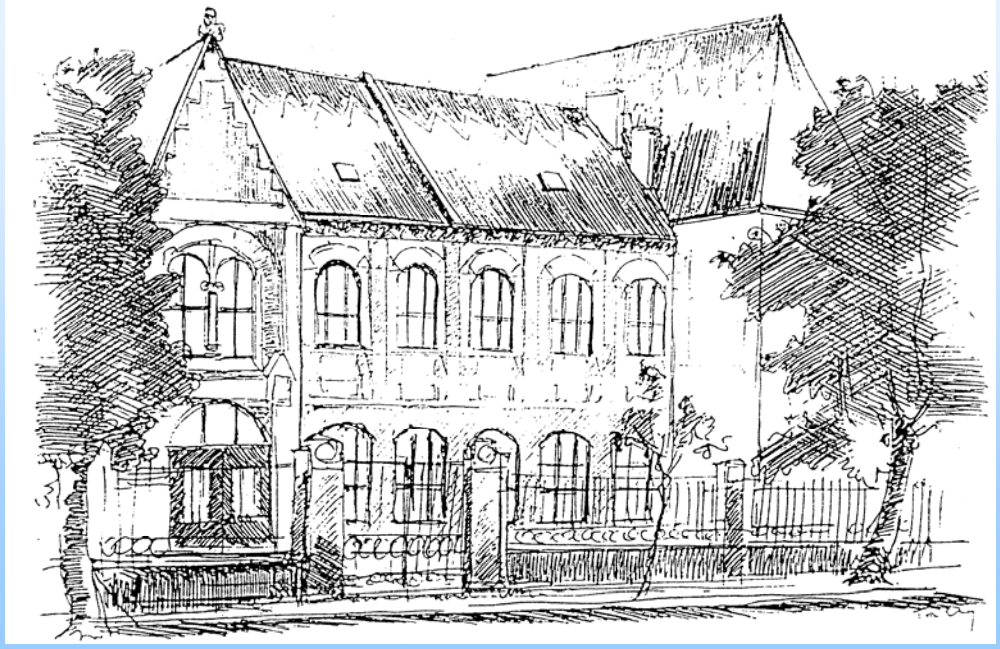
How delay equations arise in Engineering?

Gábor Stépán
Department of Applied Mechanics
Budapest University of Technology and Economics



Contents

Answer: Delay equations arise in Engineering…

… by the information system (of control), and
by the contact of bodies.

- **Linear stability & subcritical Hopf bifurcations**
- Robotic position and force control
- Balancing – human and robotic
- Contact problems
- Shimmying wheels (of trucks and motorcycles)
- Machine tool vibrations

Stability of linear RFDEs of n DoF systems

Delayed mechanical systems include 2nd derivatives:

$$M\ddot{x}(t) + \int_{-h}^{0} \mathrm{d}_{\vartheta} B(t,\vartheta)\dot{x}(t+\vartheta) + \int_{-h}^{0} \mathrm{d}_{\vartheta} K(t,\vartheta) x(t+\vartheta) = 0$$

Autonomous systems: $B(t,\vartheta) \equiv B(\vartheta),\ K(t,\vartheta) \equiv K(\vartheta)$

Trial solution: $\boxed{x(t) = A\mathrm{e}^{\lambda t}}$ $A \in R^n$

Characteristic roots: Re $\lambda_j < 0$, j=1,2,… ⇔ stability

$$D(\lambda) = \det(M\lambda^2 + \int_{-h}^{0} \lambda \mathrm{e}^{\lambda\vartheta} \mathrm{d}_{\vartheta} B(t,\vartheta) + \int_{-h}^{0} \mathrm{e}^{\lambda\vartheta} \mathrm{d}K(\vartheta))$$

D-curves: $R(\omega) = \operatorname{Re} D(i\omega),\ S(\omega) = \operatorname{Im} D(i\omega),\ \omega \in [0,\infty)$

$$\left.\begin{array}{r} R(\rho_k) = 0, k = 1,\ldots,r: \quad S(\rho_k) \neq 0, k = 1,\ldots,r \\ \sum_{k=1}^{r} (-1)^k \operatorname{sgn} S(\rho_k) = (-1)^n n \end{array}\right\} \Leftrightarrow \text{stability}$$

Examples with 1 DoF, $n = 1$

$$\ddot{x}(t) + c_0 x(t) = c_1 \int_{-1}^{0} w(\vartheta) x(t+\vartheta)\,\mathrm{d}\vartheta, \quad w(\vartheta) \equiv 1$$

$$D(\lambda) = \lambda^2 + c_0 - c_1 \int_{-1}^{0} e^{\lambda\vartheta}\,\mathrm{d}\vartheta = \lambda^2 + c_0 - c_1 \frac{1 - e^{-\lambda}}{\lambda}$$

$$R(\omega) = -\omega^2 + c_0 - c_1 \frac{\sin\omega}{\omega} \quad \Rightarrow \lim_{\omega \to +\infty} R(\omega) = -\infty$$

$$S(\omega) = c_1 \frac{1 - \cos\omega}{\omega} \quad \Rightarrow S(\omega) > 0 \text{ for } \boxed{c_1 > 0},$$

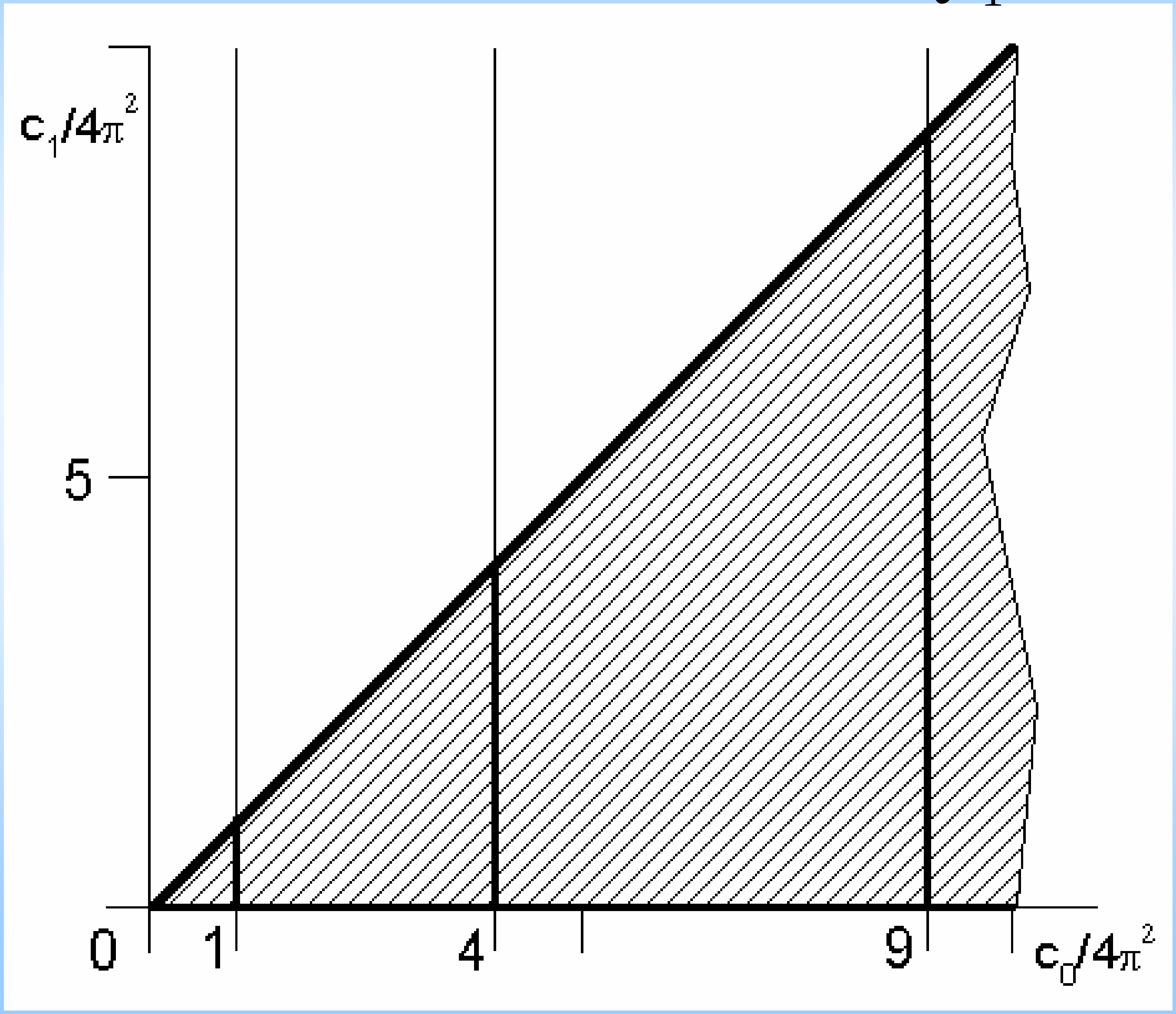
$$\omega \neq 2k\pi, k = 0, 1, \ldots$$

$$S(\rho_k) \neq 0, k = 1, \ldots, r \quad \Rightarrow R(2k\pi) = \boxed{-4k^2\pi^2 + c_0 \neq 0}$$

$$\sum_{k=1}^{r} (-1)^k \underbrace{\operatorname{sgn} S(\rho_k)}_{+1} = \underbrace{(-1)^n}_{-1} n \quad \Rightarrow r \text{ odd}$$

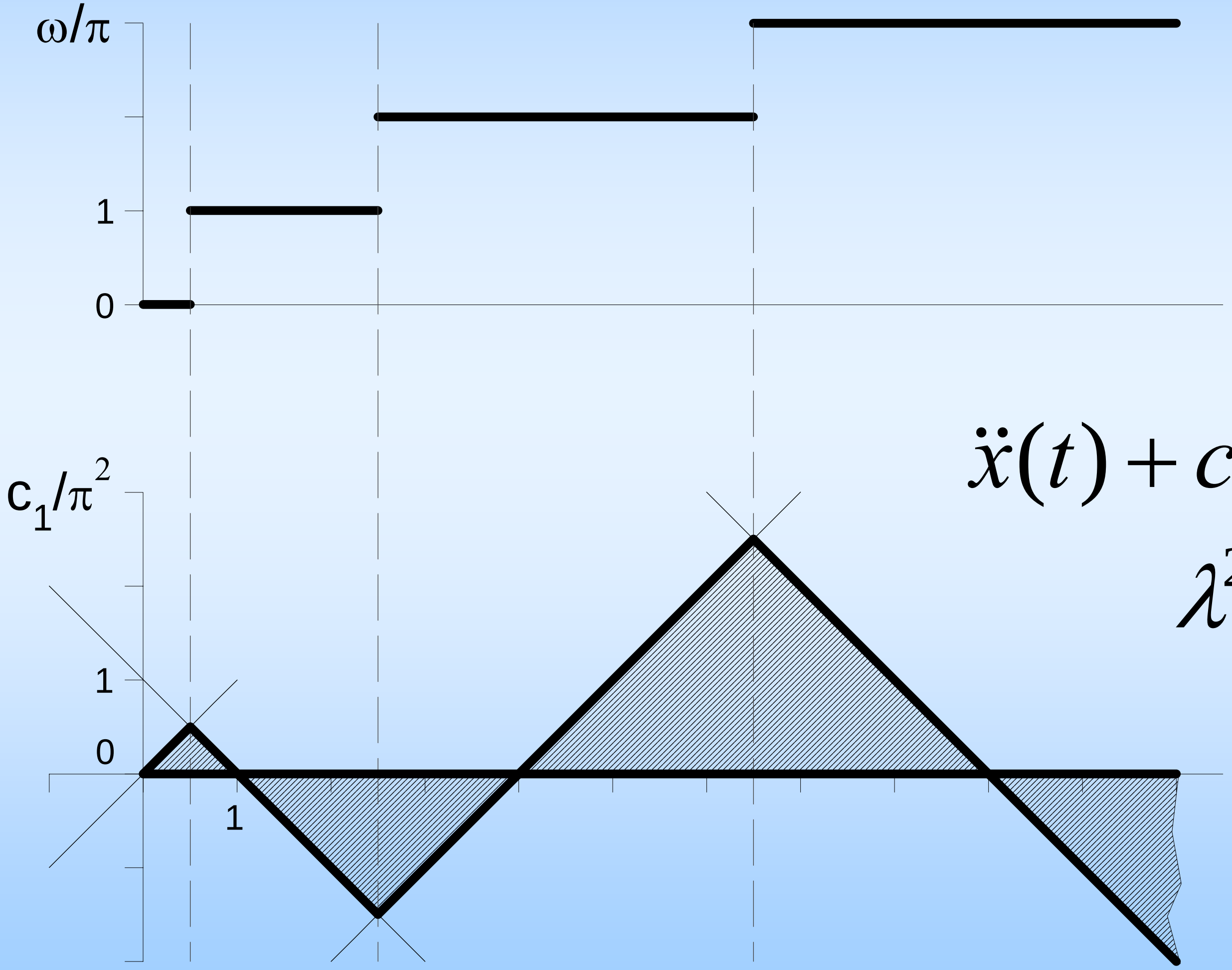
$$\Rightarrow \mathrm{R}(0) = \boxed{\mathrm{c}_0 - c_1 > 0}$$

Stability chart $\ddot{x}(t) + c_0 x(t) = c_1 \int_{-1}^{0} x(t+\vartheta)\,\mathrm{d}\vartheta$



Delayed oscillators

$$w(\vartheta) = \delta(\vartheta + 1)$$



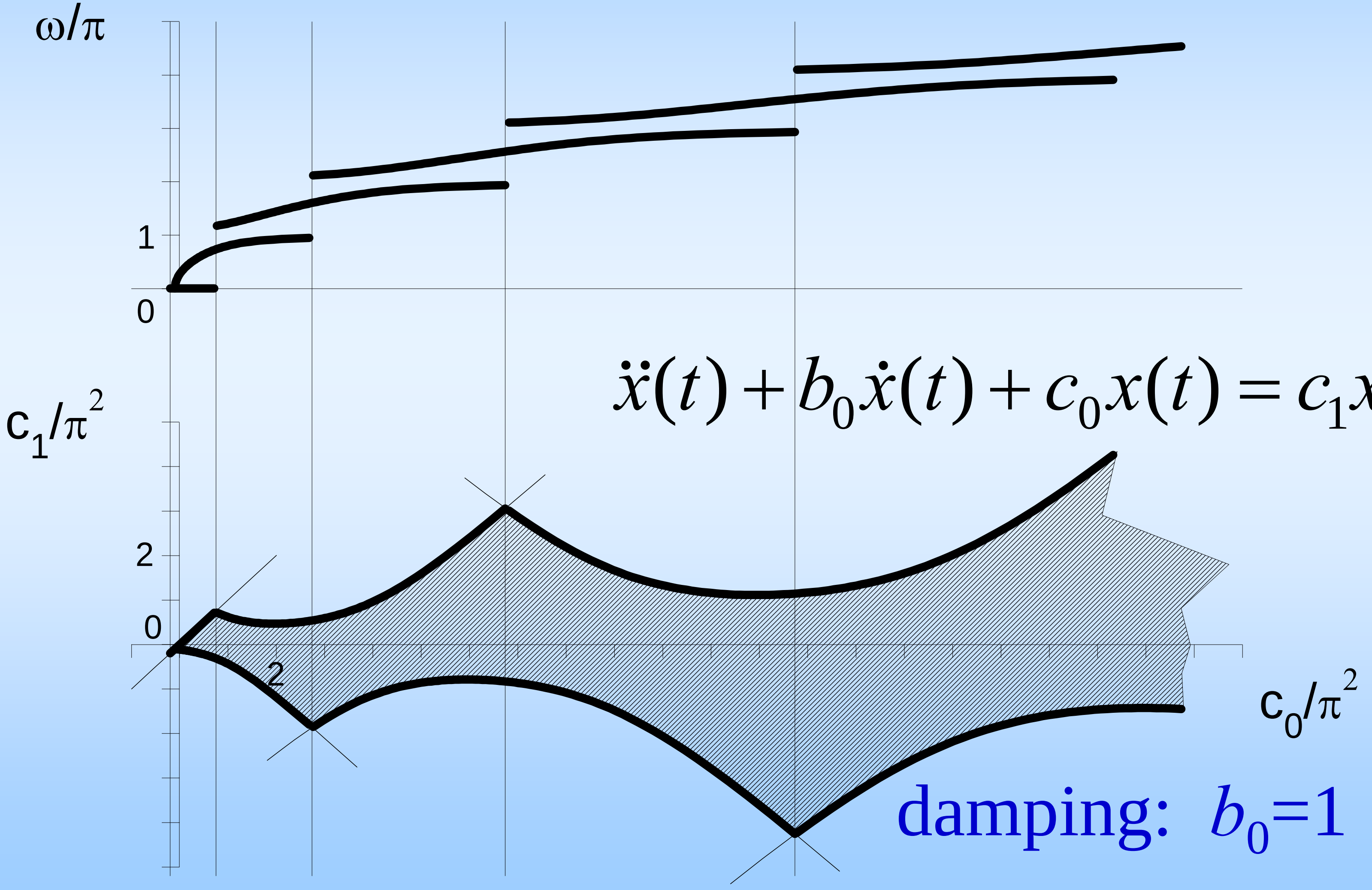
vibration frequencies

$$\ddot{x}(t) + c_0 x(t) = c_1 x(t-1)$$

$$\lambda^2 + c_0 - c_1 e^{-\lambda} = 0$$

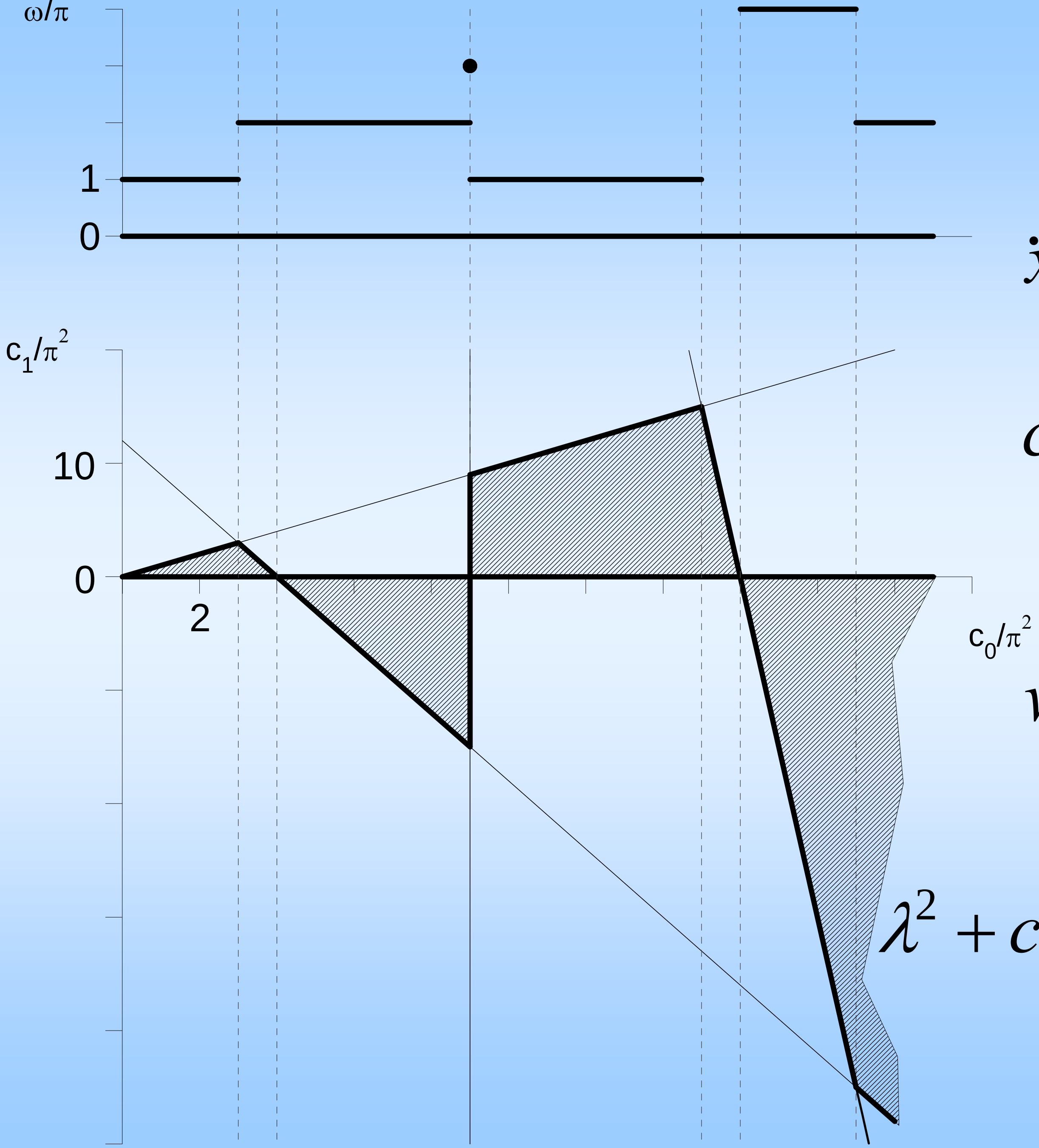
stability chart

Delayed oscillator with damping



$$\ddot{x}(t)+b_0\dot{x}(t)+c_0x(t)=c_1x(t-1)$$

damping: $b_0=1$

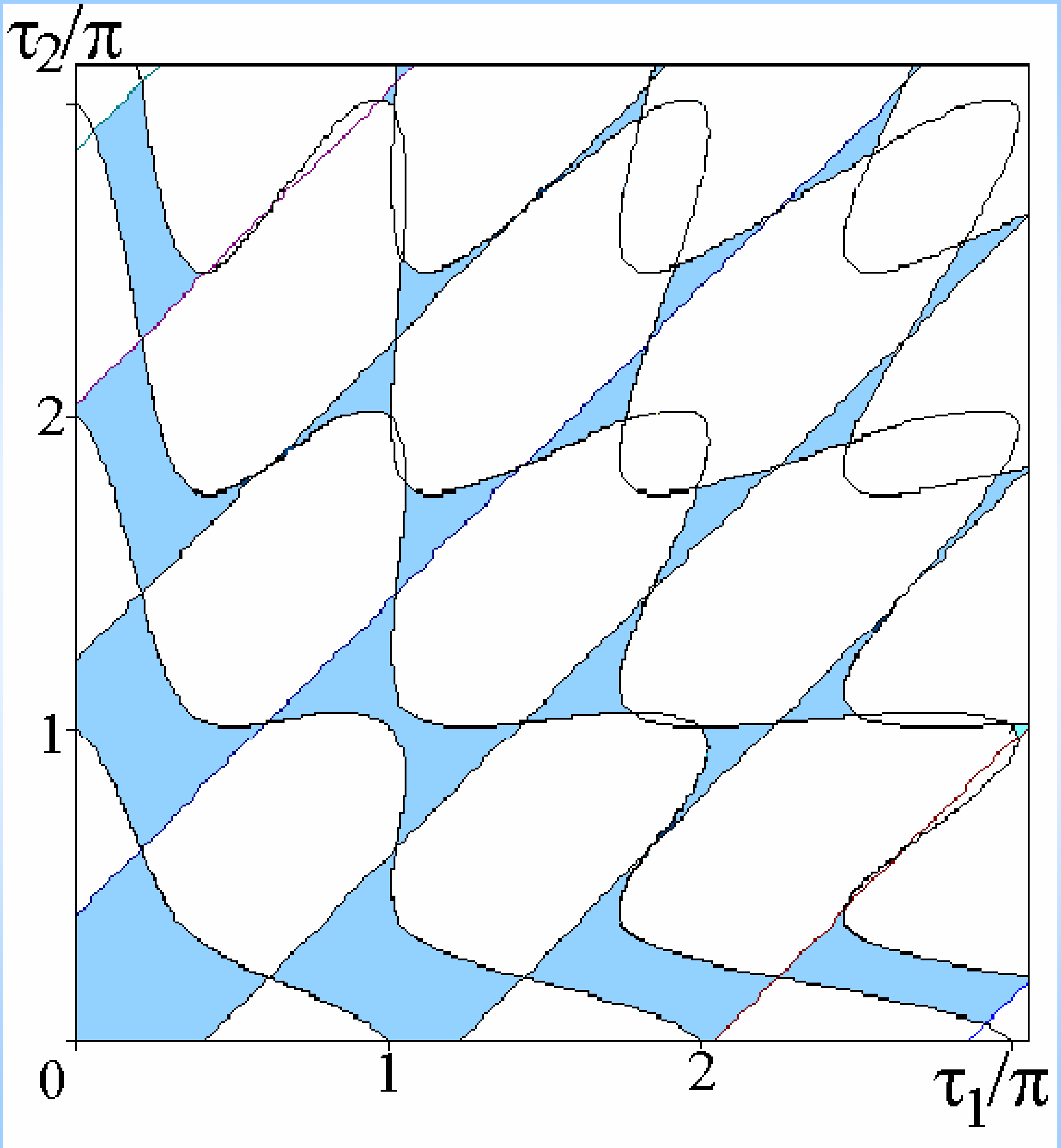


$$\ddot{x}(t) + c_0 x(t) =$$

$$c_1 \int_{-1}^{0} w(\vartheta) x(t+\vartheta) \mathrm{d}\vartheta$$

$$w(\vartheta) = -\frac{\pi}{2} \sin(\pi\vartheta)$$

$$\lambda^2 + c_0 - c_1 \frac{\pi^2}{2} \frac{1 + \mathrm{e}^{-\lambda}}{\lambda^2 + \pi^2} = 0$$



$$\ddot{x}(t) + 6x(t) =$$
$$x(t-\tau_1) + x(t-\tau_2)$$

Non-autonomous linear RFDEs

$$M\ddot{x}(t)+\int_{-h}^{0}\mathrm{d}_{\vartheta}B(t,\vartheta)\dot{x}(t+\vartheta)+\int_{-h}^{0}\mathrm{d}_{\vartheta}K(t,\vartheta)x(t+\vartheta)=0$$

Time-periodic systems: $B(t+T,\vartheta)=B(t,\vartheta)$

Trial solution: $\boxed{x(t)=p(t)\mathrm{e}^{\lambda t}}$ $K(t+T,\vartheta)=K(t,\vartheta)$

$$p(t+T)=p(t)=\sum_{k=0}^{+\infty}\left(A_k\cos\left(k\tfrac{2\pi}{T}t\right)+B_k\sin\left(k\tfrac{2\pi}{T}t\right)\right)$$

Hill's infinite dimensional determinant $\Rightarrow$

characteristic function $\Rightarrow$ characteristic roots λ

Re $\lambda_j < 0$, j=1,2,… $\Leftrightarrow$ stability $\Leftrightarrow$ $|\mu_j|<1$, j=1,2,…

for characteristic multipliers $\mu=\mathrm{e}^{\lambda T}$ of fund. op. at T

The delayed Mathieu equation

$$\ddot{x}(t) + (\delta + \varepsilon \cos t)\, x(t) = b\, x(t - 2\pi)$$

$$x(t) = \sum_{k=0}^{\infty} \left(A_k\, \mathrm{e}^{\mathrm{i}kt} + B_k\, \mathrm{e}^{-\mathrm{i}kt} \right) \mathrm{e}^{\lambda t} + \sum_{k=0}^{\infty} \left(\overline{A}_k\, \mathrm{e}^{-\mathrm{i}kt} + \overline{B}_k\, \mathrm{e}^{\mathrm{i}kt} \right) \mathrm{e}^{\overline{\lambda} t}$$

$$x(t) = \sum_{k=-\infty}^{\infty} C_k\, \mathrm{e}^{(\lambda + \mathrm{i}k)t} + \overline{C}_k\, \mathrm{e}^{(\overline{\lambda} - \mathrm{i}k)t}$$

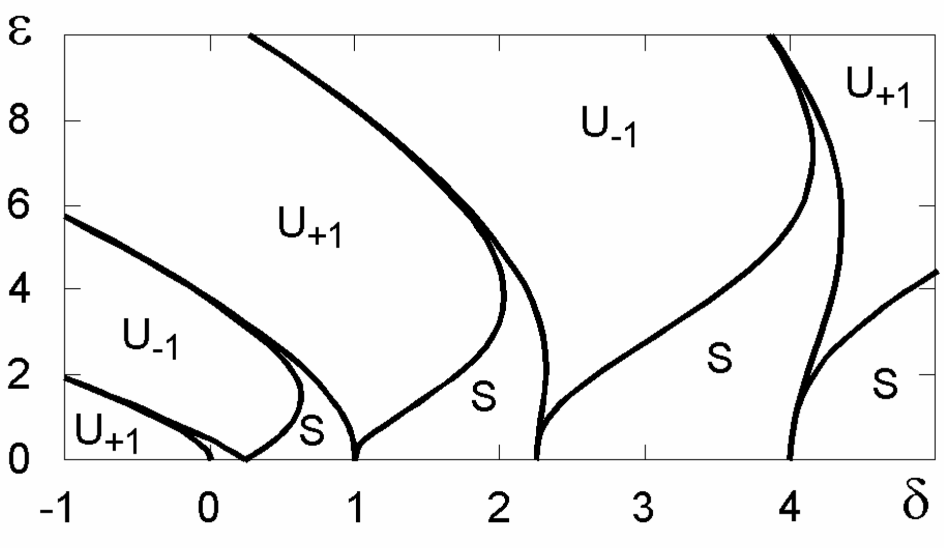
Harmonic balance

$\Rightarrow$ Hill's determinant

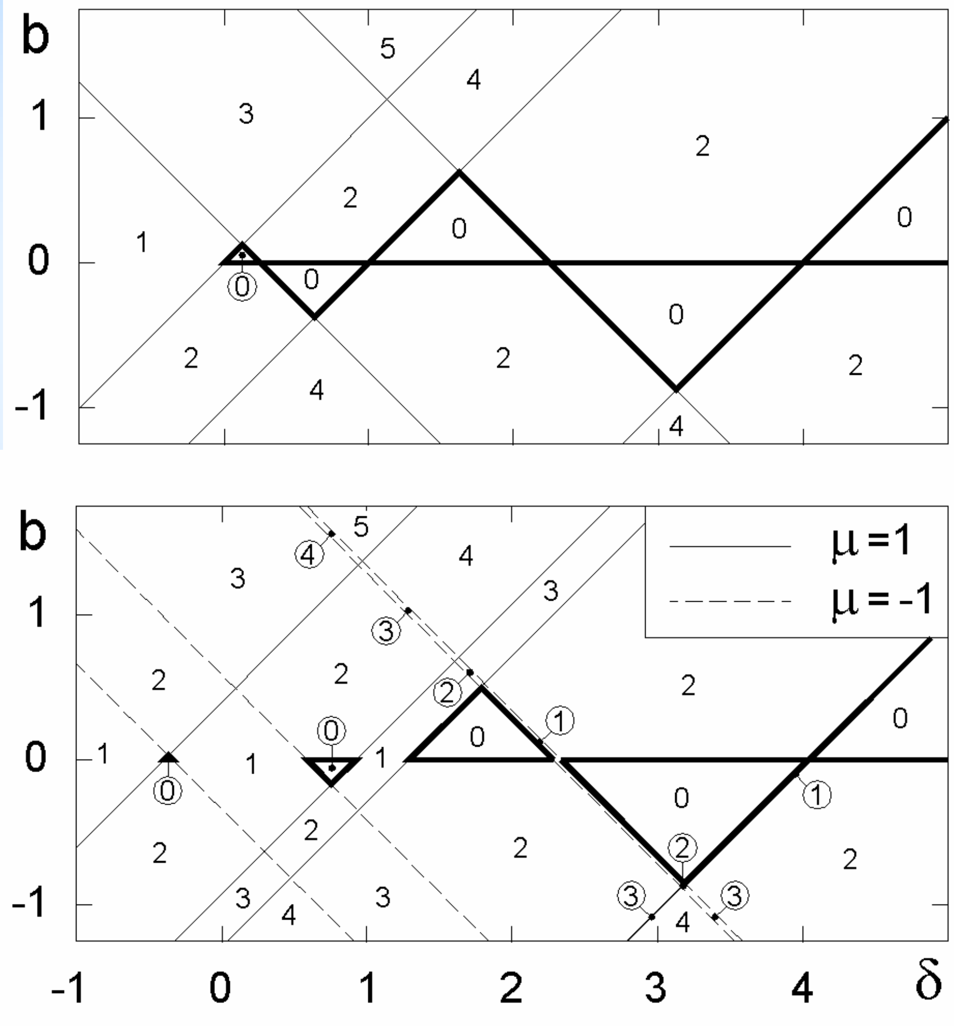
$$\det \begin{pmatrix} \ddots & \ddots & \ddots & 0 & 0 \ldots \\ \ldots 0 & \frac{\varepsilon}{2} & \delta + (\lambda + \mathrm{i}k)^2 - b\,\mathrm{e}^{-2\pi(\lambda + \mathrm{i}k)} & \frac{\varepsilon}{2} & 0 \ldots \\ \ldots 0 & 0 & \ddots & \ddots & \ddots \end{pmatrix} = 0$$

The delayed Mathieu – stability charts

$$\ddot{x}(t) + (\delta + \varepsilon \cos t)\,x(t) = b\,x(t - 2\pi)$$



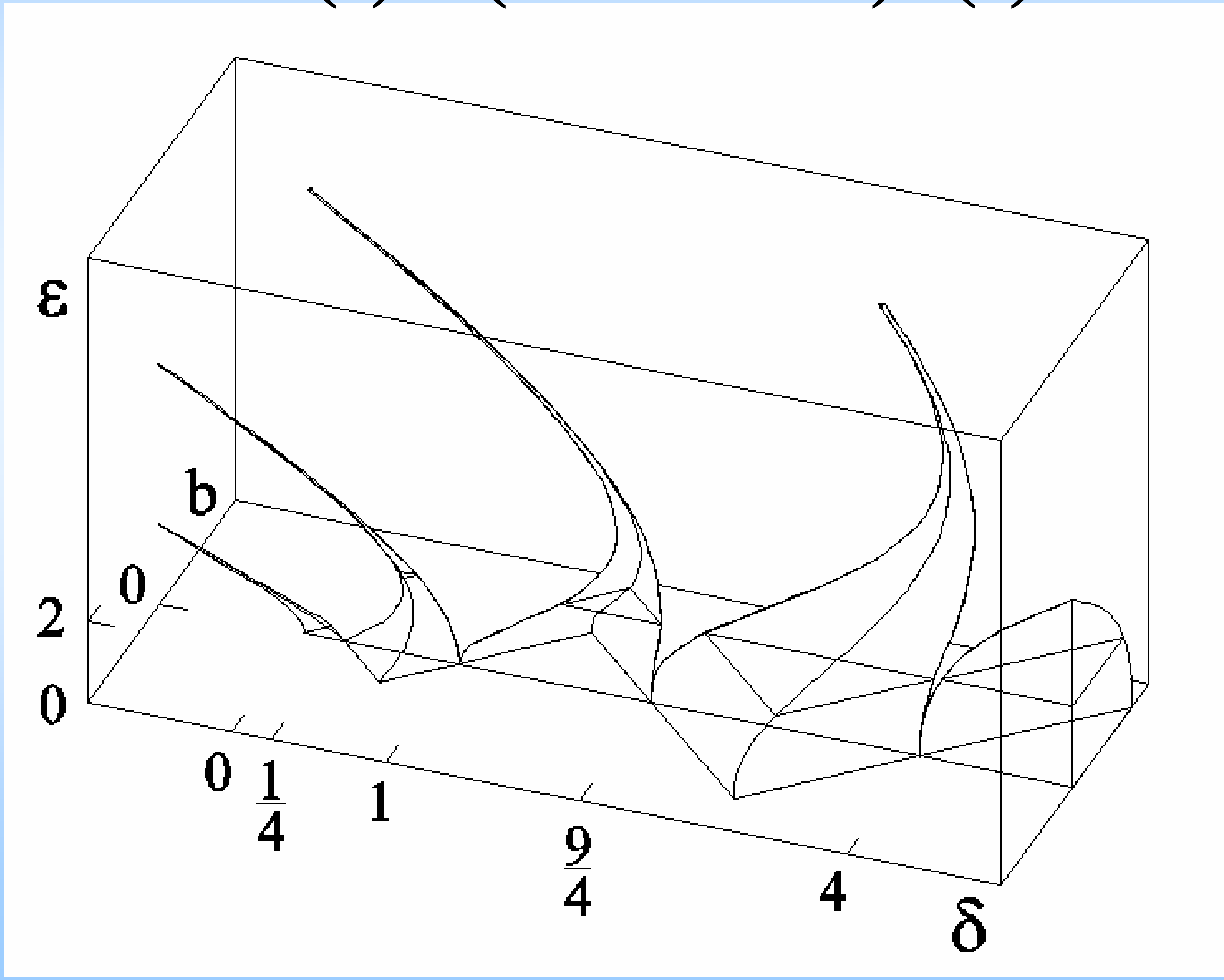
b=0 (Strutt-Incze, 1928)



ε=1

Stability chart of delayed Mathieu

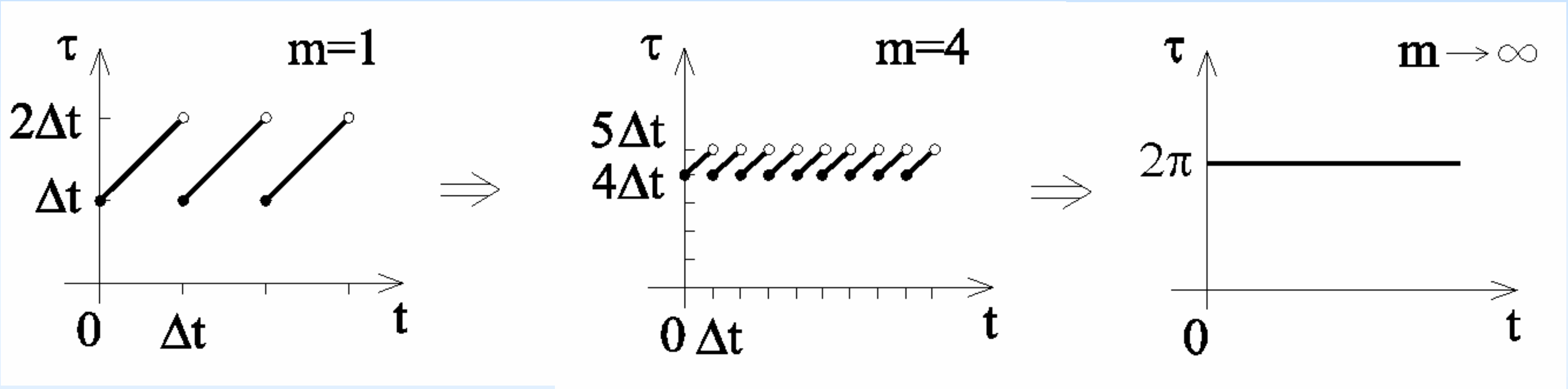
$$\ddot{x}(t) + (\delta + \varepsilon \cos t)x(t) = b\,x(t - 2\pi)$$



Insperger, Stépán (2002)

Semi-discretization method – introduction

$$\ddot{x}(t) + c_0 x(t) = c_1 x(t-\tau) \qquad \tau = 2\pi$$



The approximating DDE is *non-autonomous*

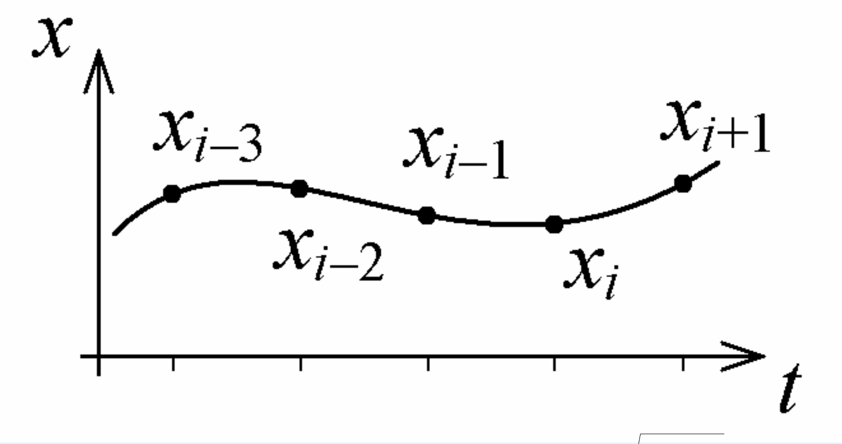
$$\ddot{x}(t) + c_0 x(t) = c_1 x(t-\tau(t)), \quad \tau(t) = t + (m - \mathrm{int}(t/\Delta t))\Delta t$$

$$\Delta t = 2\pi/(m+1/2)$$

$$t \in [t_i, t_{i+1}) = [i\Delta t, (i+1)\Delta t)$$

$$\Rightarrow \quad x(t-\tau(t)) \equiv x((i-m)\Delta t) = x_{i-m}$$

Introduction to SDM – delayed oscillator



$$\ddot{x}(t) + c_0 x(t) = c_1 x_{i-m}$$

$$x(t_i) = x_i$$

$$\dot{x}(t_i) = \dot{x}_i$$

$$x(t) = K_{1i}\cos(\sqrt{c_0}t) + K_{2i}\sin(\sqrt{c_0}t) + c_1 x_{i-m}/c_0$$

$$\dot{x}(t) = -K_{1i}\sqrt{c_0}\sin(\sqrt{c_0}t) + K_{2i}\sqrt{c_0}\cos(\sqrt{c_0}t)$$

$$x_{i+1} = a_{00}\,x_i + a_{01}\,\dot{x}_i + a_{0m}\,x_{i-m}$$

$$\dot{x}_{i+1} = a_{10}\,x_i + a_{11}\,\dot{x}_i + a_{1m}\,x_{i-m}$$

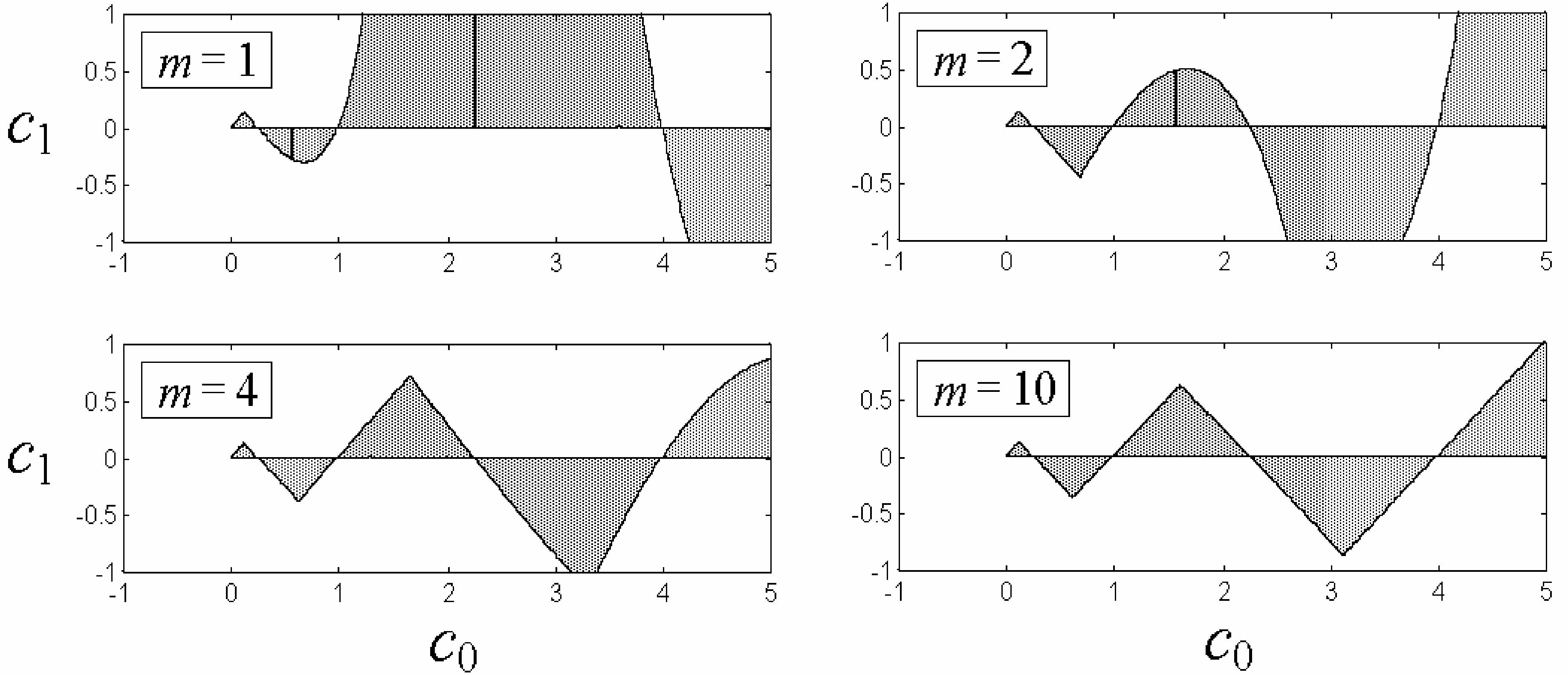
$$\mathbf{y}_i = \mathrm{col}(\dot{x}_i \;\; x_i \;\; x_{i-1} \;\ldots\; x_{i-m})$$

$$\boxed{\mathbf{y}_{i+1} = \mathbf{A}\,\mathbf{y}_i}$$

$$\det(\mu\mathbf{I} - \mathbf{A}) = 0 \;\Rightarrow\; |\mu_{1,2,\ldots,m+2}| < 1 \Leftrightarrow \text{stability}$$

Delayed oscillator – stability chart by SDM

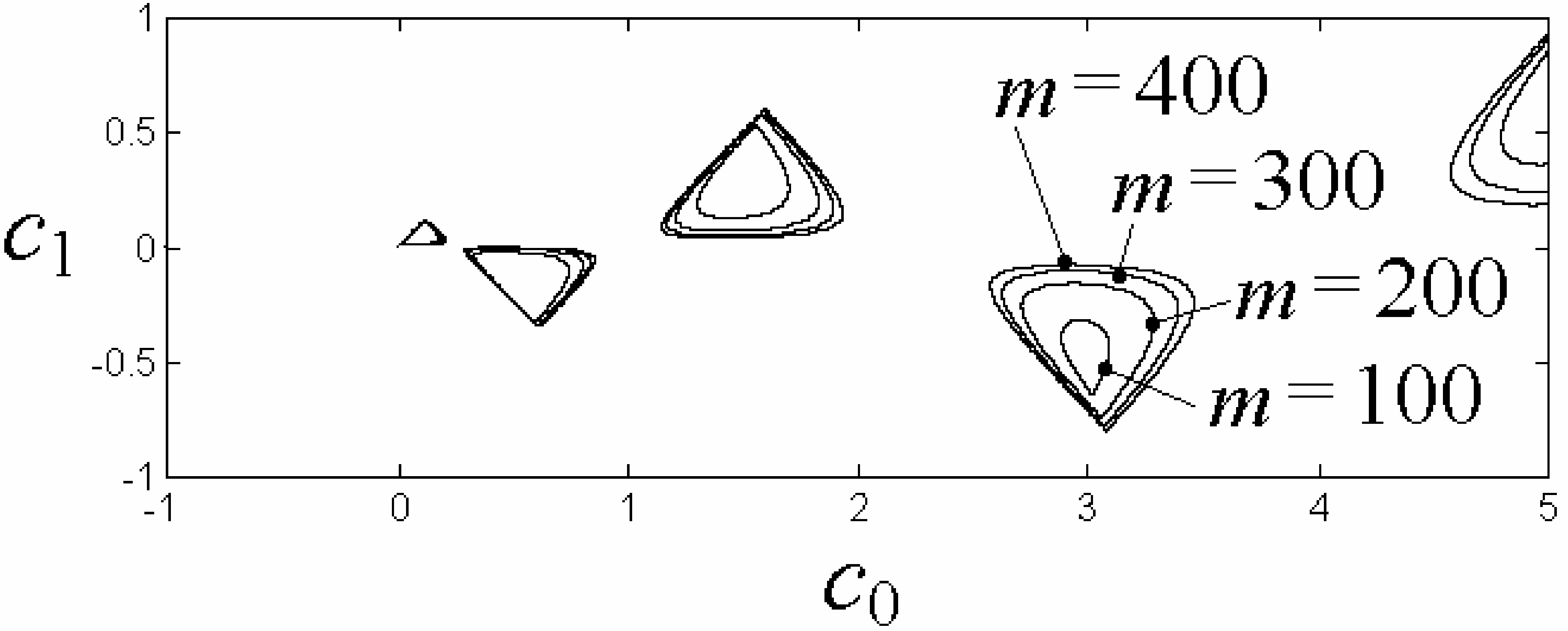
$$\ddot{x}(t) + c_0 x(t) = c_1 x(t - \tau(t)), \quad \tau(t) = t + (m - \mathrm{int}(t/\Delta t))\Delta t$$



Full discretization - comparison

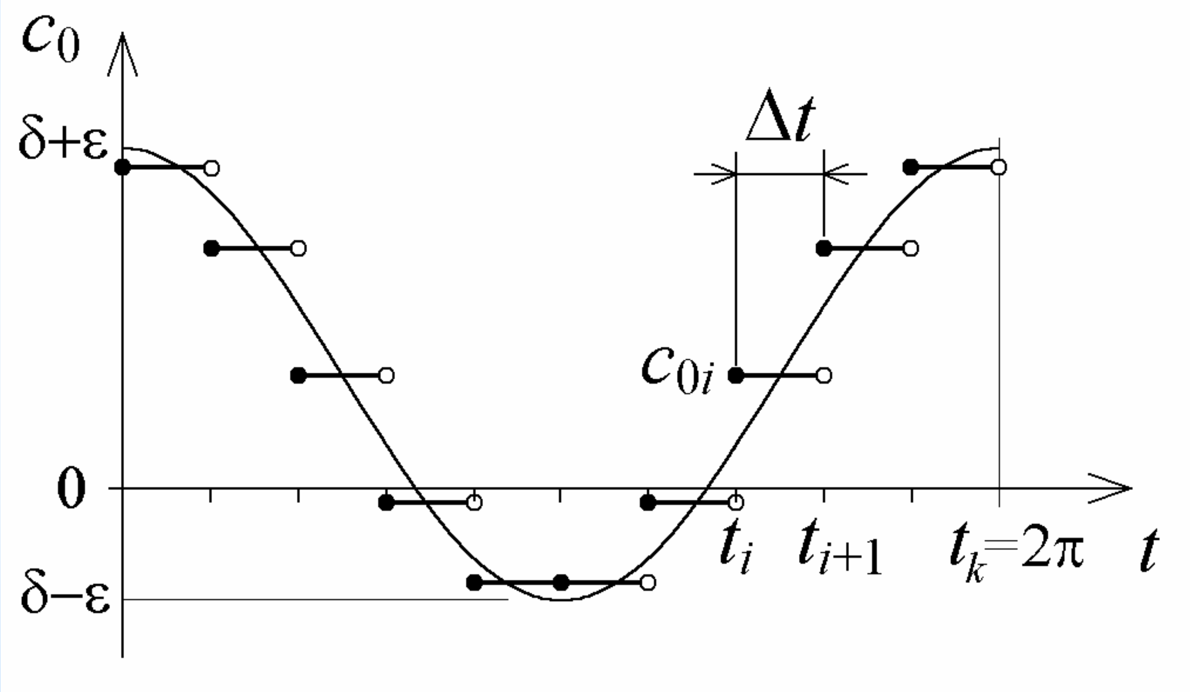
Discretization also w.r.t. time derivatives

– slow convergence



Introduction to SDM – Mathieu equation

$$\ddot{x}(t) + c_0(t)x(t) = 0 \qquad c_0(t) = \delta + \varepsilon \cos t$$



$$t \in [t_i, t_{i+1})$$

$$\ddot{x}(t) + c_{0i}x(t) = 0$$

$$x(t_i) = x_i$$

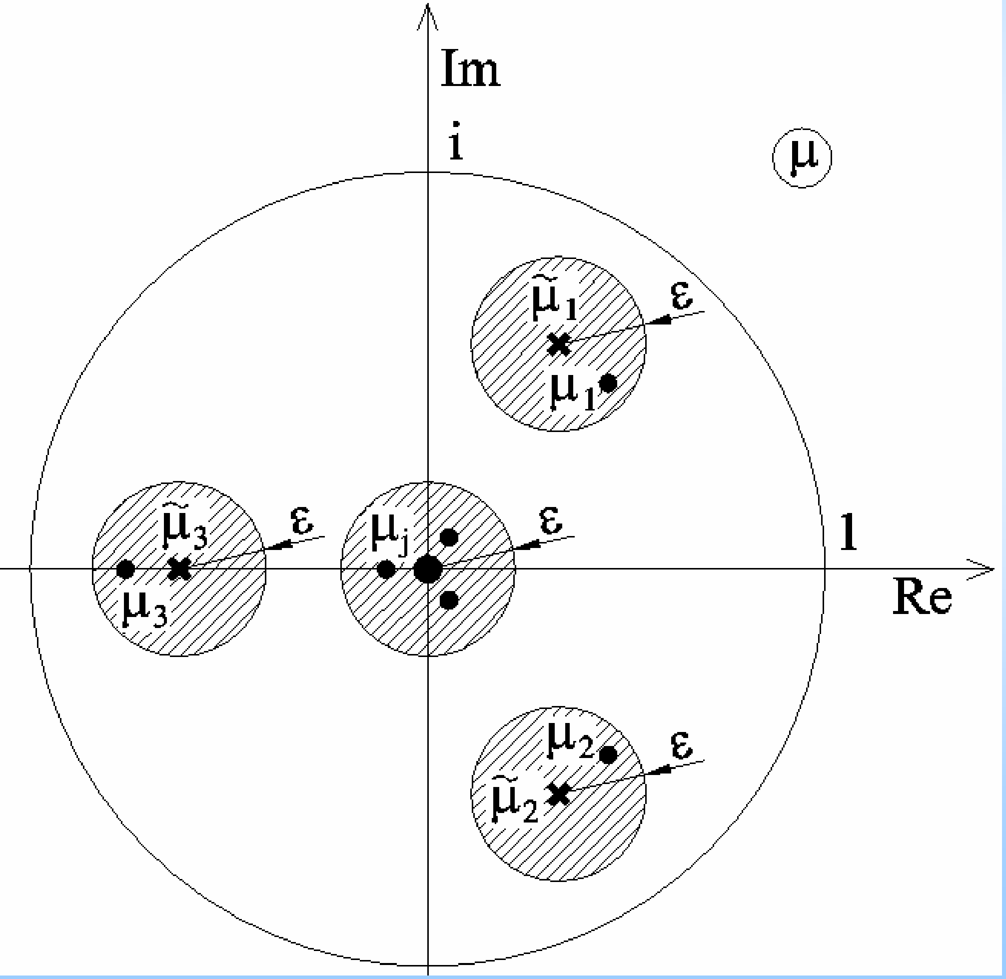
$$\dot{x}(t_i) = \dot{x}_i$$

$$i = 0,1,\ldots,k-1$$

$$x(t) = x_i \cos\left(\sqrt{c_{0i}}(t - t_i)\right) + \frac{\dot{x}_i}{\sqrt{c_{0i}}} \sin\left(\sqrt{c_{0i}}(t - t_i)\right)$$

Semi-discretization – general case

$$\dot{x}(t)=\int_{-\tau}^{0}\mathrm{d}_{\vartheta}\eta(t,\vartheta)x(t+\vartheta),\ \ \eta(t+T,\vartheta)=\eta(t,\vartheta)$$



$$\forall\varepsilon>0,\ \ \exists M(\varepsilon),\ \ \forall m>M(\varepsilon)$$

$$\Rightarrow$$

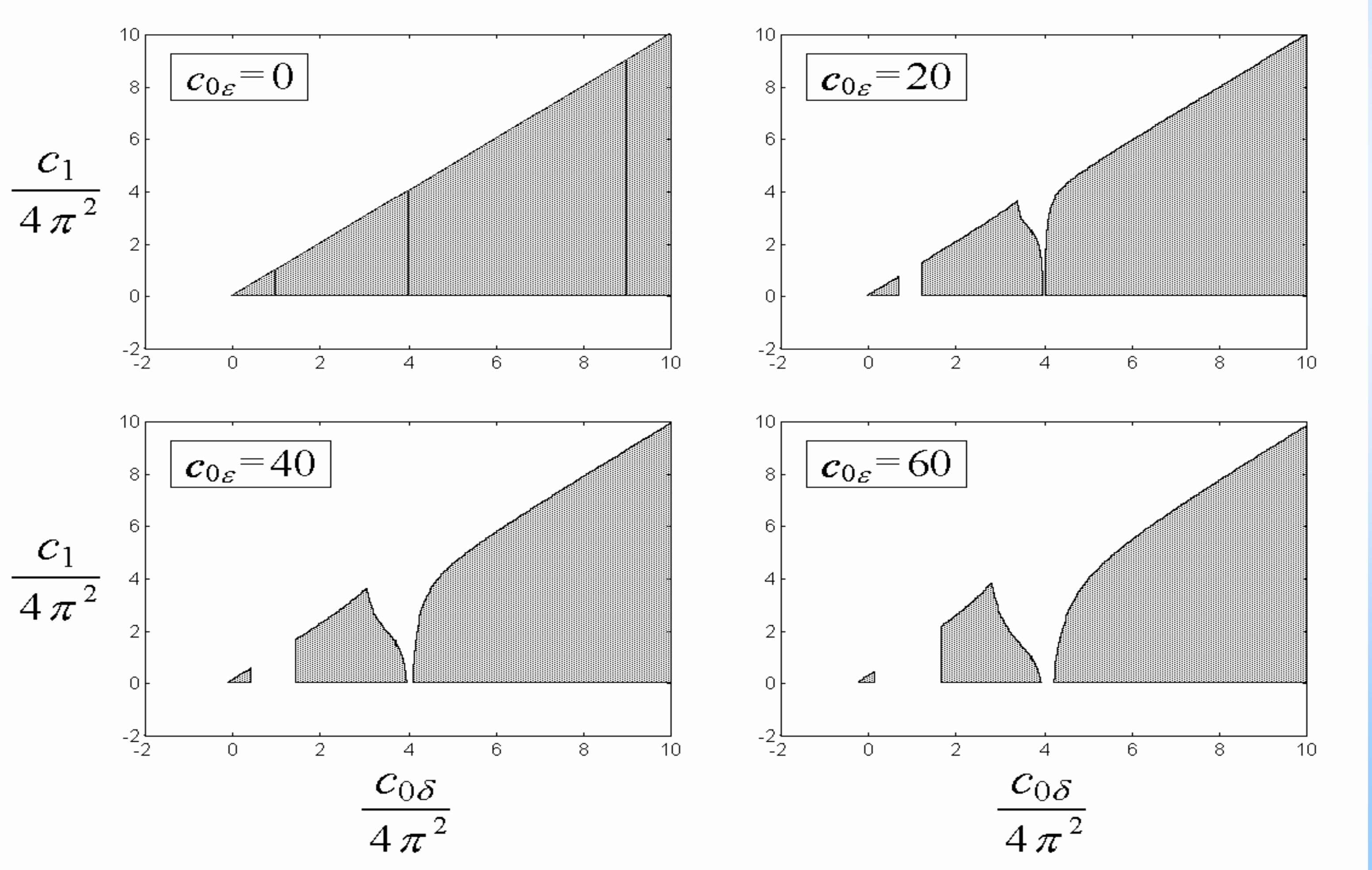
$$\mu_j\in\bigcup_{j=1}^{mn}S_{\tilde{\mu}_j,\varepsilon},\ j=1,\ldots,mn$$

$$|\mu_j|<\varepsilon,\qquad\qquad j=mn+1,\ldots$$

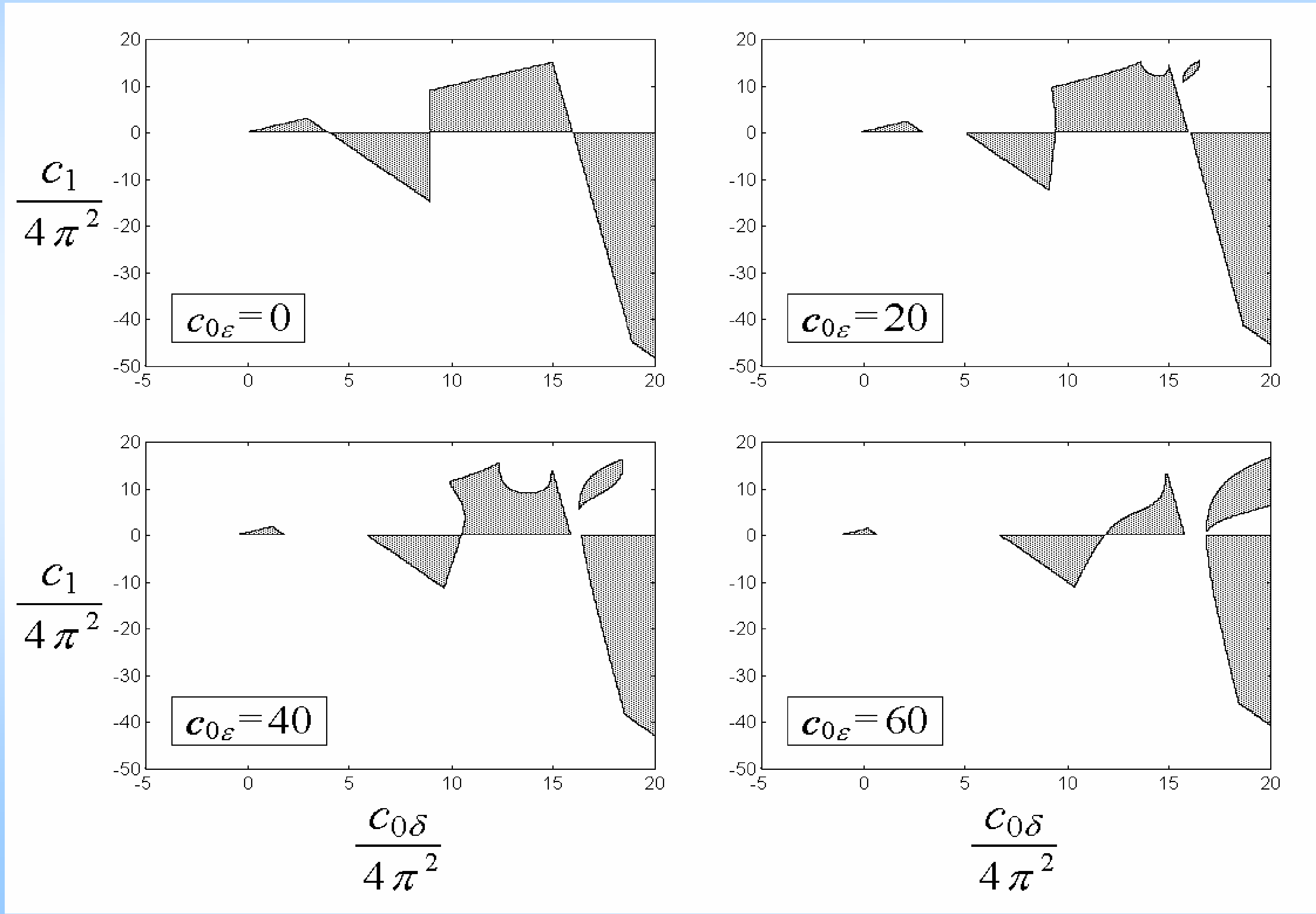
Insperger, Stepan: *Int. J. of Numerical Methods in Engineering* (2002)

Examples – test on delayed Mathieu

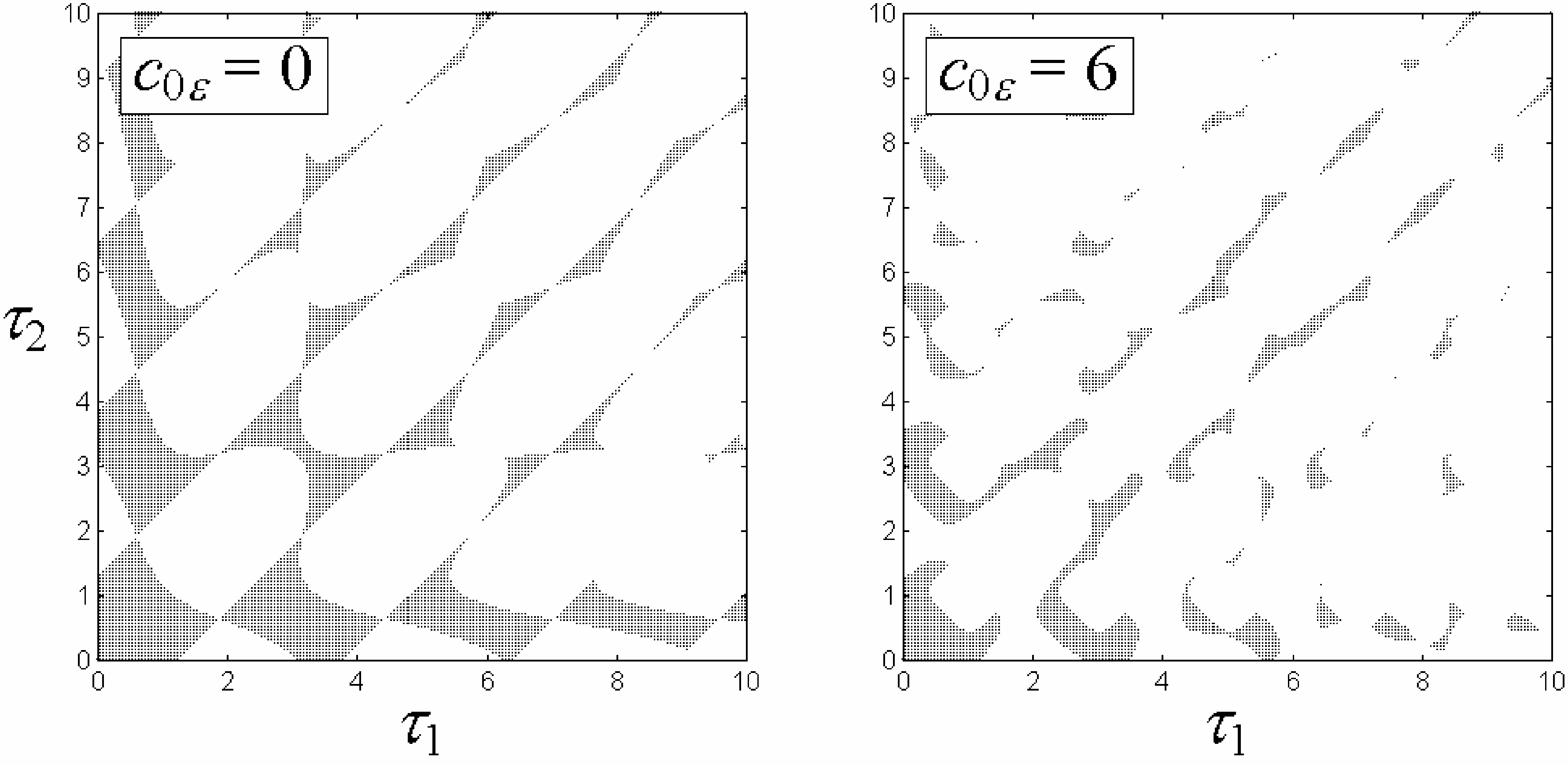
$$\ddot{x}(t) + (c_{0\delta} + c_{0\varepsilon}\cos(4\pi t))x(t) = c_1 \int_{-1}^{0} x(t+\vartheta)\mathrm{d}\vartheta$$



$$\ddot{x}(t)+(c_{0\delta}+c_{0\varepsilon}\cos(4\pi t))x(t)=c_1\frac{\pi}{2}\int_{-1}^{0}\sin(\pi\vartheta)x(t+\vartheta)\mathrm{d}\vartheta$$



$$\ddot{x}(t) + (6 + c_{0\varepsilon} \cos(2\pi t))x(t) = x(t - \tau_1) + x(t - \tau_2)$$



Nonlinear RFDEs in Engineering

Stability analysis of steady-states is followed by bifurcation analysis

Hopf bifurcation – self-excited vibrations

Supercritical case: easy to avoid vibrations by knowing the linear stability behaviour

Subcritical case: the unstable periodic solutions mean a limited domain of attraction for the desired steady-state behaviour – *cannot* be predicted by linear stability analysis.

Stick&slip – unstable periodic motion

akad_csu.mpg

Unstable limit cycle – “ghost” vibration