{H}(F, d))\ . \end{aligned} \tag{5.36}$$

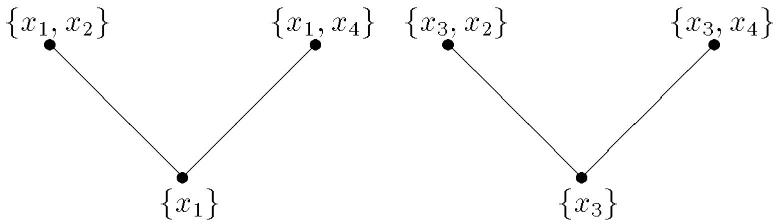

Figure 14:
The covering forest of the circle poset $P_4(S^1)$.

Notice that for the defector $d(p_1) = d(p_2) = 1$ and $d(q) = 0$ we get $\mathcal{H}(F, d) = \mathcal{H}_1 \oplus \mathcal{H}_2$ and $\mathcal{A} = \mathbb{C}\mathbb{I}_1 + \mathbb{C}\mathbb{I}_2 + \mathcal{K}_{12}$ and thus recover the algebra we got for the $\bigvee$ poset via the Bratteli construction in section 4.1.

5.2 The Behncke-Leptin Construction for Posets

To generalize the procedure of the last section to an arbitrary poset P with defector d, we have first to introduce a forest $\overline{P}$, uniquely determined by P.

Let P be a finite poset. A rope r of P is a (not necessarily maximal) chain in P starting from a minimal element and ending at some $x \in P$. The set $\overline{P}$ of all ropes of P ordered by inclusion is a poset. One can show that $\overline{P}$ is in fact a forest. Let $\varphi : \overline{P} \to P$ denote the surjective map which assigns to each rope $r \in \overline{P}$ its end point $\varphi(r) \in P$. Following [11], we will call the pair $(\overline{P}, \varphi)$ the covering forest of P. An example is given in Figure 14, which shows the covering forest of the circle poset $P_4(S^1)$ of Figure 2.

Given a defector d on P we define a defector $\overline{d}$ on $\overline{P}$ in a natural way via the pull-back:

$$\overline{d} = d \circ \varphi \; . \tag{5.37}$$

Then, since $\overline{P}$ is a forest, we can construct the algebra $\mathcal{A}(\overline{P}, \overline{d})$ following section 5.1.

Finally, to identify the C^*-algebra $\mathcal{A}(P, d)$ associated to the poset P and the defector d, we proceed to realize $\mathcal{A}(P, d)$ as a subalgebra of $\mathcal{A}(\overline{P}, \overline{d})$. In order to do so, we need to point out a simple property of the covering forest $(\overline{P}, \varphi)$.

Let r, $s \in \overline{P}$ be in the inverse image $\varphi^{-1}(x)$ of $x \in P$. Then, the subforest $(\overline{P})_r$ (see

(5.28)) is naturally isomorphic to $(\overline{P})_s$. Indeed, $(\overline{P})_r$ and $(\overline{P})_s$ consist of all extensions of the rope r and s respectively. By hypothesis, r and s have the same end point $x \in P$, so that $(\overline{P})_r \sim (\overline{P})_s$. Thus

$$\mathcal{K}\left(\mathcal{H}((\overline{P})_r, \overline{d}_r)\right) \simeq \mathcal{K}\left(\mathcal{H}((\overline{P})_s, \overline{d}_s)\right) \equiv \mathcal{K}_x \ ,$$

so that the algebras $\mathcal{R}_s$, $\mathcal{R}_r \in \mathcal{A}(\overline{P}, \overline{d})$ are given by

$$\mathcal{R}_r = \mathbb{C}\mathbb{I}_{\mathcal{H}_r} \otimes \mathcal{K}_x \ ,$$

$$\mathcal{R}_s = \mathbb{C}\mathbb{I}_{\mathcal{H}_s} \otimes \mathcal{K}_x \ .$$

For each $x \in P$ we define the algebra A_x

$$A_x = \bigoplus_{r \in \varphi^{-1}(x)} \mathcal{R}_r \tag{5.38}$$

and a subalgebra $\mathcal{R}_x \subset A_x$ given by all elements $a \in A_x$ of the form

$$a = (\lambda_{r_1} \mathbb{I}_{\mathcal{H}_{r_1}} \otimes k) + (\lambda_{r_2} \mathbb{I}_{\mathcal{H}_{r_2}} \otimes k) + \ldots + (\lambda_{r_n} \mathbb{I}_{\mathcal{H}_{r_n}} \otimes k) \ ,$$

where $r_i \in \varphi^{-1}(x)$, $\lambda_{r_j} \in \mathbb{C}$ and $k \in \mathcal{K}_x$. Thus

$$\mathcal{R}_x = \{a \in A_x \,|\, a = \bigoplus_{r \in \varphi^{-1}(x)} (\lambda_r \mathbb{I}_{\mathcal{H}_r} \otimes k) \ , \ \lambda_r \in \mathbb{C} \text{ and } k \in \mathcal{K}_x\} \ . \tag{5.39}$$

The C^*-algebra $\mathcal{A}(P, d)$ that satisfies

$$\hat{\mathcal{A}}(P, d) = P \tag{5.40}$$

is then generated by all $\mathcal{R}_x$ with $x \in P$.

There is an intuitive interpretation for (5.39). The poset P can be obtained from $\overline{P}$ by identifying any two ropes r and s that have the same ending point. Equation (5.39) simply expresses this identification at an algebraic level.

For example, for the circle poset $P_4(S^1)$ these rules give the following algebras, acting on $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3 \oplus \mathcal{H}_4$:

$$\begin{aligned}
\mathcal{A}_{x_4} &= \mathbb{C}\mathbb{I}_1 + \mathbb{C}\mathbb{I}_3 \\
\mathcal{A}_{x_2} &= \mathbb{C}\mathbb{I}_2 + \mathbb{C}\mathbb{I}_4 \\
\mathcal{A}_{x_1} &= \mathbb{C}\mathbb{I}_1 + \mathbb{C}\mathbb{I}_2 + \mathcal{K}_{12} \\
\mathcal{A}_{x_3} &= \mathbb{C}\mathbb{I}_3 + \mathbb{C}\mathbb{I}_4 + \mathcal{K}_{34} \ ,
\end{aligned} \tag{5.41}$$

if one chooses the defector $d(x_1) = d(x_2) = 1$, $d(x_3) = d(x_4) = 0$. Thus the algebra associated to $P_4(S^1)$ is

$$\mathcal{A} = \mathbb{C}\mathbb{I}_1 + \mathbb{C}\mathbb{I}_2 + \mathbb{C}\mathbb{I}_3 + \mathbb{C}\mathbb{I}_4 + \mathcal{K}_{12} + \mathcal{K}_{34} \ . \tag{5.42}$$

As before this is the algebra one gets for $P_4(S^1)$ by means of the Bratteli construction explained in section 4.3.

Equivalent defectors give rise to isomorphic C^*-algebras, whereas by choosing different non-equivalent defectors one can construct non-isomorphic C^*-algebras that all have P as structure space. In this way one can obtain *all* C^*-algebras $\mathcal{A}$ whose (finite) dual $\hat{\mathcal{A}}$ is homeomorphic to the poset P, as it is established in [11], which we quote in conclusion of this section:

Proposition 5.2

(i) Every separable C^*-algebra $\mathcal{A}$ with finite dual $\hat{\mathcal{A}} = P$ is isomorphic to some $\mathcal{A}(P, d)$.

(ii) $\mathcal{A}(P, d)$ is isomorphic to $\mathcal{A}(P, d')$ if and only if d and d' are equivalent.

6 Final remarks

In this article, we have seen how a finite poset is truly a "noncommutative space" or "noncommutative lattice", since it can be described as the structure space of a noncommutative C^*-algebra $\mathcal{A}$, which turns out to be always a postliminal AF algebra. We have also seen as this correspondence is not one-to-one, more than one non-isomorphic C^*-algebra leading to the same poset. This relation between posets and C^*-algebras was used in [17] to give a dualization of the approximation method for topological spaces introduced in [1].

In our previous work [5] we have showed how it is possible to construct a quantum theory on posets, by making use of the corresponding C^*-algebra. We have also seen how important topological properties of the continuum, such as homotopy, can be captured by the poset approximation and manifest themselves in the corresponding quantum mechanics.

We are thus naturally led to examine how one can construct further geometric structures on posets, as it suggested by Connes' noncommutative geometry [6]. First of all, we are interested in the construction of bundles and characteristic classes over a poset and, as a first step in this direction, one should examine the K-theory of these noncommutative lattices. This is the topic discussed in [12], where we present a study of the algebraic K-theory of AF algebras associated to a poset.

Then one would like to construct bundles, and notably nontrivial bundles, over a poset, and consider, for instance, the analogue of the monopole bundle over the lattice approximating the two-dimensional sphere and of nontrivial "topological charges". Work in this direction is in progress.

Acknowledgments

This work was initiated while the authors were at Syracuse University. We thank A.P. Balachandran, G. Bimonte, F. Lizzi e G. Sparano for many fruitful discussions and useful advice. The final version was written while G.L. and P.T-S were at ESI in Vienna. They would like to thank G. Marmo and P. Michor for the invitation and all people at the Institute for the warm and friendly atmosphere.

We thank the 'Istituto Italiano per gli Studi Filosofici' in Napoli for partial support. The work of P.T-S. was also supported by the Department of Energy, U.S.A. under contract number DE-FG-02-84ER40173. The work of G.L. was partially supported by the Italian 'Ministero dell' Università e della Ricerca Scientifica'.

References

[1] R.D. Sorkin, Int. J. Theor. Phys. **30** (1991) 923.

[2] P.S. Aleksandrov, *Combinatorial Topology*, Vol. 1-3 [Greylock, 1960].

[3] A.P. Balachandran, G. Bimonte, E. Ercolessi and P. Teotonio-Sobrinho, Nucl. Phys. **B** 418 (1994) 923.

[4] A.P. Balachandran, G. Bimonte, G. Landi, F. Lizzi and P. Teotonio-Sobrinho, *Lattice Gauge Fields and Noncommutative Geometry*, Preprint ESI (1995) 299 (hep-th/9604012).

[5] A.P. Balachandran, G. Bimonte, E. Ercolessi, G. Landi, F. Lizzi, G. Sparano and P. Teotonio-Sobrinho, Nucl. Phys. **B** 37C Proc. Suppl. (1995) 20; J. Geom. Phys. **18** (1996) 163 (hep-th/9510217).

[6] A. Connes, *Noncommutative Geometry* [Academic Press, 1994].

[7] J.C. Várilly and J.M. Gracia-Bondía, J. Geom. Phys. **12** (1993) 223.

[8] O. Bratteli, Trans. Amer. Math. Soc. **171** (1972) 195; J. Functional Analysis **16** (1974) 192.

[9] K.R. Goodearl, *Notes on Real and Complex C^*-algebras* , [Shiva Publishing Limited].

[10] G.A. Elliot, J. Alg. **38** (1976) 29.
E.G. Effros, *Dimension and C^*-algebras* , [American Mathematical Society, 1981].

[11] H. Behncke and H. Leptin, J. Functional Analysis **14** (1973) 253; **16** (1974) 241.

[12] E. Ercolessi, G. Landi and P. Teotonio-Sobrinho, *K-theory of Noncommutative Lattices*, Preprint (1996), q-alg/9607017.

[13] J. Dixmier, *C^*-algebras* , [North-Holland, 1982].

[14] J.M.G. Fell and R.S. Doran, *Representations of *-Algebras, Locally Compact Groups and Banach *-Algebraic Bundles* [Academic Press, 1988].

[15] J.L. Kelley, *General Topology*, [Springer-Verlag, 1955]; J. Hocking, G. Young, Topology (Dover, 1988)

[16] R.P. Stanley, *Enumerative Combinatorics*, Vol. 1 [Wordsworth and Brooks/Cole Advanced Books and Software, 1986].

[17] G. Bimonte, E. Ercolessi, G. Landi, F. Lizzi, G. Sparano and P. Teotonio-Sobrinho, *Lattices and their Continuum Limits*, J. Geom. Phys., in press; (hep-th/9507147).
G. Bimonte, E. Ercolessi, G. Landi, F. Lizzi, G. Sparano and P. Teotonio-Sobrinho, *Noncommutative Lattices and their Continuum Limits*, J. Geom. Phys., in press; (hep-th/9507148).