Part II

Model reduction and regularized fluid equations

Outline

1. A general framework for model reduction
2. Large eddy simulations and Reynolds averaging
3. Lagrangian mean flow approximation and regularized Lagrangian fluid equations
4. Time-stepping induced regularizations

References: LC Berselli, T. Iliescu, WJ Layton, Mathematics of large eddy simulation of turbulent flows.

Work on pressure regularization in collaboration with Andrew Staniforth, Nigel Wood (both UK Met Office) and Tobias Hundertmark (Uni Potsdam).

Ongoing work with MPI Hamburg (Peter Korn, Marco Restelli, Marco Giorgetta) on combining LES with pressure regularization.

1. A general framework for model reduction

Consider the rapidly forced differential equation

$$\frac{dz}{dt} = f(z, t, t/\varepsilon),$$

where $\varepsilon \ll 1$. We assume furthermore that $f(z, t, \tau)$ is 2π-periodic in $\tau = t/\varepsilon$.

The slowly varying mean $\bar{z}(t)$ can be characterized by the averaged equation

$$\begin{aligned} \frac{d\bar{z}}{dt} = \bar{f}(\bar{z}, t) &= \lim_{T\to\infty} T^{-1} \int_0^T f(\bar{z}, t, t'/\varepsilon)\, dt' \\ &= (2\pi)^{-1} \int_0^{2\pi} f(\bar{z}, t, \tau)\, d\tau. \end{aligned}$$

The last equality is a (trivial) example of ergodicity (time average equal to ensemble average).

Consider now a vector-valued differential equation

$$\frac{d\mathbf{z}}{dt} = f(\mathbf{z}; \varepsilon), \qquad \mathbf{z} \in S,$$

with fast and slow (resolved and unresolved) solution components $\mathbf{z}_f$ and $\mathbf{z}_s$, respectively, i.e., we may transform the given system into

$$\frac{d\mathbf{z}_s}{dt} = f_s(\mathbf{z}_s, \mathbf{z}_f; \varepsilon), \qquad \frac{d\mathbf{z}_f}{dt} = f_f(\mathbf{z}_s, \mathbf{z}_f; \varepsilon)$$

and $\varepsilon > 0$ is a scale parameter.

A 'naive' model reduction strategy would be to set $\mathbf{z}_f = 0$ in the differential equation for $\mathbf{z}_s$, i.e.

$$\frac{d\bar{\mathbf{z}}_s}{dt} = f_s(\bar{\mathbf{z}}_s, 0, \varepsilon).$$

A more refined model reduction is to evoke the ergodicity principle (time average is equal to a space average).

Hence we may eliminate the fast contribution to f_s in terms of a conditional expectation

$$\bar{f}_s := \mathbb{P} f_s = \mathbb{E}[f_s | \mathbf{x}_s].$$

The projector $\mathbb{P}$ turns any function g of $\mathbf{x}_s$ and $\mathbf{z}_f$ into a function $\bar{g}$ of $\mathbf{z}_s$ alone by taking the expectation value with respect to $\mathbf{z}_f$ conditioned on $\mathbf{z}_s$, i.e., we have a conditioned density $\rho(\mathbf{z}_f|\mathbf{z}_s)$ such that

$$(\mathbb{P}g)(\mathbf{z}_s) = \int g(\mathbf{z}_s, \mathbf{z}_f)\, \rho(\mathbf{z}_f|\mathbf{z}_s)\, d\mathbf{z}_f.$$

We have assumed that ρ is independent of time (stationarity).

A slow manifold approximation

$$\mathcal{M} = \{(\mathbf{x}_s, \mathbf{x}_f) : \mathbf{x}_f = f(\mathbf{x}_s)\}$$

would, for example, lead to

$$\rho(\mathbf{z}_f|\mathbf{z}_s) = \delta(\mathbf{z}_f - f(\mathbf{z}_s)).$$

The hydrostatic and geostrophic approximations fall into this category.

In constant temperature molecular dynamics, free energy calculations in a reduced set of variables $\mathbf{x}_s$ lead to

$$\rho(\mathbf{z}_f|\mathbf{z}_s) = \frac{1}{C}\,\mathsf{exp}(-\beta H(\mathbf{z}_s, \mathbf{x}_f))$$

$\beta = 1/k_B T$.

Both the 'naive' model reduction strategy as well as the refined strategy with respect to a conditional expectation value lead to a projector $\mathbb{P}$ and its complement $\mathbb{Q} = I - \mathbb{P}$ and we may write the full equation for $\mathbf{z}_s$ in abstract form

$$\frac{d\mathbf{z}_s}{dt} = \underbrace{\mathbb{P} f_s}_{\text{resolved}} + \underbrace{\mathbb{Q} f_s}_{\text{unresolved}}$$

A 'clever' choice of $\mathbb{P}$ should minimize the unresolved contribution $\mathbb{Q} f_s$.

The $\mathbb{Q} f_s$ term needs then to be parameterized (e.g. in terms of the fluctuation-dissipation terms from the Mori-Zwanzig formalism). It also often leads to non-Markovian terms (approximations depending on the past and not just the current state of the system).

We now discuss a sequence of examples from fluid dynamics.

2. Reynolds averaging and large eddy simulations

The Euler equations for a rotating fluid in 3D Cartesian geometry are given by

$$\begin{aligned}\frac{D\mathbf{u}}{Dt} + f\mathbf{k}\times\mathbf{u} + \frac{1}{\rho}\nabla p + g\mathbf{k} &= 0,\\ \rho_t + \nabla\cdot(\rho\mathbf{u}) &= 0,\\ \theta_t + \mathbf{u}\cdot\nabla\theta &= 0,\end{aligned}$$

where $\mathbf{u}\in\mathbb{R}^3$ is the velocity (wind) field, θ is the potential temperature, $k=(0,0,1)^T$, $f=2\Omega\sin\phi_0$ is the Coriolis parameter, g the gravitational constant, and the ideal gas law becomes

$$\theta = T(p_0/p)^{R/c_p}.$$

These equations constitute the model for the dry atmosphere.

Remarks:

- The Reynolds number is very large and molecular viscosity effects can be ignored!
- Direct numerical simulations (DNS) are impossible and filtered equations are being used. Two main approaches:
  - asymptotic scale analysis leading to "slow manifold" approximations and simplified equations (e.g. hydrostatic approximation $p_z + \rho g = 0$),
  - Spatially (Large Eddy Simulations) or temporally (Reynolds) averaged equations in filtered variables:

$$\bar{f} = \mathcal{S} * f = f - f',$$

    $\mathcal{S}$ an appropriate spatial or temporal (low pass) filter, i.e. $\bar{f} \rightarrow \mathbf{x}_s,\ f' \rightarrow \mathbf{x}_f$.

What are useful averages?

- Consider a smooth function $g(t)$ with the properties $g(t) \geq 0$, $g(t) = g(-t)$, and $\int_{\mathbb{R}} g(t)dt = 1$. Then we define

$$\bar{f} = \mathcal{S} * f := \gamma^{-1} \int_{\mathbb{R}} f(\tau)\, g((\tau - \cdot)/\gamma)\, d\tau,$$

  where γ is a given time-scale, which "distinguishes" between slow and fast. The idea of time filtering goes back to Reynolds.

- The same idea applied to spatial dimensions leads to the large eddy simulation (LES) spatial averages

$$\bar{f} = \mathcal{S} * f = \alpha^{-d} \int_{\mathbb{R}^d} f(\mathbf{x}')\, g((\mathbf{x}' - \cdot)/\alpha)\, d\mathbf{x}',$$

  where $\alpha > 0$ is now a length scale. The function g might be explicitly defined (e.g. Gaussian) or implicitly defined as the solution of an appropriate elliptic PDE.

Applying the averaging operator for simplicity to the incompressible Euler equations leads to

$$\begin{aligned} \bar{\mathbf{u}}_t + \nabla \cdot (\overline{\mathbf{u} : \mathbf{u}}) + \nabla \bar{p} &= 0, \\ \nabla \cdot \bar{\mathbf{u}} &= 0. \end{aligned}$$

and to the closure problem

$$\begin{aligned} \bar{\mathbf{u}}_t + \nabla \cdot (\bar{\mathbf{u}} : \bar{\mathbf{u}}) + \nabla \bar{p} &= R(\mathbf{u}, \mathbf{u}), \\ \nabla \cdot \bar{\mathbf{u}} &= 0 \end{aligned}$$

with (unknown) Reynolds stress tensor

$$R(\mathbf{u}, \mathbf{u}) = \nabla \cdot (\bar{\mathbf{u}} : \bar{\mathbf{u}}) - \nabla \cdot (\overline{\mathbf{u} : \mathbf{u}}).$$

The key challenge is now to find an appropriate representation

$$R(\mathbf{u}, \mathbf{u}) = \mathbb{P}R\,(\bar{\mathbf{u}}) + \mathbb{Q}R\,(\bar{\mathbf{u}}, \mathbf{u}').$$

Traditionally one has used $\mathbb{P}R = 0$ and $\mathbb{Q}R$ has been modeled by viscosity (eddy viscosity).

More recently, models with non-vanishing $\mathbb{P}R$ have been proposed. Stochastic contributions to $\mathbb{Q}R$ have also been considered (backscattering).

We discuss the basic ideas of a non-vanishing $\mathbb{P}R$ for the conservative Bardina model.

Based on self-similarity considerations we make the approximations:

$$\begin{aligned} R(\mathbf{u},\mathbf{u}) &\approx R(\bar{\mathbf{u}},\bar{\mathbf{u}}) = \nabla\cdot(\bar{\bar{\mathbf{u}}}:\bar{\bar{\mathbf{u}}}) - \nabla\cdot(\overline{\bar{\mathbf{u}}:\bar{\mathbf{u}}}) \\ &\approx \nabla\cdot(\bar{\mathbf{u}}:\bar{\mathbf{u}}) - \nabla\cdot(\overline{\bar{\mathbf{u}}:\bar{\mathbf{u}}}). \end{aligned}$$

We obtain the averaged equations in $\mathbf{w}=\bar{\mathbf{u}}$ and $q=\bar{p}$:

$$\begin{aligned} \mathbf{w}_t + \nabla\cdot(\overline{\mathbf{w}:\mathbf{w}}) + \nabla q &= 0, \\ \nabla\cdot\mathbf{w} &= 0. \end{aligned}$$

Global existence and uniqueness of weak solutions has been shown by Layton & Lewandowski, 2006, and Lunasin & Titi, 2006, for the filter

$$\mathcal{S} = (I - \alpha^2 \nabla^2)^{-1}$$

which implies conservation of the total (kinetic) energy

$$T(t) = \frac{1}{2}\left[\|\mathbf{w}\|^2 + \alpha^2 \|\nabla \mathbf{w}\|^2\right].$$

Numerically it has been found that the model needs to be complemented by a large eddy viscosity term, i.e.

$$\mathbb{Q}R\,(\bar{\mathbf{u}}, \mathbf{u}') \approx \frac{1}{2}\nabla \cdot (\mu(\nabla \bar{\mathbf{u}}) + \mu(\nabla \bar{\mathbf{u}})^T),$$

to yield "realistic" results (e.g. kinetic energy spectra), where μ depends on $\bar{\mathbf{u}}$.

More accurate LES models can be obtained. Consider the filter

$$\bar{\phi} = \mathcal{S}\phi := (I - \alpha^2 \nabla^2)^{-1} \phi$$

and its approximative inverse (deconvolution)

$$\mathcal{S}_N^I := \sum_{i=0}^{N} (Id - \mathcal{S})^i$$

in terms of a truncated (non-converging) Neumann series. We are talking about an ill-posed inverse problem!

We obtain

$$\phi - \mathcal{S}_N^I \bar{\phi} = (-1)^N \; \alpha^{2N+2} \; \nabla^{2N+2} \bar{\phi}$$

and the Stolz-Adams LES models

$$\begin{aligned} \mathbf{w}_t + \nabla \cdot (\overline{S_N^I \mathbf{w} : S_N^I \mathbf{w}}) + \nabla q &= 0, \\ \nabla \cdot \mathbf{w} &= 0. \end{aligned}$$

It is important to study the behavior of the regularized equations (and in particular the regularized advection operator) under small perturbations $\mathbf{w}'$ about a constant mean flow $\mathbf{b}$, i.e., we set

$$\mathbf{w} = \mathbf{b} + \mathbf{w}'.$$

The standard linearization approach leads to the linear advection equation

$$\mathbf{w}'_t = -\mathbf{b} \cdot \nabla \mathbf{w}'$$

for the non-regularized advection operator and to

$$\mathbf{w}'_t = -\mathbf{b} \cdot \nabla \bar{\mathbf{w}}'$$

for the Bardina model under the assumption that $\nabla \cdot \mathbf{w}' = 0$.

We now consider $\mathbf{b} = [1, 0, 0]^T$ and

$$\mathbf{w}' = e^{\mathrm{i}(\omega t - kx)} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

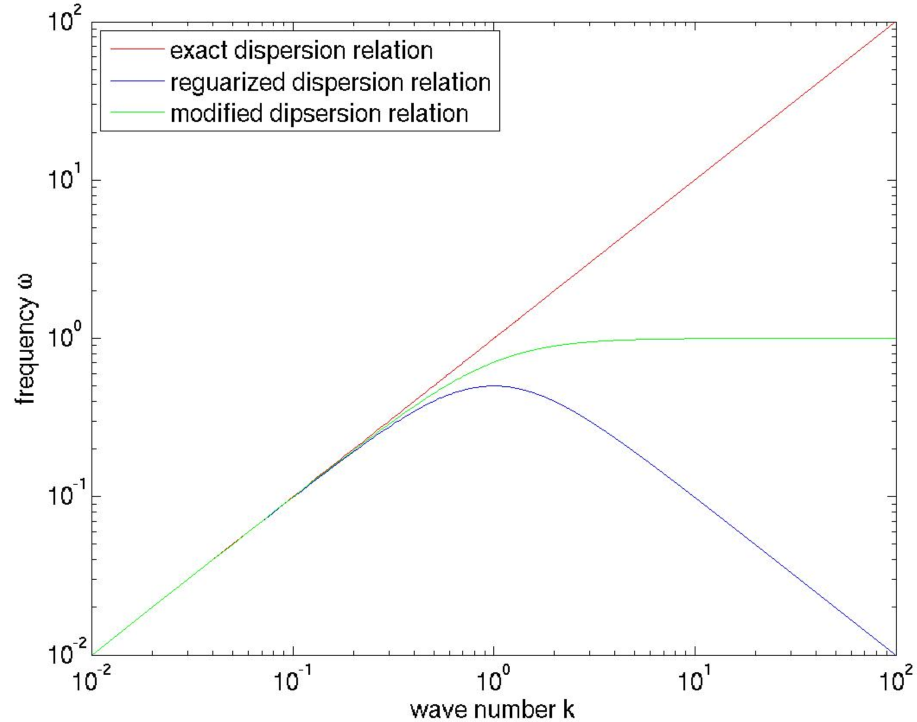
to obtain the dispersion relation

$$\omega = k$$

for the non-regularized operator and

$$\omega = \frac{k}{1 + \alpha^2 k^2}$$

for the Bardina model.



The blue dispersion relation (Bardina model) is problematic since waves with a very short wavelength become very slow. The green line is better and will be discussed under section 9.

3. Mean flow approximations for compressible Euler equations

Consider a barotropic (pressure depends on density only) ideal fluid in its Lagrangian (mechanical) formulation

$$\rho_0 \frac{D\mathbf{u}}{Dt} + \nabla p = 0, \qquad \frac{D\mathbf{X}}{Dt} = \mathbf{u}$$

with the equation of state $p = p(\rho)$ and density

$$\rho(\mathbf{x}, t) = \int \rho_0(\mathbf{x}_0)\, \delta(\mathbf{x} - \mathbf{X}(\mathbf{x}_0, t))\, dA(\mathbf{x}_0). \qquad (1)$$

Note that the material time derivative satisfies

$$\frac{Df}{Dt} = f_t + \mathbf{u} \cdot \nabla f$$

and that (1) is equivalent to the continuity equation

$$\rho_t = -\nabla \cdot (\rho \mathbf{u})$$

for sufficiently regular flow maps $\mathbf{X}(\mathbf{x}_0, t)$.

Let us now drop the assumption that $\mathbf{X}(\mathbf{a}, t)$ is one-to-one for all times t (Brenier's generalized solutions) and let us look at mean flow quantities and the formal equations

$$\begin{aligned} \rho_0 \frac{D\bar{\mathbf{u}}}{Dt} + \nabla \bar{p} &= 0, \\ \frac{D\bar{\mathbf{X}}}{Dt} &= \bar{\mathbf{u}}, \end{aligned}$$

where contributions from the Reynolds stress terms have been ignored ($\overline{Du/Dt} \neq D\bar{u}/Dt$, in general).

The key issue is now to approximate the mean density $\bar{\rho}$, which then also implies a mean pressure $\bar{p}$ under the variational principle. We will discuss this aspect later when we consider particle methods in Part III.

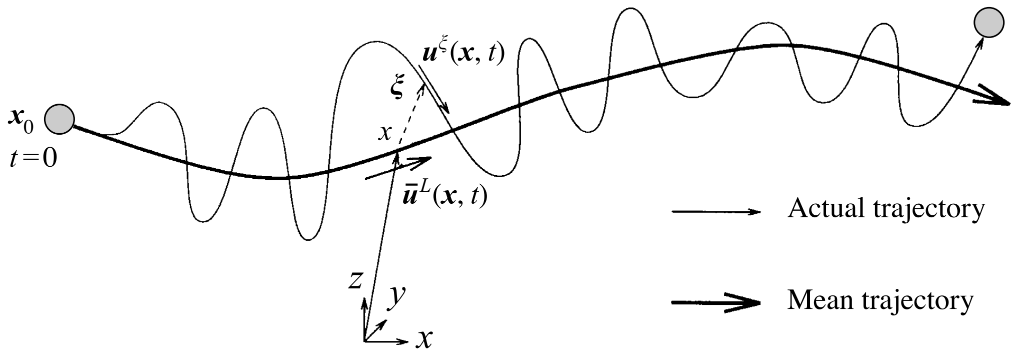


FIGURE 3. Mean and actual particle trajectories, which are supposed to have started from the same position $\boldsymbol{x}_0$. The position $\boldsymbol{x} + \boldsymbol{\xi}(\boldsymbol{x}, t)$ is the *actual* position of the particle whose *mean* position is $\boldsymbol{x}$ at time t.

The mean is to be understood in a "Lagrangian" sense, i.e., we make use of a mean path and fluctuation decomposition

$$\mathbf{X} = \bar{\mathbf{X}} + \xi$$

and obtain for the density

$$\rho(\mathbf{x}, t, \xi) = \int \rho_0(\mathbf{x}_0)\, \delta(\|\mathbf{x} - \bar{\mathbf{X}}(\mathbf{x}_0, t) - \xi\|)\, dA(\mathbf{x}_0).$$

Given a conditioned probability distribution function of the form $\pi(\xi|\bar{\mathbf{X}}) = \psi(\|\xi\|)$ for the fluctuations ξ, we obtain

$$\bar{\rho}(\mathbf{x},t) := [\mathbb{P}\rho]\,(\mathbf{x},t) = \int \rho_0(\mathbf{x}_0)\,\psi(\|\mathbf{x} - \bar{\mathbf{X}}(\mathbf{x}_0,t)\|)\,dA(\mathbf{x}_0)$$

for the slowly advected mean density $\bar{\rho}$.

Now take ψ as the Greens function of the Helmholtz operator $(1-\alpha^2\Delta)^p$. Global existence and uniqueness of weak solutions can be shown under the regularization (closure)

$$\bar{\rho} = (I - \delta^2\nabla^2)^{-p}\rho, \qquad p > 1,$$

(Oelschläger, 1991, Di Lisio, Grenier & Pulvirenti, 1998).

This regularization was motivated by particle methods such as SPH. Note that particle paths may cross and viscosity needs to be added to the model to prevent this from happening.

4. Time stepping induced regularizations

Step (i). Consider Reynolds averaging of an evolution equation

$$\frac{\partial \mathbf{u}}{\partial t} = F(\mathbf{u})$$

under the box filter

$$\bar{f}(t) := \frac{1}{\Delta t} \int_{-\Delta t/2}^{\Delta t/2} f(t+\tau)\, d\tau.$$

Note that we have

$$\overline{\frac{\partial f}{\partial t}}(t) = \frac{1}{\Delta t}(f(t+\Delta t/2) - f(t-\Delta t/2))$$

for the box filter and hence the filtered equation becomes

$$\frac{\mathbf{u}(t+\Delta t/2) - \mathbf{u}(t-\Delta t/2)}{\Delta t} = \frac{1}{\Delta t} \int_{-\Delta t/2}^{\Delta t/2} F(\mathbf{u}(t+\tau)\, d\tau.$$

Step (ii). Closure is achieved by applying an appropriate quadrature rule, e.g., the trapezoidal rule:

$$\frac{\mathbf{w}(t+\Delta t/2)-\mathbf{w}(t-\Delta t/2)}{\Delta t} \approx \underbrace{\frac{1}{2}\left[F(\mathbf{w}(t+\Delta t/2))+F(\mathbf{w}(t-\Delta t/2))\right]}_{\text{closure}}$$

Step (iii). Find an equivalent explicit time-stepping method, e.g.

$$\frac{\mathbf{w}(t+\Delta t)-\mathbf{w}(t-\Delta t)}{2\Delta t} = \tilde{F}(\mathbf{w}(t))$$

for a set of averaged equations

$$\frac{\partial \mathbf{w}}{\partial t} = \tilde{F}(\mathbf{w}).$$

5. A linear example

Consider the linear wave equation

$$u_t = -p_x + f(x,t),$$
$$p_t = -u_x,$$

where f is a given function. Time filtering is equivalent to the Crank-Nicolson (CN) method and leads to:

$$\frac{u^{n+1} - u^n}{\Delta t} = -p_x^{n+1/2} + f^{n+1/2},$$
$$\frac{p^{n+1} - p^n}{\Delta t} = -u_x^{n+1/2}$$

with, for any X, the temporal average

$$X^{n+1/2} \equiv \frac{1}{2}(X^{n+1} + X^n).$$

We now introduce the filtered wave equation:

$$u_t = -\widehat{p}_x + f, \qquad p_t = -u_x,$$

with

$$\left[1 - \alpha^2 \partial_x^2\right] (\widehat{p} - p) = \alpha^2 \left[p_{xx} - f_x\right].$$

These equations are discretized by an explicit method:

$$\frac{u^{n+1/2} - u^{n-1/2}}{\Delta t} = -\widehat{p}_x^n + f^n,$$
$$\frac{p^{n+1} - p^n}{\Delta t} = -u_x^{n+1/2}.$$

The CN and the filtered explicit method are equivalent in terms of the propagation of p provided that $\alpha = \Delta t/2$, i.e., the CN time-average is equivalent to a spatial average.

6. Example: Compressible flows

We start from a Crank-Nicolson discretization of the Lagrangian formulation, i.e.

$$\begin{aligned}
\frac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\Delta t} &= -\frac{1}{2\rho_0}\left[\nabla p^{n+1} + p^n\right], \\
\frac{\mathbf{X}^{n+1} - \mathbf{X}^n}{\Delta t} &= \frac{1}{2}\left[\mathbf{u}^{n+1} + \mathbf{u}^n\right]
\end{aligned}$$

and re-interpret this as the Störmer-Verlet method applied to the regularized equations

$$\begin{aligned}
\mathbf{u}^{n+1/2} &= \mathbf{u}^n - \frac{\Delta t}{2\rho_0}\nabla \bar{p}^n, \\
\mathbf{X}^{n+1} &= \mathbf{X}^n + \Delta t \mathbf{u}^{n+1/2}, \\
\mathbf{u}^{n+1} &= \mathbf{u}^{n+1/2} - \frac{\Delta t}{2\rho_0}\nabla \bar{p}^{n+1}.
\end{aligned}$$

The relation between p and $\bar{p}$ is best explained in the Eulerian formulation

$$\begin{aligned} \mathbf{m}_t + \nabla \cdot (\mathbf{u} : \mathbf{m}) + \nabla \bar{p} &= 0, \\ \rho_t + \nabla \cdot \mathbf{m} &= 0, \end{aligned}$$

with momentum density $\mathbf{m} = \rho \mathbf{u}$, gas law $p = p(\rho)$, speed of sound $c_s^2 = \frac{dp}{d\rho}(\rho_{ref})$, and

$$\left[I - \left(\frac{c_s \Delta t}{2} \right)^2 \nabla^2 \right] (\bar{p} - p) = \left(\frac{c_s \Delta t}{2} \right)^2 \nabla \cdot \{ \nabla p + \nabla \cdot (\mathbf{u} : \mathbf{m}) \}.$$

With $p = p_{inc} + p'$, where p_{inc} satisfies

$$0 = \nabla \cdot \mathbf{m}_t = \nabla \cdot \{ \nabla p_{inc} + \nabla \cdot (\mathbf{u} : \mathbf{m}) \}$$

we get

$$\bar{p}_{inc} = p_{inc}, \qquad \bar{p}' - p' = \left(\frac{c_s \Delta t}{2} \right)^2 \nabla^2 \bar{p}'.$$

This works well for small Mach numbers $Ma = U_{ref}/c_s$ and leads to schemes with stability determined by advection.

Higher accuracy can be achieved by approximate deconvolution:

(i) Define:

$$
\begin{aligned}
R^0 &:= \left(\frac{c_s \Delta t}{2}\right)^2 \nabla \cdot \{\nabla p + \nabla \cdot (\mathbf{u} : \mathbf{m})\}, \\
\mathcal{S} &:= \left[I - \left(\frac{c_s \Delta t}{2}\right)^2 \nabla^2 \right]^{-1}.
\end{aligned}
$$

(ii) for $i = 1 : N$, compute $R^i := \mathcal{S} R^{i-1}$.

(iii) Obtain filtered pressure $\bar{p}$ from

$$
\bar{p} - p = \mathcal{S} \left\{ R^0 - \sum_{i=1}^{N} (-1)^{i-1} \left(\left(\frac{c_s \Delta t}{2}\right)^2 \nabla^2 \right)^{i-1} R^i \right\}.
$$

Properties of the regularized equations:

Reversibility: The regularized Euler equations are still time reversible.

Linear stability: Linearization about motionless stationary state yields

$$|\omega| \leq \frac{2}{\Delta t}$$

for all frequencies $\omega(k)$ and wave numbers k.

Incompressibility: The incompressible limit for $Ma \to 0$ is maintained, i.e., the filtered equations reduce to the incompressible Euler equations if initialized appropriately ($p' \approx 0$, $p'_t \approx 0$).

7. Lighthill radiation

Let us assume a small Mach number limit (nearly incompressible behavior on length scales of order l_{ref}).

Recall that $p = p_{inc} + p'$ where p_{inc} satisfies

$$\nabla \cdot \mathbf{m}_t = \nabla \cdot [\nabla \cdot (\mathbf{u} : \mathbf{m}) + \nabla p_{inc}] = 0,$$

and

$$p'_{tt} - c_s^2 \Delta p' = S(x, t, M)$$

is the Lighthill equation [Lighthill, 1952] with source term S determined from the incompressible Euler equations.

The Lighthill equation

$$p'_{tt} - c_s^2 \Delta p' = S(x, t, M)$$

allows for resonant interaction of slow vortical (incompressible) motion on a length scale l_{ref} with sound waves of sufficiently large scales l_{sound}, i.e.

$$l_{sound}/l_{ref} \sim Ma^{-1} = c_s/u_{ref}.$$

The wave generation due to S is weak but of polynomial order in M. This effect has also been investigated by Einstein in the context of gravity wave generation.

Lighthill radiation implies that the 'naive' projector $\mathbb{P}$ characterized by divergence $\delta = 0$ is not sufficient in general. Fortunately, the pressure regularization just does the right "projection".

Lighthill radiation under regularization:

We obtain $\bar{p} \approx p$ for $Ma \sim 1$. For $Ma \to 0$ recall that $p_2 = p_{inc} + p'$ and the regularization reduces to

$$\left[1 - \left(\frac{c_s \Delta t}{2}\right)^2 \Delta\right] \bar{p}' \approx p'.$$

Since

$$c_s \frac{\Delta t}{l_{sound}} = c_s \frac{\Delta t}{l_{ref}} \frac{l_{ref}}{l_{sound}} \sim Ma^{-1} \frac{l_{ref}}{l_{sound}},$$

we have either

(i) $\bar{p}' \approx 0$ for $l_{sound}/l_{ref} = \mathcal{O}(Ma^0)$ (incompressible regime)

or

(ii) $\bar{p}' \approx p'$ for $l_{ref}/l_{sound} = \mathcal{O}(Ma)$ (Lighthill regime).

8. Example: Stratified flow and vertical slice model

The following regularized vertical slice model has been proposed by Dubal, Wood, Staniforth, Reich (2006):

$$
\begin{aligned}
\frac{Du}{Dt} &= -c_p\theta\frac{\partial\tilde{\pi}}{\partial x}, \\
\left(1+\alpha^2\bar{\mathcal{N}}^2\right)\frac{Dw}{Dt} &= -c_p\theta\frac{\partial\tilde{\pi}}{\partial z} - g, \\
\frac{D\pi}{Dt} &= -\pi\left(\frac{\kappa}{1-\kappa}\right)\left(\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}\right), \\
\frac{D\theta}{Dt} &= 0,
\end{aligned}
$$

where $\bar{\mathcal{N}} \equiv g(d\ln\bar{\theta})/dz$ and $\alpha > 0$ is a parameter.

The regularized Exner function $\tilde{\pi}$ is determined by

$$[1 + \alpha^2 \mathcal{D}](\tilde{\pi} - \pi) = \alpha^2 \mathcal{R}_\pi,$$

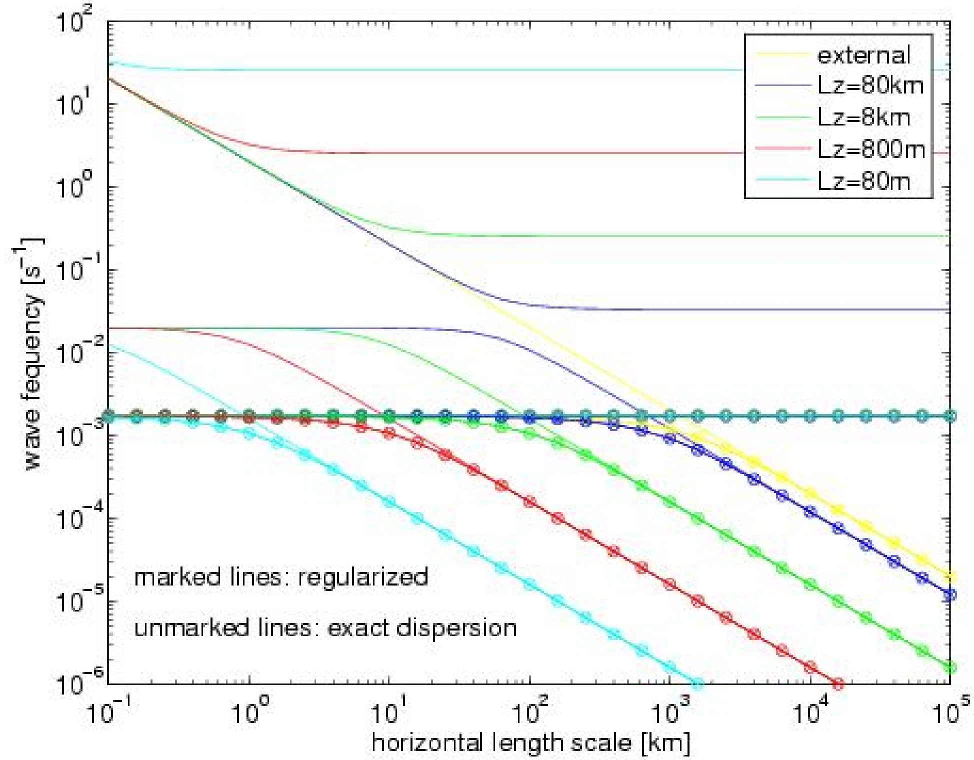
which is an elliptic PDE. Key properties:

- Linear stability and linear equivalence to SI scheme.
- Nonlinear balance, i.e., $\tilde{\pi} = \pi$ ($\mathcal{R}_\pi = 0$, respectively), if

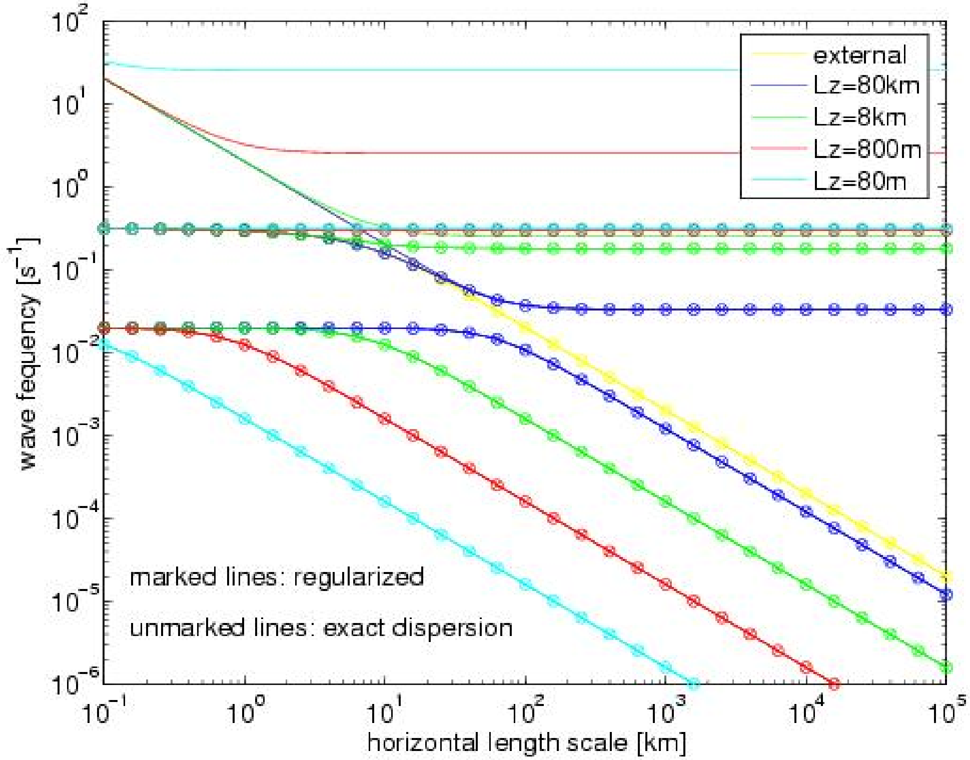
  $$\frac{\partial}{\partial x}\left(\rho\theta\frac{\partial u}{\partial t}\right) + \frac{\partial}{\partial z}\left(\rho\theta\frac{\partial w}{\partial t}\right) = 0$$

  ("pseudo-incompressibility") (Hundermark & Reich, 2007).
- The elliptic problem does not need to be solved to full accuracy (multi-grid smoother).
- Higher-order approximations can be obtained by approximative "deconvolution".

Linear dispersion for vertical slice model with $\Delta t = 30$ min.



Linear dispersion for vertical slice model with $\Delta t = 10$ sec.



9. Example: Advection

Consider advection under a given velocity field $\mathbf{b}(\mathbf{x}, t)$:

$$\mathbf{u}_t = -\mathbf{b} \cdot \nabla \mathbf{u} = -\nabla \cdot (\mathbf{b} : \mathbf{u}), \qquad \nabla \cdot \mathbf{b} = 0,$$

which becomes

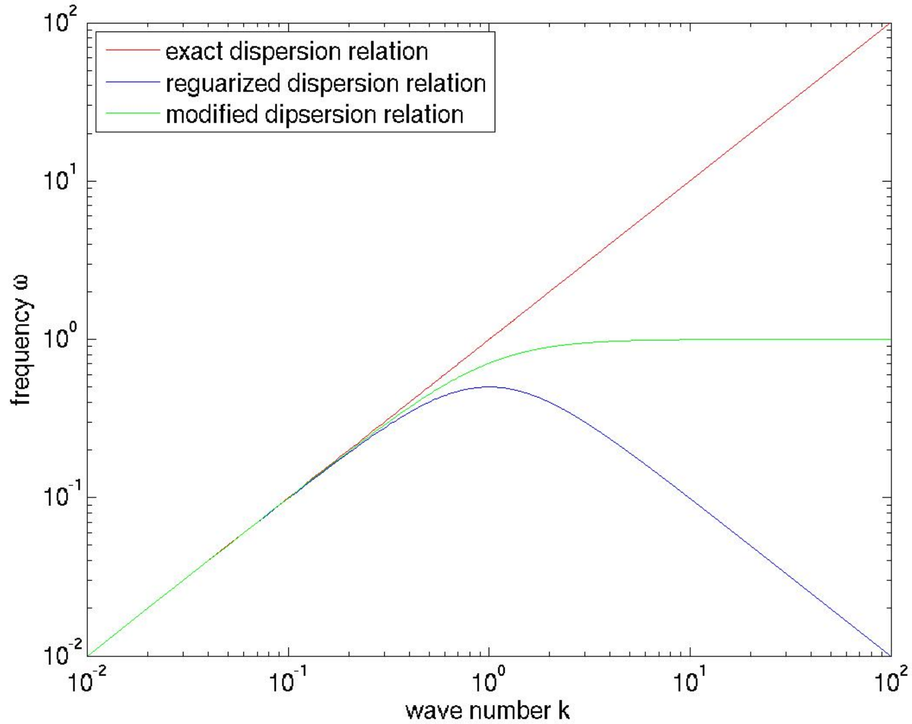
$$\frac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\Delta t} = -\frac{1}{2}\left[\nabla \cdot (\mathbf{b}^{n+1} : \mathbf{u}^{n+1}) + (\mathbf{b}^n : \mathbf{u}^n)\right]$$

under the CN scheme. A linear analysis for $\mathbf{b} = constant$ suggests then a Bardina model

$$\begin{aligned} \mathbf{w}_t + \nabla \cdot (\overline{\mathbf{w} : \mathbf{w}}) + \nabla q &= 0, \\ \nabla \cdot \mathbf{w} &= 0 \end{aligned}$$

with filter

$$\bar{f} = \mathcal{S} * f, \qquad \mathcal{S} = \left[I - \Delta t^2 (\mathbf{u} \cdot \nabla)^2\right]^{-1/2}.$$



The green line corresponds to the linear dispersion relation under modified Bardina model. One could also consider LES models such as Leray and α-Euler which do not modify linear advection at all.

The incompressible α-Euler equations can be written in the form

$$\mathbf{w}_t + \bar{\mathbf{w}} \cdot \nabla \mathbf{w} + \nabla \bar{\mathbf{w}}^T \cdot \mathbf{w} + \nabla q = 0$$

where

$$\nabla \bar{\mathbf{w}}^T \cdot \mathbf{w} = \sum_{j=1}^{3} w_j \nabla \bar{w}_j .$$

This implies a Reynolds stress tensor approximation

$$\begin{aligned} R(\mathbf{u},\mathbf{u}) &= \nabla \cdot (\bar{\mathbf{u}} : \bar{\mathbf{u}}) - \nabla \cdot (\overline{\mathbf{u} : \mathbf{u}}) \\ &\approx \nabla \cdot (\bar{\mathbf{u}} : \bar{\mathbf{u}}) - \underbrace{\bar{\bar{\mathbf{u}}} \cdot \nabla \bar{\mathbf{u}}}_{\text{Leray model}} - \underbrace{\nabla \left[\bar{\bar{\mathbf{u}}} - \bar{\mathbf{u}} \right]^T \cdot \bar{\mathbf{u}}}_{\text{Stokes drift}} . \end{aligned}$$

The α-Euler equations can be derived from a variational principle (Holm, Marsden, Ratiu, 1998) and satisfy conservation of circulation. We are currently investigating extension to compressible flows (with Marco Restelli, MPI Hamburg).