Threefolds in $\mathbb{P}^5$ with a 3-dimensional Family of Plane Curves

Emilia Mezzetti and *Dario Portelli*

Dipartimento di Scienze Matematiche, Università di Trieste,

Piazzale Europa 1,

34127 Trieste, ITALY

e-mail: mezzette@univ.trieste.it, porteda@univ.trieste.it

A classification theorem is given of smooth threefolds of $\mathbb{P}^5$ covered by a family of dimension at least three of plane integral curves of degree $d \geqslant 2$. It is shown that for such a threefold X there are two possibilities:

(1) X is any threefold contained in a hyperquadric;

(2) $d \leqslant 3$ and X is either the Bordiga or the Palatini scroll.

Introduction

Let S be a non-degenerate surface of $\mathbb{P}^n$, $n \geqslant 4$. A classical theorem of Corrado Segre ([cS], 1921) states that, if S contains a family of dimension 2 of irreducible plane curves not lines, then these curves are conics and S is a Veronese surface of $\mathbb{P}^5$ or a projection of its. It has been remarked recently ([MP]) that this theorem is strictly related to two classical theorems on surfaces of $\mathbb{P}^4$, namely the Severi theorem and the Franchetta theorem. The first one ([fS], 1901) asserts that a smooth surface of $\mathbb{P}^4$ is linearly normal, unless it is a projected Veronese surface. The theorem of Franchetta ([aF], 1947) says that the projected Veronese surface is the unique smooth surface of $\mathbb{P}^4$ whose general projection into $\mathbb{P}^3$ has a reducible double curve.

In the study of 3-dimensional manifolds of $\mathbb{P}^5$ the analogous problems are still open. It has been proved that all smooth 3-folds of $\mathbb{P}^5$ are linearly normal. About 2-normality, it has been conjectured by Peskine and Van de Ven that the unique non 2-normal smooth 3-fold of $\mathbb{P}^5$ is the Palatini scroll, of degree 7 and sectional genus 4. Moreover, when considering general projections of 3-folds of $\mathbb{P}^5$ into $\mathbb{P}^4$, it results that the triple locus is a curve, and in all known examples, except the Palatini scroll, this curve is irreducible.

In this paper we are interested in stating the analogous of Segre' s theorem, i.e. we study 3-dimensional manifolds of $\mathbb{P}^5$ containing a family of dimension 3 of integral plane curves. Our results, which are contained in Theorems 1.4 and 1.6, are the following.

Let X be a smooth threefold of $\mathbb{P}^5$ covered by a family of dimension at least 3 of plane integral curves not lines. Then there are two possibilities:
(1) X is any threefold contained in a hyperquadric (X is called in this case *not of isolated type*);
(2) X is either the Bordiga or the Palatini scroll and it contains in both cases two families of dimension 3 of plane curves, one of conics and one of cubics.

The methods that we use are mainly the adjunction theory, properties of normal bundles and the classification of smooth surfaces of $\mathbb{P}^4$ in low degree. In particular, in several situations we heavily use the assumption that X is smooth.

The paper is organized as follows. In the first section we treat the case of threefolds not of isolated type and we sketch the proof of the classification theorem for those of isolated type. In particular we recall the results of adjunction theory that we shall need in the sequel. The second section is devoted to the study of the normal bundle $\mathcal{N}_{C/X}$. The third section contains the rather involved proof that a threefold of isolated type is log-special. The key point here is to bound the self-intersection C^2 of C on a hyperplane section of X. In Lemma 3.1 we show that $C^2 = -1$ when C is a conic and $-1 \leqslant C^2 \leqslant 0$ when C is a cubic. This is the more technical part of the paper. Finally, in sections 4 and 5 we perform the analysis of the log-special threefolds not contained in a quadric and different from the Bordiga or the Palatini scrolls, to rule out the possibility that some of them contain a family of dimension 3 of conics or plane cubics.

This work has been done in the framework of the activities of Europroj. Both authors have been supported by funds of MURST and are members of GNSAGA.

1.- Preliminaries and threefolds not of isolated type.

Let $X \subset \mathbb{P}^5$ be an integral projective variety of dimension 3, and degree d. We will always assume that X is non-degenerate, i.e. it is not contained in any hyperplane.

We suppose that X contains an algebraic family $\mathcal{F}$ of dimension at least 3 of plane, integral curves. It is immediate to remark that, if the union of the curves of $\mathcal{F}$ does not cover X, then this union is a surface containing a 3-dimensional family of plane curves, hence a union of surfaces of $\mathbb{P}^3$. We will always exclude this situation, so from now on *we assume that the curves of $\mathcal{F}$ cover X*. Our aim is to classify such varieties X.

If the curves of $\mathcal{F}$ are lines, then the answer is classical and is given by the following theorem.

Theorem 1.1. *Let $X \subset \mathbb{P}^5$ be an integral non-degenerate variety of dimension 3 covered by a family $\mathcal{F}$ of dimension 3 of lines. Then X is ruled by planes over a curve. If moreover X is smooth, then deg $X = 3$ and $X = \mathbb{P}^1 \times \mathbb{P}^2$.*

Proof. The first assertion follows from the following theorem of B.Segre, the second one from [LT]. $\square$

Theorem 1.2.(B.Segre) *Let $X \subset \mathbb{P}^n$ be a s-dimensional integral projective variety, and let $\Sigma \subset \mathbb{G}(k,n)$ be a component of maximal dimension of the variety of linear spaces of dimension k contained in X. Then:*

(i) dim $\Sigma \leqslant (k+1)(s-k)$ and, if equality holds, X is linear;

(ii) if dim $\Sigma < (k+1)(s-k)$, then dim $\Sigma \leqslant (k+1)(s-k)-k$ and, if equality holds, then either X is a scroll in $\mathbb{P}^{s-1}$'s or $k=1$ and X is a quadric;

(iii) if $k>1$ and dim $\Sigma < (k+1)(s-k)-k$, then dim $\Sigma \leqslant (k+1)(s-k)-2k+1$ and, if equality holds, X is a quadric.

Proof. See [bS] and [eR]. $\square$

From now on we assume that *the curves of $\mathcal{F}$ have degree $\geqslant 2$*, so each of them generates a plane. Under this hypothesis, we are in position to apply the following theorem of classification of varieties containing a high dimensional family of degenerate subvarieties :

Theorem 1.3. *Let $X \subset \mathbb{P}^n$ be an integral variety of dimension s containing a family $\mathcal{F}$ of dimension $c = h+1$ of integral subvarieties of dimension $s-h$. Let Y be a general variety of $\mathcal{F}$ and assume that Y spans a $\mathbb{P}^{n-h-1}$. Then one of the following happens:*

(i) there exists an integer r, $1 \leqslant r < n-s$, such that X is contained in a variety V of dimension at most $n-r$ containing ∞^{h+1} varieties of dimension $n-h-r$, each one contained in a linear space of dimension $n-h-1$;

(ii) deg Y is bounded by a function of h and $n-s$.

Proof. See [eM], Theorem (1.3). $\square$

We point out that it follows from the proof of the above theorem that, if *dim* $X = 3$, then possibility (i) happens precisely when *the planes of the curves of $\mathcal{F}$ do not fill the ambient space $\mathbb{P}^5$.*

If possibility (ii) happens, then X is called *of isolated type.* In our case, i.e. if $n=5$, $s=3$, $h=2$, then the upper bound on the degree of the curves of $\mathcal{F}$ can be computed to be 3 ([eM], §3), so the curves are conics or cubics.

Next theorem gives the classification of smooth threefolds which are not of isolated type.

Theorem 1.4 *Let X be a smooth threefold of $\mathbb{P}^5$ of degree d covered by a family $\mathcal{F}$ of dimension at least 3 of plane integral curves not lines. Then X is not of isolated type if and only if it is contained in a quadric.*

Proof. Assume that X is not of isolated type; by Theorem 1.3, X is contained in a variety V of dimension 4 containing a 3-dimensional family of planes. Then V is either a quadric or a scroll in $\mathbb{P}^3$'s over a curve, by Theorem 1.2 (in the second case V contains a family of planes of dimension 4). Let us assume that V is a scroll over a curve B and let $p\colon V \longrightarrow B$ be the map which gives it the scroll's structure. Since $p|_X$ cannot be constant, it is surjective and the fibers are surfaces of $\mathbb{P}^3$. Hence, if H is a general hyperplane, $S := X \cap H$ is a smooth surface of $\mathbb{P}^4$ fibered by plane curves. By a theorem of Ranestad [kR] there are three possibilities:

(a) S is contained in a quadric;
(b) S is an abelian surface of degree 10;
(c) S is a bielliptic surface of degree 10.

In cases (b) and (c) the irregularity q of S is positive: this possibility is excluded by the assumption that X is smooth, in view of the theorem of Barth–Lefschetz. In case (a), if $d > 4$ also X is contained in a quadric by Roth's theorem (see for instance [MR]); if $d = 3$ then $X = \mathbb{P}^1 \times \mathbb{P}^2$ (up to projective equivalence), so it is contained in a quadric; if $d = 4$ then S must be a Del Pezzo surface, complete intersection of two quadrics, and again X is contained in a quadric.

Conversely, if X is any threefold of $\mathbb{P}^5$ contained in a quadric Q, then it intersects the planes of Q along plane curves forming an algebraic family of dimension at least 3. □

Remarks 1.5.

(i) With notations as in 1.4, if X is allowed to have isolated singularities, then S is still smooth, but it could have $q > 0$, so cases (b) and (c) are not excluded.

(ii) The classification of smooth threefolds which are contained in a quadric is complete; in fact they are all arithmetically Cohen-Macaulay and either complete intersection or linked to a $\mathbb{P}^3$ in a complete intersection (see for instance [DP]).

It remains to classify the threefolds of isolated type.

Theorem 1.6 *Let X be a smooth, non-degenerate threefold of $\mathbb{P}^5$ of degree d, covered by a family $\mathcal{F}$ of dimension at least 3 of plane integral curves, not lines. Assume that X is of isolated type, i.e. X is not contained in a quadric. Then X is the Bordiga's scroll or X is the Palatini's scroll. In either cases X contains both a 3-dimensional family of conics and one of elliptic cubics.* (For the definitions of these scrolls see e.g. [gO].)

Outline of the Proof. The main tool for proving Theorem 1.6 is adjunction theory; while giving this sketch of the proof we will recall some fact from this theory we shall freely use in the sequel (for more informations see, e.g., [DP] and the references given there).

Let H and K denote respectively a hyperplane divisor and a canonical divisor on X. Since X is not contained in a quadric, X is different from the Segre scroll $\mathbb{P}^1 \times \mathbb{P}^2$. Therefore, the linear system $| K + 2H |$ is base point free and the rational map ϕ associated to it, the *adjunction map*, is defined on the whole X. Moreover, being X not contained in a quadric, we have necessarily

$dim\ \phi(X) \geqslant 2$.

If $dim\ \phi(X) = 2$, then X is a scroll over a smooth surface. Since X is not contained in a quadric, there are the following possibilities: X is the Bordiga's scroll or X is the Palatini's scroll or

(E1) X is a scroll over a $K3$ surface, of degree $d = 9$ and sectional genus $\pi = 8$.

If $dim\ \phi(X) = 3$ then, in general, ϕ is the contraction of the exceptional planes contained in X. More precisely, ϕ is a closed imbedding unless

(E2) X is of degree $d = 7$ and sectional genus $\pi = 5$, isomorphic to the blowing-up of a complete intersection of three quadrics in $\mathbb{P}^6$, not contained in a quadric.

If X is such that ϕ is a closed imbedding, then the divisor $K + H$ is considered. It turns out that $K + H$ is nef and big unless, always under the assumption that X is not contained in a quadric,

(E3) X is a Del Pezzo fibration over $\mathbb{P}^1$ via $\mid K + H \mid$, with general fibre a complete intersection of type $(2,2)$ in $\mathbb{P}^4$. Moreover, $d = 8$ and $\pi = 7$.

(E4) $d = 9$, $\pi = 9$, and X is a conic bundle over $\mathbb{P}^2$;

(E5) $d = 12$, $\pi = 15$, and X is a conic bundle over a quartic surface of $\mathbb{P}^3$.

In both cases (E4), (E5) the structure of conic bundle to X is given by the map associated to $\mid K + H \mid$.

If $K + H$ is not nef and big, then X is said to be of *log-special type*. Otherwise it is called of *log-general type*; in this last case, a general hyperplane section of X is a minimal surface of general type. Moreover, a suitable multiple of $K + H$ defines a birational morphism.

The key step in the proof of Theorem 1.6 is to show that, under the given assumptions, *X must be log-special*. As remarked after Thm. 1.3, in the isolated case the curves of $\mathcal{F}$ can be only conics or cubics. Then, the possibilities (E1),..., (E5) are ruled out case by case by a careful analysis of the families of conics and cubic curves lying on these threefolds. The last statement of the theorem is proved in [eM], §3. The proof is accomplished.

We conclude this section with one remark.

Remark 1.7. If X is of isolated type, then *the family $\mathcal{F}$ cannot have a base point*. Otherwise, take such a base point P and project X into $\mathbb{P}^4$ from P. The image is a 3-fold X' containing a 3-dimensional family of lines, hence it is a quadric or a scroll in planes, by 1.2. Since X is contained in the cone of vertex P over X', it is not of isolated type.

2.- Some preliminaries on the normal sheaf.

If C is a closed subvariety of a variety X, let $\mathcal{I} \subset \mathcal{O}_X$ denote the coherent ideal which defines C on X. Recall that the *normal sheaf* of C on X is defined as the dual of the $\mathcal{O}_C$-module $\mathcal{I}/\mathcal{I}^2$. We shall denote this sheaf by $\mathcal{N}_{C/X}$.

It is well known that, if C is locally a complete intersection on X, e.g. if both C and X are smooth, then $\mathcal{N}_{C/X}$ is a vector bundle on C of rank equal to the codimension of C in X.
In the case C is an arbitrary (even singular !) plane curve on a smooth X, then $\mathcal{N}_{C/X}$ is a vector bundle on C. In fact, the following lemma allows us to apply the above mentioned theorem.

Lemma 2.1 *Assume that $C \subset X \subset \mathbb{P}^n$ are as above. Assume, moreover, that X is smooth and that C is (locally) a complete intersection in $\mathbb{P}^n$. Then C is locally a complete intersection as a subvariety of X .*

Proof. Being the problem local, let us consider an arbitrary point $x \in C$. We have a surjective map $\mathcal{O}_{X,x} \to \mathcal{O}_{C,x}$. Since $\mathcal{O}_{X,x}$ is a regular local ring and $\mathcal{O}_{C,x}$ is a complete intersection local ring, the result follows from [BH], Thm. 2.3.3. $\square$

Let $X \subset \mathbb{P}^5$ be a threefold satisfying the assumptions of Thm. 1.6. Let $C \in \mathcal{F}$ be a general element and let $h \in \mathbb{Q}[t]$ be the Hilbert polynomial of C. We will identify $\mathcal{F}$ with an irreducible component of the Hilbert scheme H parametrizing the curves on X having h as Hilbert polynomial. It is well known that, for $C \in H$, the Zariski tangent space to H at C is isomorphic to $H^0(\mathcal{N}_{C/X})$. Therefore, from $dim(\mathcal{F}) \geqslant 3$ we get $dim(H) \geqslant 3$, hence the useful numerical information $h^0(\mathcal{N}_{C/X}) \geqslant 3$ for any $C \in \mathcal{F}$.
From now on, we shall denote by δ the degree of a curve $C \in \mathcal{F}$; then $\delta = 2$ or $\delta = 3$.

Another feature of $\mathcal{N}_{C/X}$ in our situation is given by the following

Lemma 2.2 *For a general $C \in \mathcal{F}$ the bundle $\mathcal{N}_{C/X}$ is spanned by its global sections.*

Proof. Use the argument in [tF], pp. 104-105. $\square$

Lemma 2.3 *Let $dim(\mathcal{F}) = 3 + c$, where $c \geqslant 0$. A general (smooth) hyperplane section S of X contains a family of dimension c of curves $C \in \mathcal{F}$.*

Proof. Let $\varphi : \mathcal{F} \dashrightarrow \mathbb{G}(2,5)$ be the rational map which associates to the general $C \in \mathcal{F}$ the linear span $\langle C \rangle$ of C. Then, the fibre of φ over a general element of its image has dimension 0. In fact, otherwise, $\langle C \rangle \subset X$ and a straightforward application of Theorem 1.3 yields $X = \mathbb{P}^3$, a contradiction. Therefore, $dim(\varphi(\mathcal{F})) = 3 + c$. If $\Omega \subset \mathbb{G}(2,5)$ represents the Schubert cycle parametrizing the planes of $\mathbb{P}^5$ contained in a general hyperplane, then Ω intersects $\varphi(\mathcal{F})$ properly, and this proves the lemma since $codim(\Omega) = 3$. $\square$

We shall denote by C^2 the self-intersection of C on S.

Lemma 2.4 *We have*

(i) $c_1(\mathcal{N}_{C/X}) = C^2 + \delta$;

(ii) $K_X \cdot C = 2\,p_a(C) - 2 - C^2 - \delta$.

Proof. From $C \subset S \subset X$ we deduce the exact sequence of $\mathcal{O}_C$-bundles

$$0 \to \mathcal{N}_{C/S} \to \mathcal{N}_{C/X} \to \mathcal{N}_{S/X}|_C \to 0 \tag{1}$$

Then (i) follows, since $\mathcal{N}_{C/S} \simeq \mathcal{O}_C(C)$ and $\mathcal{N}_{S/X}|_C \simeq \mathcal{O}_C(1)$.
From the exact sequence

$$0 \to \mathcal{T}_X \to \mathcal{T}_{\mathbb{P}^5}|_X \to \mathcal{N}_{X/\mathbb{P}^5} \to 0$$

(where $\mathcal{T}_X$ and $\mathcal{T}_{\mathbb{P}^5}$ denote the tangent bundles to X and $\mathbb{P}^5$, respectively) and from $c_1(\mathcal{T}_X) = -K_X$ we get

$$c_1(\mathcal{N}_{X/\mathbb{P}^5}) = 6\,H \cdot X + K_X\ . \tag{2}$$

From the inclusions $C \subset X \subset \mathbb{P}^5$ we deduce the exact sequence of $\mathcal{O}_C$-bundles

$$0 \to \mathcal{N}_{C/X} \to \mathcal{N}_{C/\mathbb{P}^5} \to \mathcal{N}_{X/\mathbb{P}^5}|_C \to 0$$

An easy computation shows that $\mathcal{N}_{C/\mathbb{P}^5} \simeq \mathcal{O}_C(\delta) \oplus \mathcal{O}_C(1)^3$, hence $c_1(\mathcal{N}_{C/\mathbb{P}^5}) = \delta^2 + 3\,\delta$. Moreover, by (2) $c_1(\mathcal{N}_{X/\mathbb{P}^5}|_C) = 6\,H \cdot C + K_X \cdot C = 6\,\delta + K_X \cdot C$, and, by the additivity of Chern classes, it follows

$$K_X \cdot C = 2\,p_a(C) - 2 - c_1(\mathcal{N}_{C/X})\ . \tag{3}$$

Finally, combining (i) with (3) we get (ii). □

For the values of δ we are interested in we get explicitly $K_X \cdot C = -C^2 - 4$ if $\delta = 2$, and $K_X \cdot C = -C^2 - 3$ for $\delta = 3$.

3.- The proof that X is log-special

In this section we will assume that X satisfies the hypotheses of Thm. 1.6.

Lemma 3.1 *On a general hyperplane section S of X we have $C^2 = -1$ if $\delta = 2$, and $C^2 = 0$ or $C^2 = -1$ if $\delta = 3$. Moreover, $dim(\mathcal{F}) = 3$.*

Proof. We shall give the proof in several steps.

STEP 1. Let H_S be a general hyperplane section of S containing C. Since S is non-degenerate, H_S is also non-degenerate. Therefore $H_S \neq C$ and $H_S = C + D$, where D is a curve on S of degree $d - \delta$. The curve H_S is connected, hence $C \cdot D \geqslant 1$. Assume $C \cdot D = 1$. Then S is a Veronese surface or a scroll in $\mathbb{P}^4$ ([VdV]). In the former case, X is a cone over S. But this would imply that X is not smooth, a contradiction. On the other hand, the smooth scrolls in $\mathbb{P}^4$ are classified; they are the rational scroll of degree 3, and the elliptic scroll of degree 5. In the former case, X would be $\mathbb{P}^1 \times \mathbb{P}^2$, which is contained in a quadric hypersurface. The latter case is ruled out because the irregularity of the elliptic quintic scroll is strictly positive, whereas we remarked in the proof of Thm. 1.4, that the irregularity of S is always zero. Therefore, we have $C \cdot D \geqslant 2$, hence

$$\delta = C \cdot H_S = C \cdot (C + D) = C^2 + C \cdot D \geqslant C^2 + 2$$

and we conclude $C^2 \leqslant 0$ for $\delta = 2$, and $C^2 \leqslant 1$ for $\delta = 3$.

STEP 2. We want to exclude, now, the possibility $C^2 = 1$ whenever $\delta = 3$. Since $p_a(C) = 1$, the "adjunction formula" on S yields $-C \cdot K_S = C^2$. By Riemann-Roch

$$\chi(\mathcal{O}_S(C)) = C^2 + \chi(\mathcal{O}_S) = C^2 + 1 + h^2(\mathcal{O}_S).$$

Assume $C^2 = 1$. From the above relation and from $h^2(\mathcal{O}_S) \geqslant h^2(\mathcal{O}_S(C))$ we get

$$h^0(\mathcal{O}_S(C)) \geqslant 2 + h^2(\mathcal{O}_S) - h^2(\mathcal{O}_S(C)) \geqslant 2$$

As a consequence, $dim|C| \geqslant 1$. If we set $h^0(\mathcal{O}_S(C)) = 2 + k$, $k \geqslant 0$, then it is easily checked that $h^1(\mathcal{O}_S(C)) \leqslant k$. From the long exact cohomology sequence associated to

$$0 \to \mathcal{O}_S(C) \to \mathcal{O}_S(1) \to \mathcal{O}_D(1) \to 0$$

we get

$$h^0(\mathcal{O}_S(C)) - h^0(\mathcal{O}_S(1)) + h^0(\mathcal{O}_D(1)) \leqslant h^1(\mathcal{O}_S(C)) \leqslant k,$$

hence $h^0(\mathcal{O}_D(H_S)) \leqslant 3$. From this inequality it follows, in particular, that *the curves D are plane curves.* Moreover, the long exact cohomology sequence associated to

$$0 \to \mathcal{O}_S(D) \to \mathcal{O}_S(1) \to \mathcal{O}_C(1) \to 0$$

jointly with $h^0(\mathcal{O}_C(1)) \geqslant 3$ (by Riemann-Roch on C), yields

$$h^0(\mathcal{O}_S(D)) \geqslant h^0(\mathcal{O}_S(1)) - h^0(\mathcal{O}_C(1)) \geqslant 2.$$

Then $|D|$ is a linear system whose curves cover S. By varing S among the hyperplane sections of X we get a family $\mathcal{G}$ of plane curves covering X, with $dim(\mathcal{G}) \geqslant 3$. Since X is of isolated type, we have $deg(D) = d - 3 \leqslant 3$, hence $d \leqslant 6$. The smooth threefolds $X \subset \mathbb{P}^5$ with $d < 6$ are all contained in a quadric. For $d = 6$ we have that X is a Bordiga's scroll and $deg(D) = 3$. But the hyperplane sections of a Bordiga's scroll contain only finitely many cubic curves. In any case, our assumption $C^2 = 1$ yields a contradiction.

STEP 3. We prove, now, that $dim(\mathcal{F}) = 3$.

In fact, $dim(\mathcal{F}) \geqslant 5$ would yield the existence on S of a family of plane curves of dimension at least 2, by Lemma 2.3. Then S would be a Veronese surface, a contradiction, since S is the hyperplane section of a smooth threefold.

If $dim(\mathcal{F}) = 4$, then the curves of $\mathcal{F}$ on S form a 1-dimensional family, and from $C^2 \leqslant 0$ it follows that this family is actually a fibration of S. Then, as in the proof of Thm. 1.4, we conclude that X is contained in a quadric, a contradiction.

STEP 4. If C is a conic, then $\mathcal{N}_{C/X} \simeq \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \mathcal{O}_{\mathbb{P}^1}(a_2)$. Since $\mathcal{N}_{C/X}$ is spanned by its global sections, then $a_1 \geqslant 0$, $a_2 \geqslant 0$. As a consequence we have $H^1(\mathcal{N}_{C/X}) = 0$, hence, by the theory of Hilbert schemes, $\mathcal{F}$ is smooth at any general point. Therefore, from the previous step and from $a_1 \geqslant 0$, $a_2 \geqslant 0$ we get

$$3 = h^0(\mathcal{N}_{C/X}) = a_1 + a_2 + 2$$

and $a_1 + a_2 = 1$. But $a_1 + a_2 = c_1(\mathcal{N}_{C/X}) = C^2 + 2$, by Lemma 2.4. Then $C^2 = -1$.

Therefore the proof is complete for the case $\delta = 2$ and, from now on, we will assume $\delta = 3$.

STEP 5. We will prove, now, $C^2 \geqslant -1$ in the cases when the general $C \in \mathcal{F}$ is a nodal plane cubic or a cuspidal plane cubic (which are, a priori, possible). In both cases we have a diagram

$$\begin{tikzcd}
\mathbb{P}^1 \arrow[d, "n"'] \arrow[dr, "f"] & \\
C \arrow[r, hook, "j"'] & X
\end{tikzcd}$$

where j is the canonical imbedding and n is the normalization map.
For a general $C \in \mathcal{F}$ by Lemma 2.2 we have that $\mathcal{N}_{C/X}$ is spanned by its global sections. Since n^* is right exact, $n^*\mathcal{N}_{C/X} \simeq \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \mathcal{O}_{\mathbb{P}^1}(a_2)$ is also generated by its global sections, hence $a_1 \geqslant 0$ and $a_2 \geqslant 0$.
Let us define $\mathcal{N}_f\check{}:= Ker(f^*\Omega_X \to \Omega_{\mathbb{P}^1})$. If R denotes the ramification divisor on $\mathbb{P}^1$ of the map $n : \mathbb{P}^1 \to C$, then we have an exact sequence

$$0 \to \mathcal{N}_f\check{} \to f^*\Omega_X \to \Omega_{\mathbb{P}^1}(-R) \to 0$$

([GK], proof of Lemma, pag. 101). It follows that $\mathcal{N}_f\check{}$ is a vector bundle, and we set $\mathcal{N}_f\check{} \simeq \mathcal{O}_{\mathbb{P}^1}(-b_1) \oplus \mathcal{O}_{\mathbb{P}^1}(-b_2)$. Moreover, the degree of $\mathcal{N}_f\check{}$ or, better, of $\mathcal{N}_f$ is easily computed to be

$$b_1 + b_2 = C^2 + 1 \qquad \text{for the nodal case;}$$

(4)

$$b_1 + b_2 = C^2 \qquad \text{for the cuspidal case.}$$

Now, we have the canonical, commutative diagram with exact rows

(5)
$$\begin{tikzcd} & n^*\mathcal{N}_{C/X}\check{} \arrow[r] & f^*\Omega_X \arrow[r] \arrow[d, equal] & n^*\Omega_C \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathcal{N}_f\check{} \arrow[r] & f^*\Omega_X \arrow[r] & \Omega_{\mathbb{P}^1} & \end{tikzcd}$$

Then there exists a map $\alpha : n^*\mathcal{N}_{C/X}\check{} \to \mathcal{N}_f\check{}$. This map is represented by a matrix

$$\begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix}$$

where the F_{ij}'s are homogeneous polynomials of degree $a_i - b_j$. For future use we want to determine the degree of the determinant of α. Of course, it is equal to $dim_k(Coker(\alpha))$; moreover, it is easily seen that the support of $(Coker(\alpha))$ is contained into $n^{-1}(Sing(C))$. Finally, let us consider the exact sequence

$$0 \to K \to n^*\Omega_C \to \Omega_{\mathbb{P}^1} \to \Omega_{\mathbb{P}^1/C} \to 0$$

Let $x \in n^{-1}(Sing(C))$; by applying the "snake's lemma" to the diagram obtained localizing (6) at x, we show that $Coker(\alpha)_x \simeq K_x$. Now, in the nodal case $\Omega_{\mathbb{P}^1/C} = 0$. Therefore $dim_k(K_x) = 1$. Since $n^{-1}(Sing(C))$ consists of two points, we get $dim_k(Coker(\alpha)) = 2$. In the cuspidal case $\Omega_{\mathbb{P}^1/C}$ is a skyscraper sheaf on the unique point $x \in n^{-1}(Sing(C))$, and $dim_k(\Omega_{\mathbb{P}^1/C,x}) = 1$. Therefore, K is a skyscraper sheaf on x, and $dim_k(K) = 2$. Hence, both in the nodal and in the cuspidal case we have

(6)
$$a_1 - b_2 + a_2 - b_1 = deg(det(\alpha)) = 2.$$

After these preparations we will deal first with the cuspidal case. If we tensorize the canonical inclusion $\mathcal{O}_C \subset n_*\mathcal{O}_{\mathbb{P}^1}$ by $\mathcal{N}_{C/X}$, then we get by the "projection formula" another inclusion, namely $\mathcal{N}_{C/X} \subset n_*n^*\mathcal{N}_{C/X}$. Therefore

$$3 \leqslant h^0(\mathcal{N}_{C/X}) \leqslant h^0(n^*\mathcal{N}_{C/X}) = a_1 + a_2 + 2,$$

$a_1 + a_2 \geqslant 1$. From (6) we get $b_1 + b_2 \geqslant -1$, and the conclusion follows by (4).

Rather surprisingly, this approach is too rough for the nodal case; in fact, it yields only $C^2 \geqslant -2$. To overcome this difficulty, we intrduce the sheaf on C

$$\mathcal{N}'_{C/X} := Ker(\mathcal{N}_{C/X} \to \mathcal{T}^1_C)$$

(see [GK]). The infinitesimal deformations of C into X which preserve the singularity of C are parametrized by $H^0(\mathcal{N}'_{C/X})$; moreover, we have $\mathcal{N}'_{C/X} \subset n_*\mathcal{N}_f$ (for all these facts on $\mathcal{N}'_{C/X}$ see [GK]). Therefore, $dim(\mathcal{F}) = 3$ implies

$$3 \leqslant h^0(\mathcal{N}'_{C/X}) \leqslant h^0(\mathcal{N}_f),$$

hence $h^0(\mathcal{O}_{\mathbb{P}^1}(b_1) \oplus \mathcal{O}_{\mathbb{P}^1}(b_2) \geqslant 3$. From this inequality it follows at once that the case $b_1 < 0$ and $b_2 < 0$ is impossible. If $b_1 \geqslant 0$ and $b_2 \geqslant 0$, then

$$b_1 + b_2 + 2 = h^0(\mathcal{O}_{\mathbb{P}^1}(b_1) \oplus \mathcal{O}_{\mathbb{P}^1}(b_2)) \geqslant 3$$

so $b_1 + b_2 \geqslant 1$ and $C^2 \geqslant 0$. Finally, assume $b_1 > 0$ and $b_2 < 0$. Then $h^0(\mathcal{O}_{\mathbb{P}^1}(b_1) \oplus \mathcal{O}_{\mathbb{P}^1}(b_2)) = b_1 + 1$ and $b_1 \geqslant 2$. Assume $a_1 \geqslant b_1$, so $a_1 = b_1 + k$ with $k \geqslant 0$. Then, from (6) we get $b_2 + 2 = a_2 + k$, hence $b_2 \geqslant -2$. From $b_1 \geqslant 2$ it follows $b_1 + b_2 \geqslant 0$, and, finally, $C^2 \geqslant -1$ by (4).
So we can assume $a_1 < b_1$ which implies $F_{11} = 0$. This forces $F_{12} \neq 0$ and $F_{21} \neq 0$, hence $a_1 - b_2 \geqslant 0$ and $a_2 - b_1 \geqslant 0$. Moreover, $a_1 - b_2 > 0$ since $b_2 < 0$. From (6) it follows that there are only two possibilities:

$$a_1 - b_2 = 1 \qquad \text{and} \qquad a_2 - b_1 = 1$$

$$a_1 - b_2 = 2 \qquad \text{and} \qquad a_2 - b_1 = 0$$

In the former case $b_2 \geqslant -1$ and $b_1 + b_2 \geqslant 1$, $C^2 \geqslant 0$. In the latter case we get $C^2 \geqslant -1$.

STEP 6. Finally, we prove that $C^2 \geqslant -1$ in the case when the general $C \in \mathcal{F}$ is elliptic. First of all, we want to show that in this case $\mathcal{N}_{C/X}$ splits.
It is well known that on an elliptic curve C any indecomposable rank 2 vector bundle $\mathcal{E}$ fits into a short exact sequence of one of the following forms

$$0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{L} \to 0 \qquad\qquad 0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{L}(P) \to 0$$

where $\mathcal{L}$ is an invertible sheaf on C and $P \in C$. The exact sequence (1) becomes

$$0 \to \mathcal{O}_C(C) \to \mathcal{N}_{C/X} \to \mathcal{O}_C(1) \to 0 \tag{7}$$

Assume $\mathcal{N}_{C/X}$ indecomposable. Then $\mathcal{O}_C(C) \simeq \mathcal{L}$. But from (7) it follows at once that $c_1(\mathcal{L}) \geqslant 2$, whereas it was already proved that $c_1(\mathcal{O}_C(C)) \leqslant 0$, a contradiction.
Therefore, we can assume $\mathcal{N}_{C/X} \simeq \mathcal{L}_1 \oplus \mathcal{L}_2$ and set $d_i := c_1(\mathcal{L}_i)$. Since $\mathcal{N}_{C/X}$ is spanned, both $\mathcal{L}_i$ are spanned and $d_i \geqslant 0$. If $d_i > 0$ for $i = 1, 2$, then

$$C^2 + 3 = c_1(\mathcal{N}_{C/X}) = c_1(\mathcal{L}_1) + c_1(\mathcal{L}_2) = h^0(\mathcal{L}_1) + h^0(\mathcal{L}_2) = h^0(\mathcal{N}_{C/X}) \geqslant 3 \ ,$$

and we can conclude as in the previous case.

Finally, assume $\mathcal{N}_{C/X} \simeq \mathcal{O}_C \oplus \mathcal{L}$ with $c_1(\mathcal{L}) > 0$. Then, from

$$C^2 + 3 = c_1(\mathcal{N}_{C/X}) = c_1(\mathcal{L}) = h^0(\mathcal{L})$$

we get

$$C^2 + 4 = h^0(\mathcal{O}_C) + h^0(\mathcal{L}) = h^0(\mathcal{N}_{C/X}) \geqslant 3$$

hence $C^2 \geqslant -1$. $\square$

Corollary 3.2 *If X satisfies the assumptions of Theorem 1.6, then X is uniruled.*

Proof. The above lemma jointly with Lemma 2.4 (ii) yield $K_X \cdot C < 0$. Since the curves C cover X, the assertion follows from [MM], Thm.1. $\square$

Lemma 3.3 *If K_X+H is big and nef, then $h^0(\mathcal{O}_X(K_X+H)) \geqslant 2$. In particular, $K_X + H$ is effective and dim $\phi_{K_X+H}(X) \geqslant 1$.*

Proof. By Kodaira's Vanishing Theorem we have

$$\chi(\mathcal{O}_X(K_X + H)) = h^0(\mathcal{O}_X(K_X + H)) \ .$$

Moreover, Serre's duality yields $\chi(K_X) = -\chi(\mathcal{O}_X)$. The following inequality, due to Sommese ([aS], Theorem (1.0)),

$$(K_X + H)^3 \leqslant 36(\chi(K_X) + \chi(\mathcal{O}_X(K_X + H)))$$

then becomes

$$h^0(\mathcal{O}_X(K_X + H)) \geqslant \frac{1}{36}(K_X + H)^3 + \chi(\mathcal{O}_X) \ .$$

Being $K_X + H$ big and nef, $(K_X + H)^3 > 0$. Therefore, the desired conclusion follows since X is uniruled and with irregularity zero, which imply $\chi(\mathcal{O}_X) = 1 + h^2(\mathcal{O}_X) \geqslant 1$. $\square$

We can prove, now, the main result of this section.

Proposition 3.4 *If X satisfies the assumptions of Theorem 1.6, then X is of log-special type.*

Proof. First of all, we deal with the case $\delta = 2$. From Lemma 2.4,(ii) we get

$$(K_X + H) \cdot C = K_X \cdot C + H \cdot C = -C^2 - 2$$

Then, from Lemma 3.1 it follows at once that $(K_X + H)$ cannot be nef.

Assume, now, $\delta = 3$ and $C^2 = 0$. If $K_X + H$ is big and nef, then for some integer $r >> 0$ the rational map $\phi_{r(K_X+H)} : X \dashrightarrow \mathbb{P}^n$ is generically finite onto its image W. Let D_1, D_2 be two general (distinct) elements of the linear system $|K_X + H|$. We have $rD_i = \phi^*_{r(K_X+H)}L_i$, where L_i is a hyperplane section of W, for $i = 1, 2$. On W we have the curve $L_1 \cap L_2$ and, since $\phi_{r(K_X+H)}$ is generically finite over W, then

$$supp(D_1) \cap supp(D_2) = supp(rD_1) \cap supp(rD_2) = \phi^{-1}_{r(K_X+H)}(L_1 \cap L_2)$$

is also a curve outside the base locus B of $|K_X + H|$. Therefore, only finitely many curves $C \in \mathfrak{F}$ can be irreducible components of $(D_1 \cap D_2)\backslash B$. Now, from $\delta = 3$ and from Lemma 2.4 it follows

$$(K_X + H) \cdot C = K_X \cdot C + H \cdot C = -C^2 \tag{8}$$

Then, assume $P \in (D_1 \cap D_2) \backslash B$. By (8) and the assumption $C^2 = 0$, we have that every curve $C \in \mathcal{F}$ such that $P \in C$ is completely contained both in D_1 and in D_2. Since we have infinitely many curves of $\mathcal{F}$ containing P, we have a contradiction.

Finally, assume $C^2 = -1$, always with $\delta = 3$. By Lemma 3.4 we have a rational map $\phi_{K_X+H} : X \dashrightarrow \mathbb{P}^n$, for some $n \geqslant 1$. Let C be a general element of $\mathcal{F}$. The divisor E on C which defines $\phi_{K_X+H}|_C$ has degree 1 by (8). Therefore, $h^0(\mathcal{O}_C(E)) = 1$ and the map $\phi_{K_X+H}|_C$ is *constant*. Let $C_0 \in \mathcal{F}$ be a fixed, general curve. We set

$$\mathcal{F}' := \{C \in \mathcal{F} \mid C \cap C_0 \neq \emptyset\}$$

$\mathcal{F}'$ is a 2-dimensional subfamily of $\mathcal{F}$.
We claim that $V := \bigcup_{C \in \mathcal{F}'} C$ *is dense in* X .
Assume the contrary. Then V is a surface containing a 2-dimensional family of plane cubic curves. By Corrado Segre's theorem ([cS]), $V \subset \mathbb{P}^3$. Then, the curves $C \in \mathcal{F}'$ belong to the linear system of the plane sections of V. In fact, otherwise the degree of V would be at least 4. Since X is fibered by surfaces like V, this would imply the existence on X of a family of dimension $\geqslant 3$ of plane curves of degree > 3. But X is of isolated type and this contradicts the bound on the degree (see remarks after Thm. 1.3.) Therefore, on V we have the 3-dimensional family of its plane sections, and the plane cubic curves on X form a family of dimension 4, which contradicts Lemma 3.1. The proof of the claim is complete.

From the claim it follows that the rational map ϕ_{K_X+H} is constant, which contradicts $n \geqslant 1$. $\square$

4.- Conics contained in the threefolds (E1),...,(E5).

We recall that, if $C \in \mathcal{F}$ is a conic on a general hyperplane section S of X, then $C^2 = -1$ by Lemma 3.1, hence C is an exceptional curve on S.

In the case X is a Del Pezzo fibration over $\mathbb{P}^1$ the general hyperplane section S of X is an elliptic, regular surface which is minimal ([cO1]). Therefore, on S we cannot have conics C with $C^2 = -1$.

In both cases (E1),(E2) the generic hyperplane section S of X is a non-minimal K3 surface. Moreover, in both cases the (-1)-lines on S are contracted by the adjunction map $\phi_{K_S+H_S}$ on S and the image W of this map is a minimal K3 surface ([AR] and [cO2]). Let us remark that, by the "adjunction formula", the restriction to S of the adjunction map ϕ_{K_X+2H} is just $\phi_{K_S+H_S}$. Then, the image of a conic $C \in \mathcal{F}$ in $\phi_{K_S+H_S}$ has degree

$$C \cdot (K_X + 2H) = -C^2 - 4 + 4 = 1$$

hence it is a line on W such that $\phi_{K_S+H_S}(C)^2 = -1$, a contradiction.

Finally, we deal with the cases when X is a conic bundle (see [BOSS]). The image W of the map ϕ_{K_X+H} is $\mathbb{P}^2$ in the case $d = 9$, while $W \subset \mathbb{P}^3$ is a quartic surface in the case $d = 12$. Let $C \in \mathcal{F}$ be a conic on X. Let $D \in |K_X + H|$, and assume $D \cap C \neq \emptyset$, or, more precisely, that $P \in D \cap C$. Then, from $(K_X + H) \cdot C < 0$, it follows that $C \subset D$. Let $R \subset W$ be the image of D in ϕ_{K_X+H}. Moreover, let $R' \subset W$ be a hyperplane section different from R and containing the point $Q := \phi_{K_X+H}(P)$. Finally, set $D' \in |K_X + H|$ the preimage of R'. Then, clearly, $C \subseteq D \cap D'$, hence C is a conic of the fibration. Therefore, $dim(\mathcal{F}) = 2$.

We can conclude that in each case (E1), ... , (E5) we cannot have on X a family of conics of dimension 3.

5.- Plane cubics contained in the threefolds (E1),...,(E5).

In this section we will prove that a 3-fold of one of the types (E1),...,(E5) cannot contain a family of dimension 3 of plane cubics.

(E1) To rule out this 3-fold, we need to study something more its geometry.

It is well known (see e.g. [C] and [AR]) that such a X is ruled by lines over a surface $Y \simeq \mathbb{G}(1,5) \cap \mathbb{P}^8$, and precisely $X \simeq \mathbb{P}(U^* \mid_Y)$ where U is the universal bundle on the Grassmannian. A general hyperplane section S is a non-minimal K3 surface with $\pi = 8$ and $K_S^2 = -5$. Therefore S contains five (-1)-lines $E_1,\dots,E_5$ which are blown-down by the adjunction map $\phi := \phi_{K_S+H_S} : S \longrightarrow \mathbb{P}^8$ (note that $K_S = E_1 + \dots + E_5$); the image of ϕ, which is a minimal K3 surface S' of degree 14 with $\pi = 8$, coincides with the previous Y.

Let us assume now that a K3 scroll X contains a 3-dimensional family of plane cubics: then S contains at least one cubic C of the family. Let $C' := \phi(C) \subset S'$. In the case $C^2 = 0$, if H' is a general hyperplane of $\mathbb{P}^8$, we have

$$C' \cdot H' = (\phi^* C') \cdot (\phi^* H') = (\phi^* C') \cdot (H + K_S) = C \cdot H + \sum (\phi^* C') \cdot E_i = C \cdot H.$$

Hence also C' is a plane cubic, birational to C. Let Γ' be a hyperplane section of S': Γ' is a canonical curve which possesses at least one trisecant line. It is easy to see that if the canonical curve Γ' has one trisecant line, then it has a 1-dimensional family of trisecants, that cut on Γ' a g^1_3; so Γ' is trigonal and its trisecants generate a rational normal scroll. In this case the homogeneous ideal of S' is generated by quadrics and cubics and the quadrics containing S' intersect along a rational normal scroll V (possibly a cone), such that the fibers of the restriction to S' of the natural map $V \to \mathbb{P}^1$ are precisely the curves of $\mid C' \mid$. The diagram

$$\begin{tikzcd} S \arrow[r, "\phi"] & S' \arrow[r, hook] \arrow[dr] & V \arrow[d] \\ & & \mathbb{P}^1 \end{tikzcd}$$

shows that also S is ruled by plane cubics. So, by [kR], S is contained in a quadric: but this is impossible by [AR] (2.12).

In the case $C^2 = -1$, by "adjunction formula" $C \cdot K = 1$, namely C intersects exactly one exceptional line. From this it follows that $\mid C' \mid$ is a pencil of elliptic quartics. This pencil cuts on the general hyperplane section Γ' of S' a g^1_4, so Γ' is a canonical tetragonal curve. By [sM], it follows that Γ' is not a linear section of $\mathbb{G}(1,5)$. This contradiction shows that also this case is impossible.

(E2) X has degree $d = 7$ and sectional genus $\pi = 5$.

The adjunction map $\phi_{K+2H} : X \to \mathbb{P}^6$ is the contraction of the exceptional plane contained in X, and the image X' is a Fano manifold, complete intersection of type $(2,2,2)$. As in the previous case, if X contains a 3-dimensional family of plane cubics and if $C^2 = 0$, then also X' contains a similar family. Hence a general hyperplane section S' of X' contains at least one plane cubic and a general hyperplane section Γ' of S' possesses at least one trisecant line. But Γ' is a canonical curve, so it is trigonal. From the classification of trigonal Fano varieties ([I]), it follows that this case is impossible.

If $C^2 = -1$, as in the previous case we may conclude that S' contains a pencil of elliptic quartics curves. The union of the $\mathbb{P}^3$'s generated by such quartics is a quadric of rank 4 in $\mathbb{P}^5$ containing S'. Since S' is a general hyperplane section of X', also X' is contained in a quadric Q of rank 4. Let $X' = Q \cap Q_1 \cap Q_2$; then an elementary computation shows that X' is singular at the four points of intersection of the plane $SingQ$ with $Q_1 \cap Q_2$: also this case is excluded.

(E3) X is a Del Pezzo fibration over $\mathbb{P}^1$.

It is shown in [BOSS], that the image of the map ϕ_{K+H} is $\mathbb{P}^1$, hence the case $C^2 = -1$ is excluded as in the proof of Prop.3.5.

Moreover the fibers are Del Pezzo surfaces of $\mathbb{P}^4$, i. e. complete intersections of type $(2,2)$. If $C^2 = 0$ then, from $(K+H) \cdot C \leqslant 0$, it follows that the plane cubics should be contained in such Del Pezzo surfaces, so each surface of $\mid K + H \mid$ should contain a family of dimension 2 of plane cubics: but this contradicts the theorem of C.Segre ([cS]).

(E4), (E5) If $C^2 = -1$, we may argue as in the proof of Prop.3.5. If $C^2 = 0$, then the discussion goes as in the case of the family of conics.

References

[AR] Aure A. B. - Ranestad K.: The smooth surfaces of degree 9 in $\mathbb{P}^4$. Complex Projective Geometry, Proceedings Bergen-Trieste, London Math. Soc. LNS **179**, 32-46 (1992)

[BOSS] Braun R. - Ottaviani G. - Schneider M. - Schreyer F.O.: Classification of log-special 3-folds in $\mathbb{P}^5$. Preprint Bayreuth (1992)

[BH] Bruns W. - Herzog J.: Cohen-Macaulay rings. Cambridge Univ. Press, 1993

[C] Chang M.C.: Classification of Buchsbaum subvarieties of codimension 2 in projective space. J. reine angew. Math. **401**, 101-112 (1989)

[DP] Decker W. - Popescu S.: On surfaces in $\mathbb{P}^4$ and 3-folds in $\mathbb{P}^5$. Preprint Saarbrücken (1993)

[aF] Franchetta A.: Sulla curva doppia della proiezione della superficie generale dell'S_4, da un punto generico su un S_3. Rend. Acc. d'Italia ser. VII-2, 282-288 (1940)

[tF] Fujita T.: Classification Theories of Polarized Varieties. London Math. Soc. Lecture Note Ser. 155, Cambridge University Press, 1990

[GK] Greuel G.M. - Karras U.: Families of varieties with prescribed singularities. Comp. Math. **69**, 83-110 (1989)

[I] Iskovskih V.A.: Fano 3-folds II. Math. U.S.S.R. Izv. **12**, 469-506 (1978)

[LT] Lanteri A. - Turrini C.: Some formulas concerning non-singular algebraic varieties embedded in some ambient variety. Atti Accad. Naz. Lincei (8) **69**, 463-474 (1980)

[eM] Mezzetti E.: Projective varieties with many degenerate subvarieties. BUMI (7) **8-B**, 807-832 (1994)

[MP] Mezzetti E. - Portelli D.: A tour through some classical theorem on algebraic surfaces. Preprint, Trieste 1995

[MR] Mezzetti E. - Raspanti I.: A Laudal-type theorem for surfaces in $\mathbb{P}^4$. Rend. Sem. Mat. Univ. Politec. Torino **48**, 529-537 (1990)

[MM] Miyaoka Y. - Mori S.: A numerical criterion for uniruledness. Annals of Math. **124**, 65-69 (1986)

[sM] Mukai S.: Curves and Symmetric Spaces. Proc. Japan Acad. **68** Ser.A, 7-10 (1992)

[cO1] Okonek C.: Flächen vom Grad 8 in $\mathbb{P}^4$. Math Z. **191**, 207-223 (1986)

[cO2] Okonek C.: Über 2-codimensionale Untermannigfaltigkeiten vom Grad 7 in $\mathbb{P}^4$ und $\mathbb{P}^5$. Math Z. **187**, 209-219 (1984)

[gO] Ottaviani G.: On 3-folds in $\mathbb{P}^5$ which are scrolls. Ann. Scuola Norm. Sup. Pisa Cl. Sci. Ser. (4) **19**, 451-471 (1992)

[kR] Ranestad K.: On Smooth Plane Curve Fibrations in $\mathbb{P}^4$. In Geometry of Complex Projective Varieties, Cetraro (Italy) 1990, Mediterranean Press 243-255 (1993)

[eR] Rogora E.: Varieties with many lines. Manuscr. Math. **82**, 207-226 (1994)

[bS] Segre B.: Sulle V_n contenenti piú di $\infty^{n-k} S_k$. I, Lincei - Rend. Sc. fis. mat. e nat. **5**, 193-197 (1948); II, Lincei - Rend. Sc. fis. mat. e nat. **5**, 275-280 (1948)

[cS] Segre C.: Le superficie degli iperspazi con una doppia infinità di curve piane o spaziali. Atti R. Acc. Scienze di Torino **56**, 75-89 (1921)

[fS] Severi F.: Intorno ai punti doppi improprii di una superficie generale dello spazio a quattro dimensioni, e ai suoi punti tripli apparenti. Rend. Circolo Matematico di Palermo **15**, 33-51 (1901)

[aS] Sommese A. J.: On the nonemptiness of the adjoint linear system of a hyperplane section of a threefold. J. reine angew. Math. **402**, 211-220 (1989)

[VdV] Van de Ven A.: On the 2-connectedness of a very ample divisor on a surface. Duke Math. J. **46**, 403-407 (1979)