Univalence of σ-harmonic mappings and applications

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1 Introduction

In this paper we study mappings U from an open set Ω of the plane R^2 into R^2 whose components u_1 and u_2 are σ-*harmonic functions* in the sense that they are weak solutions to the divergence form elliptic equation

$$\text{div } (\sigma \nabla u_i) = 0 \quad \text{in} \quad \Omega, \quad i = 1, 2 \tag{1.1}$$

where σ is a symmetric, uniformly elliptic matrix with measurable entries. Our starting point for this investigation has its origin in applications to homogenization. In order to describe such applications and the results of the present paper, it is necessary to introduce some notation. We shall denote by $\mathcal{M}^s$ the class of real, two by two symmetric matrices and by $\mathcal{M}^s_K$, $K \geq 1$, the subclass of matrices $\sigma = \{\sigma_{ij}\} \in \mathcal{M}^s$ satisfying the uniform ellipticity condition

$$K^{-1} \mid \xi \mid^2 \leq \sigma_{ij} \xi_i \xi_j \leq K \mid \xi \mid^2 \quad \text{for} \quad \text{every} \quad \xi \in R^2 \quad .$$

Let Ω be an open set in R^2, we shall refer to any given $\sigma \in L^\infty(\Omega, \mathcal{M}^s_K)$ as a *conductivity* and a mapping $U \in W^{1,2}_{\text{loc}}(\Omega, R^2)$ will be said *σ-harmonic* if its components u_1 and u_2 are weak solutions to (1.1). In places, we shall consider $\Omega = R^2$. We set $Q = (0,1) \times (0,1)$ and we shall deal with functions which are 1-periodic with respect to each of its variables x_1 and x_2, which we will call *Q-periodic*, or for short, *periodic*. This will be indicated by the subscript $\sharp$ in the relevant function spaces. For instance

$$L^\infty_\sharp(R^2, \mathcal{M}^s_K) \equiv \{\sigma \in L^\infty(R^2, \mathcal{M}^s_K) \quad |$$

$$\sigma(x_1 + m, x_2 + n) = \sigma(x_1, x_2) \quad \text{for} \quad \text{a.e.} \quad (x_1, x_2) \in R^2, \quad \forall m, n \in Z\} \quad ,$$

$$W^{1,2}_\sharp(R^2, R^2) \equiv \{U \in W^{1,2}_{\text{loc}}(R^2, R^2) \quad |$$

$$U(x_1 + m, x_2 + n) = U(x_1, x_2) \quad \text{for} \quad \text{a.e.} \quad (x_1, x_2) \in R^2, \quad \forall m, n \in Z\} \quad .$$

It is also convenient to define, for any two by two matrix A,

$$W^{1,2}_{\sharp,A}(R^2, R^2) \equiv \{U \in W^{1,2}_{\text{loc}}(R^2, R^2) \mid U - Ax \in W^{1,2}_\sharp(R^2, R^2)\} \quad . \tag{1.2}$$

We are especially interested in boundary conditions of periodic type because of their central role in homogenization and in particular in the so-called G-closure problems. Let us review some very basic facts in homogenization theory.

1.1 Connections with homogenization

We recall the definition of effective (or homogenized) conductivity restricting our attention to dimension two. Let $\sigma \in L^\infty_\sharp(R^2, \mathcal{M}^s_K)$ be given and let Ω be a bounded open and simply connected set with Lipschitz boundary. Let $f \in W^{-1,2}(\Omega, R)$ and set $\sigma_\epsilon(x) = \sigma(\frac{x}{\epsilon})$. Consider the problem

$$\begin{cases} -\text{div}(\sigma_\epsilon(x) \nabla u_\epsilon(x)) = f \quad \text{in} \quad \Omega \\ u_\epsilon \in W^{1,2}_0(\Omega, R) \quad\quad . \end{cases}$$

Then it is well known (see for instance [10]) that $u_\epsilon \rightharpoonup u_0$ in $W^{1,2}(\Omega, R)$ where u_0 solves the following (homogenized) problem:

$$\begin{cases} -\text{div}(\sigma_{\text{hom}} \nabla u_0(x)) = f \quad \text{in} \quad \Omega \\ u_0 \in W^{1,2}_0(\Omega, R) \quad\quad . \end{cases}$$

The new (constant) matrix $\sigma_{\rm hom}$, called *homogenized conductivity*, belongs to $\mathcal{M}^s_K$ and it is determined by the following rule

$$\forall \xi \in R^2 \quad , \quad \langle \sigma_{\rm hom}\xi, \xi\rangle = \inf_{u-\langle \xi, x\rangle \in W^{1,2}_\sharp(R^2,R)} \int_Q \langle \sigma(y)\nabla u(y), \nabla u(y)\rangle dy \; . \tag{1.3}$$

A topic of great interest in material science and in optimal design is the so-called G-closure problem. The simplest non trivial example is the so-called *two-phase problem* which we now describe. Assume that

$$\sigma(x) = (K\chi_E(x) + K^{-1}(1-\chi_E(x)))I$$

where E is a measurable subset of Q. The G-closure problem in this case can be roughly described as follows. The given data are the conductivities in each phase (KI and $K^{-1}I$), the *volume fractions* p, $1-p$ with $p \in [0,1]$. The unknown is the set E, called the *microgeometry*. As E varies, so does the homogenized matrix $\sigma_{\rm hom}$. The goal is to characterize the exact range of $\sigma_{\rm hom}$ as the measurable set E varies in the family of *all* possible measurable subsets of Q satisfying the constraint $\mid E \mid = p$ (more precisely its closure in the space of symmetric matrices equipped with its natural norm).

The study of the two-phase problem has been initiated by Hashin and Shtrikman [32] and it has been completely solved only twenty years later by Tartar and Murat [65], [54] and by Cherkaev and Lurie [44]. Many interesting G-closure problems are still open. For instance the *three-phase problem* in which σ takes three distinct values with prescribed volume fractions, has attracted considerable attention.

To outline the role played by our results in this context, we need some further notation and preliminary background.

We denote the set of real two by two matrices by $\mathcal{M}$ and the subsets of matrices with strictly positive and non-zero determinant, by $\mathcal{M}_+$ and $\mathcal{M}_{\neq}$ respectively.

We change the definition (1.3) to an equivalent but more convenient one:

$$\forall A \in \mathcal{M} \quad , \quad {\rm tr}(A\sigma_{\rm hom}A^T) = \inf_{U \in W^{1,2}_{\sharp,A}(R^2,R^2)} \int_Q {\rm tr}\left[DU(y)\sigma(y)DU(y)^T\right] dy. \tag{1.4}$$

The infimum in (1.4) is taken on a class of vector fields rather than functions. We use the notation D (rather than ∇) to denote the gradient of vector valued mappings.

Our convention is that for $F=(f,g)$,

$$DF=\begin{pmatrix} f_{x_1} & f_{x_2} \\ g_{x_1} & g_{x_2} \end{pmatrix} \quad .$$

Obviously (1.4) implies (1.3). By the linearity of the Euler-Lagrange equations associated to the variational principle (1.3), the latter implies (1.4). Hence the two definitions are indeed equivalent.

Given $A\in\mathcal{M}$, denote by U^A a solution (unique up to an additive constant) of

$$\begin{cases} \operatorname{Div}\left[\sigma(y)(DU^A(y))^T\right]=0 \quad \text{in} \quad R^2 \\ U^A\in W^{1,2}_{\sharp,A}(R^2,R^2) \qquad\qquad , \end{cases} \tag{1.5}$$

where for any matrix B, $\operatorname{Div}B$ is the vector whose i-th component is the divergence of the vector whose components form the i-th column of B.

Note that by (1.5), U^A is a solution of the Euler-Lagrange equations associated to (1.4). It is easily seen using by integration by parts that

$$\forall A\in\mathcal{M} \quad , \quad \sigma_{\text{hom}}A^T=\int_Q \sigma(y)DU^A(y)^T dy \quad . \tag{1.6}$$

The auxiliary problem (1.5) is usually called the cell problem. Solutions to (1.5) will be called, with a slight abuse of language, *periodic* σ-harmonic mappings. Recall that they are unique up to translation by a constant vector. We now state some of the main results of the present paper.

In the sequel, $K\geq 1$ and $\sigma\in L^\infty_\sharp(R^2,\mathcal{M}^s_K)$ are given.

Theorem A Set $A\in\mathcal{M}_+$ and let U^A be a solution to (1.5). Then
(i) U^A is a homeomorphism of R^2 onto itself.
Moreover
(ii) we have that

$$\det DU^A>0 \quad \text{almost} \quad \text{everywhere} \quad \text{in} \quad R^2. \tag{1.7}$$

See Theorems 2.1 and 3.1 for details.

The next results are applications to the study of G-closure problems. We first state them and then make some comments about their significance and potential for applications.

Theorem B Set

$$d_m := \operatorname{ess}\inf_{x\in Q} \sqrt{\det \sigma} \quad .$$

Then

$$\det \sigma_{\text{hom}} \geq d_m^2 \tag{1.8}$$

and, for every $\lambda \in (-d_m, d_m)$ and for every $A \in \mathcal{M}$,

$$\frac{\operatorname{tr}(A\sigma_{\text{hom}}A^T) - 2\lambda \det A}{\det \sigma_{\text{hom}} - \lambda^2} =$$

$$\inf_{U\in W^{1,2}_{\sharp,A}(R^2,R^2)} \frac{1}{\mid Q \mid} \int_Q \frac{\operatorname{tr}[DU(y)\sigma(y)DU(y)^T] - 2\lambda \det DU(y)}{\det \sigma(y) - \lambda^2} dy \quad . \tag{1.9}$$

To state the next result it is convenient to make the following definitions. Let $S \in \mathcal{M}^s_K$, set $s = \sqrt{\det S}$ and let $\lambda \in [0, s]$. We define a set of *quasiconformal matrices* and a corresponding function space as follows:

$$m(S,\lambda) \equiv \begin{cases} \{G \in \mathcal{M}^s \ : \ (s^2+\lambda^2)\det G > \lambda \operatorname{tr}(G\operatorname{Adj}(S)G^T)\} & \text{if } \lambda \in [0,s) \\ \{G \in \mathcal{M}^s \ : \ (s^2+\lambda^2)\det G = \lambda \operatorname{tr}(G\operatorname{Adj}(S)G^T)\} & \text{if } \lambda = s \quad ; \end{cases} \tag{1.10}$$

$$W(\sigma,\lambda,A) \equiv \{U \in W^{1,2}_{\sharp,A}(R^2,R^2) \ : \ DU(x) \in m(\sigma(x),\lambda) \text{ a.e. } x \in Q\} \ . \tag{1.11}$$

It is easy to check that $S^{-\frac{1}{2}} \in m(S,\lambda)$ for every $\lambda \in [0,s]$, hence

$$\forall \lambda \in [0,s] \quad , \quad m(S,\lambda) \neq \emptyset \quad . \tag{1.12}$$

We continue to adopt the notation of Theorem B and we set

$$Q_m = \{x \in Q \quad : \quad \sqrt{\det \sigma(x)} = d_m\} \quad . \tag{1.13}$$

Theorem C If we assume that

$$0 <\mid Q_m \mid< 1 \quad , \tag{1.14}$$

then

$$\det \sigma_{\text{hom}} > d_m^2 \tag{1.15}$$

and, for every $A \in m(\sigma_{\rm hom}, d_m)$,

$$
\frac{\operatorname{tr}(A\sigma_{\rm hom}A^T) - 2d_m \det A}{\det \sigma_{\rm hom} - d_m^2} =
$$

$$
\inf_{U \in W(\sigma, d_m, A)} \int_Q \left\{ \chi_{Q_m}(y) \frac{\operatorname{tr}[DU(y)\sigma(y)DU(y)^T]}{2\det \sigma(y)} \right. +
$$

$$
\left. (1 - \chi_{Q_m}(y)) \frac{\operatorname{tr}[DU(y)\sigma(y)DU(y)^T] - 2d_m \det DU(y)}{\det \sigma(y) - d_m^2} \right\} dy \quad . \tag{1.16}
$$

We now outline the role of these results in the study of the G-closure problems focusing, to fix ideas, on the issue to prove bounds on $\sigma_{\rm hom}$ in the two-phase problem. To be useful, these bounds must depend solely on the two given numbers K and p. The first elementary observation is that for any admissible λ, one achieves a particular bound by choosing $U = Ax$ in (1.9). We call the latter choice the *trivial test field.*

Proceeding in this way one obtains a family of bounds depending on the parameter λ. Optimization over the admissible values of λ delivers a so-called *translation* (or *compensated compactness*) bound. It is well known that for suitable choices of the matrix A these (lower) bounds are *optimal.* In other words, for each chosen value of $\sigma_{\rm hom}$ satisfying the bounds as an equality, one can find a corresponding microgeometry (or at least a corresponding sequence of microgeometries) with explicitly computable homogenized matrix matching (or at least approaching with arbitrary precision) that chosen value.

However, the optimality of the trivial test field, is somewhat accidental. It doesn't hold as soon as the G-closure problem has a slightly more complicated structure. For instance, in the three-phase problem, the bounds obtained by such a simple use of the variational principle are known to be optimal only under undesirable restrictions on the volume fraction of the phases, see [52], [56], [16], [15] and [29].

To explain the progress made with Theorems B and C, we need to say a bit more about the so-called *translation method.* It may seem rather surprising that the choice of the trivial test field in (1.9) leads to an optimal bound. This little miracle, can be partially explained. The variational principle (1.9) is build in such a way to use already what (in the slightly different context of multi-well problems) are called the "minor relations". In other words, implicit use has already been made of the fundamental fact that given any $A \in \mathcal{M}$ and any $U \in W^{1,2}_{\sharp,A}(R^2, R^2)$ one has

$$\int_Q \mathrm{trace} DU(x)dx = \mathrm{trace} A$$

and

$$\int_Q \det DU(x)dx = \det A \quad . \tag{1.17}$$

This fact is often expressed by saying that $A \to \mathrm{tr} A$ and $A \to \det A$ are *null-lagrangians* on the space $W^{1,2}_{\sharp,A}(R^2, R^2)$. The equality (1.17) is a special instance of a much more general phenomenon leading to the notion of *quasiconvexity* : a continuous functions F on the space of two by two matrices is quasiconvex if $U \in W^{1,2}_{\sharp,A}(R^2, R^2)$ implies

$$\int_Q F(DU)dx \geq F(A) \quad .$$

By Jensen's inequality, convex functions have this property and if the target space of U has dimension one, the set of quasiconvex functions, reduces itself to the set of convex functions [67]. However, if both the domain and the target space have dimension greater than one, there exist quasiconvex functions which are not convex. The compensated compactness developed by Murat and Tartar [66], [54] is the natural mathematical tool to find bounds on homogenized coefficients by using the existence of these functions. Due to its elegance, simplicity and generality, the method has been a tremendous source of stimulus and results in material sciences, optimal design and their connections to certain branches of the calculus of variations. The task to give a list of these results and connections will not be attempted here. We refer to the book of G. W. Milton which includes a very detailed and inspired overview [51] (see also [70]).

Unfortunately, in its general form, the compensated compactness method faces another difficulty, namely that very little is known about the set of quasiconvex functions. In practice, in two dimensional conductivity, all the bounds obtained with this approach select only the determinant (or not relevant modifications of it) in the (unknown) class of all quasiconvex functions and use it as efficiently as possible. This is what we will call the *conventional translation method.* Use of different quasiconvex functions (*unconventional translation method*) is in principle possible but, at present, no other efficient candidates are available, at least in dimension two.

Here, a crucial point is that the bound obtained using the conventional translation method can be seen to be equivalent to the bound that one obtains from (1.9) after inserting the trivial field first and then optimizing over λ. The question of improving upon this choice when the latter is not optimal is an important and interesting one. This issue motivates the results of Section 7 in this paper. Theorem 7.1 provides a more detailed formulation than Theorem B.

Indeed, Theorem 7.1, in conjunction with Corollary 7.1, states that, under a mild condition on the matrix A, the class of relevant test fields in (1.9) can be restricted to a narrow subset of the classical space $W^{1,2}_{\sharp,A}$, namely the space of quasiconformal mappings which have dilation bounded by a certain constant depending only on K and λ. (For the precise statement see Section 6).

Let us emphasize that Theorem B is the "periodic" version of a result of K. Astala and M. Miettinen ([7], Proposition 2.1). The fundamental advantage of the new version relies in the *quasiconformality* of the minimizers outlined in the previous observation. Such property is not known for minimizers of the variational principle of Astala and Miettinen. (The precise definition of quasiconformality can be found in §1.3).

An example in which a non affine (but quasiconformal) test field in the above class delivers optimal bounds can be found in [57]. Corollary 7.1 in Section 7 shows that the latter has a quite general character: under an assumption on A, but no further assumptions on σ, the minimizer in (1.9) is quasiconformal even if the microgeometry is not optimal for the G-closure problem. Moreover one also has an a priori bound on the *dilatation* of this quasiconformal map which, again, depends only upon K and λ.

The reason why one can make an efficient use of this information, as observed in [57], is a celebrated result by K. Astala [5], [6] later refined by Eremenko and Hamilton [22], which allows for a very good control of the so-called area distortion of quasiconformal mappings.

Let us now stress the qualitative difference between Theorem B and Theorem C, assuming that (1.14) holds. The variational principle (1.16) is rather unconventional. By (1.12), the set $m(\sigma_{\text{hom}}, d_m)$ is non empty and correspondingly, as proved in Lemma 7.4, for any matrix $A \in m(\sigma_{\text{hom}}, d_m)$, the functional space $W(\sigma, d_m, A)$ is also non empty.

Next we note that, for fixed λ, the quadratic form associated to the variational principle (1.9) is bounded above and below by positive constants depending on λ.

Therefore, for fixed λ, the corresponding functional is strictly convex. However as $\lambda \to d_m^-$, the quadratic form degenerates: its maximum eigenvalue diverges. In fact, one can check that this happens (almost everywhere) in the set Q_m which, by assumption, has positive measure. In contrast, the quadratic form associated with (1.16) is bounded above and below (uniformly with respect to x).

Therefore Theorem C achieves the following tasks. It effectively shows that the limit (in the variational sense) as $\lambda \to d_m^-$ of the family of functionals given by Theorem B exists and it computes it. At the same time, it shows that, despite the degeneracy mentioned before, the limit is a quadratic form which is uniformly bounded above and below. Its explicit form depends on position in an unconventional but instructive way. An attempt to fully exploit this new formula will be the subject of future work.

We emphasize the fact that Theorem C uses the full strength of property (1.7) of periodic σ-harmonic mappings.

Theorem C is proven in Section 7 and is complemented by Corollary 7.2, which treats the extremal cases $\mid Q_m \mid = 0, 1$ not comprised in the assumption (1.14). We hope that these results may provide a useful tool for bounding the effective conductivity. Let us describe some of the examples which are covered by Theorem C and Corollary 7.2 in order to put our work into context.

Assume that σ is not *isotropic* in the sense that there exists a set of positive measure where the two eigenvalues, (called the *principal conductivities*) of σ are distinct.

This category of examples includes the following well studied cases.
First: the *single-phase polycrystal problem.*

Here the only data is $K > 1$. The conductivity has the form $R^T(x)\mathrm{diag}(K, K^{-1})R(x)$ where $x \to R(x)$ is a measurable field of matrices in $SO(2)$ (which is called a *rotation* of the original *crystal*). Obviously $\sqrt{\det \sigma} = 1$ at almost every point, hence $d_m = 1$ and $\mid Q_m \mid = 1$. Therefore one needs Corollary 7.2 rather than Theorem C. The corresponding classical literature includes [34], [21] and [48] and it is based on the idea of *duality.* In the language we will introduce later in this section, these are the first papers in the field of composites where the idea of *stream function* shows its power. In this case Corollary 7.2 reproduces the known results and it adds an extra information. Indeed, under a condition on A, the minimizer of the corresponding variational principle is a K-quasiconformal mapping with dilatation constantly equal to K.

Second: *two-phase polycrystal problem.*

Here the data are two *crystals*, i.e. two constant diagonal and positive matrices σ_a and σ_b, (giving the conductivity of the pair of given crystals). To distinguish this example from the first we assume $\det \sigma_a \neq \det \sigma_b$.

The conductivity has the form $\sigma(x) = R^T(x)(\sigma_a \chi_a + \sigma_b \chi_b)R(x)$, where $x \to R(x)$ is a measurable field of matrices in $SO(2)$ and χ_a and χ_b are characteristic functions summing up to one. Phase a is characterized as the set where the principal conductivities of σ and those of σ_a are the same.

If the volume fraction of each phase is not prescribed, the problem is simpler but non trivial. Its study was initiated in [45] and completed later by Francfort and Murat [25] in an interesting paper which, unfortunately, has not been fully appreciated.

Several papers deal with examples using various form of duality including [36], [61] and [62]. Some other work partly focusing on the case with prescribed volume fraction, uses duality in conjunction with more complex arguments [24], [31] and [55] (section 7.3). More recently, quasiconformal mapping are having an increasing impact on the two-phase polycrystalline problem, [57], [7] and [53].

For the case of prescribed volume fraction, Theorem C gives several extensions of results that one can prove by duality. For instance it imposes the following very constrained behavior: the (quasiconformal) minimizer of (1.16) has *constant and prescribed* dilation in the "weakest phase" (i.e. the phase where $\det \sigma$ is smaller).

Moreover, still in this category of examples, there are some cases in which Theorem C implies that the minimizer cannot be the trivial field, hence showing, in a very direct way, the lack of optimality of bounds obtained using the conventional translation method.

In Section 8 we discuss on further applications to homogenization of our results. The first is a partial solution to a conjecture made by G. W. Milton [50]. These are statements about higher integrability the gradient of σ-harmonic functions. The higher integrability for planar quasiconformal mapping dates back to the work of Bojarski [14], see also [12] and [11]. It was later extended to σ-harmonic functions in any dimension in the work of N. Meyers, [55]. Gehring established higher integrability for quasiconformal mappings in any dimensions [27]. However these approaches did not give a precise evaluation of the *best exponent.* Gehring and Milton made conjectures about the best exponent in any dimension for the case of quasiconformal mappings and σ-harmonic functions respectively. Milton made an even more challenging conjecture about integrability of a precise power of $|\nabla u|^{-1}$ for σ-harmonic mappings.

Gehring's conjecture was proved for dimension two, by Astala [5] in a fundamental advance. Using this result Leonetti and Nesi [42] proved Milton's conjecture in dimensions two. However the result for $|\nabla u|^{-1}$ depends very strongly on boundary conditions and in [42], the authors were only able to treat Dirichlet and Neumann boundary conditions making very heavy use of results by Alessandrini and Magnanini, [2]. In Theorem 8.1, we extend the result to periodic σ-harmonic mappings which are the most interesting from the point of view of applications to homogenization.

We also prove in Proposition 8.1, that σ-harmonic mappings satisfy the change of variable formula and we give some example in this direction.

We summarize this subsection with an observation. Some recent advances obtained in two dimensional G-closure problems, have been obtained by two apparently distinct arguments. The first [56], was based on work by Bauman, Marini and Nesi [9], exploiting the properties of positivity of the Jacobian determinant of σ-harmonic mappings with affine boundary data. The second, [57], was based on work by K. Astala [5] (later refined in [22]) on quasiconformal mappings. In this paper, we show that these two contributions are very tightly linked by a number of structural properties.

1.2 Univalence

Let us now put into context Theorem A above. The ancestor of such a result is a Theorem due to Radó [60] (see proofs by Kneser [35] and Choquet [17]) which states that if U is a harmonic mapping (that is σ-harmonic with $\sigma = I$) on a disk B, whose boundary data $\Phi = U|_{\partial B}$ form a homeomorphism of ∂B onto a closed *convex* curve Γ, then U is univalent. As a consequence of a result by H. Lewy [43], one also obtains that the Jacobian determinant of U is nonvanishing inside B.

We refer to the survey by Duren, [20] about developments on the study of univalent harmonic mappings in the plane. Let us also quote the recent interesting counterexamples by Melas [47] and by Laugesen [46] to the extension of Radó's Theorem to higher dimensions.

Let us mention also generalizations to harmonic mappings between certain Riemannian two-dimensional manifolds, Schoen and Yau [63], Jost [33], and to mappings whose components are solutions to quasilinear degenerate elliptic equations of the type of the p-laplacian, Alessandrini and Sigalotti [4].

But, here, we are especially interested at the generalization of Radó's Theorem to σ-harmonic mappings. A result in this direction has been proven by Bauman, Marini and Nesi [9] for the case of a smooth conductivity σ, and was already applied to issues of homogenization. In Theorem 2.1 we give a proof to the part (i) of Theorem A, that is of the univalence of the periodic σ-harmonic mapping U^A solving the problem (1.5). To the best of our knowledge, this is the first available result of univalence in the periodic setting, also in the case of a smooth σ. Our approach is based on the analysis of the structure of the level lines of the σ-harmonic functions obtained by linear combinations of the components of U^A . This analysis relies on arguments and concepts introduced in Alessandrini and Magnanini [2] which enable to treat, in a generalized sense, critical points of a σ-harmonic function u and develop a corresponding index calculus also when the conductivity σ is discontinuous and the gradient ∇u is not defined in a pointwise sense. We review these concepts at the beginning of Section 2. Moreover, as a by-product of this approach, we also obtain a generalization of Radó's Theorem to σ-harmonic mappings with nonsmooth σ, Theorem 2.3, and some results on necessary and sufficient conditions for local univalence of σ-harmonic mappings, Theorem 2.2.

Part (ii) of Theorem A, namely (1.7), requires a different analysis. In fact, once univalence is proven, it can be seen rather easily, for instance by a regularization argument , that U^A satisfies

$$\det DU^A \geq 0 \quad \text{almost} \quad \text{everywhere}$$

see also, for an alternative derivation, Remark 2.3 below. It seems less trivial to show that, for a locally univalent σ-harmonic mapping U, $\det DU$ is non vanishing almost everywhere. The main subject of Section 3 is in proving that indeed, for any such U, $\log(\det DU)$ is locally a BMO function, see Theorem 3.1. This, as is well-known from the theory of Muckenhoupt weights (see for instance García - Cuerva and Rubio de Francia [26]), implies that, locally, for some small $\epsilon > 0$, $(\det DU)^{-\epsilon}$ is integrable. The main tools that we use for the proof of Theorem 3.1 are a reverse Hölder inequality for nonnegative solutions of the adjoint equation for a nondivergence elliptic operator due to Bauman [8], later refined by Fabes and Strook [23], see Theorem 3.4, and the results by Reimann [60] about the transformation rules of the space BMO under quasiconformal mappings, see Theorems 3.2, 3.3. In fact we shall show that, modulo a suitably chosen quasiconformal change of coordinates, $\det DU$ is the solution of an adjoint equation for a nondivergence elliptic operator. Hence, from the above mentioned results by Bauman-Fabes-Strook and Reimann, it will turn out that, independently of the quasiconformal change of coordinates, $\log(\det DU)$ locally belongs to BMO.

1.3 Connections with quasiregular mappings

We recall that, given $f \in W^{1,2}_{\mathrm{loc}}(\Omega, R^2)$, the *dilatation quotient* for f is defined for almost every $x \in \Omega$ as

$$\mathcal{D}_f(x) = \frac{\max_{|\xi|=1} |\partial_\xi f(x)|}{\min_{|\xi|=1} |\partial_\xi f(x)|} \tag{1.18}$$

where ∂_ξ denotes directional derivative in the direction ξ, and that, for a given $K \geq 1$, $f \in W^{1,2}_{\mathrm{loc}}(\Omega, R^2)$ is said to be a (sense preserving) K-*quasiregular* mapping if

$$\mathcal{D}_f(x) \leq K \ , \ \text{and } \det Df \geq 0 \ , \ \text{for almost every } x \in \Omega, \tag{1.19}$$

where Df denotes the jacobian matrix of f. A mapping f will be said K-*quasiconformal* if in addition it is injective. We also recall that equivalent conditions to (1.19) are given by

$$\mathrm{tr}(Df Df^T) \leq (K + K^{-1}) \det Df \text{ almost everywhere in } \Omega, \tag{1.20}$$

or else

$$|f_{\bar{z}}| \leq \frac{K-1}{K+1} |f_z| \text{ almost everywhere in } \Omega, \tag{1.21}$$

where the standard identification $z = x_1 + ix_2$ is used. See, as a basic reference for quasiregular mappings in the plane, Lehto and Virtanen [41].

The connections between σ-harmonic and quasiregular mappings are many-sided.

First of all, the components u_1 and u_2 of a σ-harmonic mapping U are also the components of quasiregular mappings. In fact, to each σ-harmonic function u (that is a solution to (1.1)) we can associate in a natural fashion, which generalizes the harmonic conjugation, a new function, the so-called *stream function* $\tilde{u}$, which is a solution to the dual equation

$$\mathrm{div}\,(\tfrac{\sigma}{\det\sigma} \nabla \tilde{u}) = 0 \quad \text{in} \quad \Omega,$$

and which is such that the mapping $f = u + i\tilde{u}$ turns out to be K-quasiregular. These facts, which can be traced back to the functional analytic approach for two-dimensional elliptic equations due to Bers and Nirenberg [12] (see also Bers, John and Schechter [11] Chapter II.6, and Vekua [69]), have been the starting point for the geometric study of σ-harmonic functions in [2], see also the discussion at the beginning of Section 2.

Second, we stress the well-known fact that quasiregular mappings are indeed σ-harmonic for some suitable σ. In fact by the Ahlfors-Bers representation [1], any quasiregular mapping f can be written as

$$f = F \circ \chi$$

where F is holomorphic and χ is quasiconformal. Being the components of F harmonic, we obtain that the components of f are σ-harmonic where σ can be chosen as follows

$$\sigma(x) = \begin{cases} \det D\chi (D\chi^T D\chi)^{-1} & \text{if} \quad \det D\chi \neq 0 \ , \\ I & \text{if} \quad \det D\chi = 0 \ . \end{cases} \tag{1.22}$$

Notice that $\sigma \in L^\infty(\Omega, \mathcal{M}^s_K)$ where K is the supremum of the dilatation of f (which is the same as the one of χ) and also that $\det\sigma = 1$ everywhere.

Another type of connection between σ-harmonic and quasiregular mappings emerges from Theorem 3.1, which we illustrated before, since σ-harmonic mappings share with quasiregular mappings the property $\log(\det DU) \in \mathrm{BMO}$.

It might seem spontaneous, then, to wonder whether locally univalent σ-harmonic mappings are indeed quasiregular. On a crude level, one could say that the answer is definitely negative, as it can be seen by the univalent harmonic mapping

$$U(x) = \left(x_1 + \frac{1}{2}(x_1^2 - x_2^2) \ , \ x_2 - x_1 x_2 \right) \ , \ x_1^2 + x_2^2 < 1 \ ,$$

whose dilatation quotient is given by

$$D_U(x) = \frac{1 + (x_1^2 + x_2^2)^{1/2}}{1 - (x_1^2 + x_2^2)^{1/2}}$$

which blows up as $(x_1^2 + x_2^2)^{1/2} \to 1$, see [20]. What we see, in this example, is that the harmonic mapping U is not quasiregular on the whole unit disk, the domain on which it is univalent, but it is such on any of its compact subdomains. Therefore, a more refined way of posing the question is whether, for a given conductivity σ, locally univalent σ-harmonic mappings are *locally* quasiregular. This question is the object of Sections 4 and 5. We wish to stress that, at least at present, this investigation is not specifically related to applications to homogenization. However we believe that it can be of some independent interest, and might present stimulating open problems.

The first significant result in this direction (see, for details, Theorem 4.1, part(iii)) is that if, for a given conductivity σ, there exists *one* locally univalent σ-harmonic mapping which is also locally quasiregular, then *every* locally univalent σ-harmonic mapping is also locally quasiregular. This fact can be expressed by saying that, for a univalent σ-harmonic mapping U, being locally quasiregular, is a property that depends on σ only and not on the specific mapping U. We shall denominate Σ_{qc} the class of those conductivities σ having such a property, see Definition 4.1.

Next, in Theorem 4.1 part (ii) and Corollary 4.1, we collect a number of necessary and sufficient conditions to have $\sigma \in \Sigma_{\mathrm{qc}}$. A practical sufficient condition is also found, namely $\det\sigma \in C^{\alpha}_{\mathrm{loc}}$, see Theorem 4.2.

Explicit examples of conductivities σ not belonging to Σ_{qc} are found in Examples 5.1, 5.2. We also provide examples of $\sigma \in \Sigma_{\mathrm{qc}}$ which can be dramatically discontinuous, see Example 5.3. It seems to us that an interesting open problem, which is suggested by these examples, is the study of the structure of Σ_{qc} as a proper non empty subset of $\cup_{K\geq 1} L^{\infty}(\Omega, \mathcal{M}^{s}_{K})$. In fact, Example 5.3, in partial contrast to what suggested by Theorem 4.2, shows that Σ_{qc} cannot be characterized only in terms of regularity properties. In this respect, we refer also to the considerations developed in Remark 5.3.

A further connection with quasiregular mappings, which we already touched upon in §1.1, arises from the variational principle (1.9). In fact, the construction of the minimizer of in (1.9), see Theorem 7.1, suggests in a natural fashion to associate to each σ-harmonic mapping U, a family of mappings $\phi_{U,\lambda}$, depending linearly on U and affinely on the real parameter λ. We shall show that, provided that $\det DU \geq 0$ almost everywhere, then the mapping $\phi_{U,\lambda}$ has the remarkable property of being *globally* quasiregular for every $\lambda > 0$, see part i) of Proposition 6.1. This fact will be crucial in the proof of Theorem C, see also Corollary 7.1. Moreover we prove interesting properties of univalence of $\phi_{U,\lambda}$. Namely, if U is locally univalent and sense preserving, so is $\phi_{U,\lambda}$ for every $\lambda > 0$ (see part ii) of Proposition 6.1), and if $U = U^A$ is the solution to (1.5) with $\det A > 0$, then $\phi_{U,\lambda}$ is a homeomorphism of R^2 onto itself for every $\lambda > 0$, see Proposition 6.2.

2 Conditions for univalence

The main subject of this Section is the proof of part (i) of Theorem A which we rephrase as follows.

Theorem 2.1 Let $\sigma \in L^\infty_\sharp(R^2, \mathcal{M}^s_K)$ and let $A \in \mathcal{M}_{\neq}$. If $U^A \in W^{1,2}_{\sharp,A}(R^2, R^2)$ is a σ-harmonic mapping, then U^A is a homeomorphism of R^2 onto itself.

Remark 2.1 The existence and uniqueness up to an additive constant vector field of such an U^A is well-known and can be obtained by standard Hilbert space arguments (see, e.g. [10]).

Remark 2.2 Theorem 2.1 is adequate for the purpose of the applications to homogenization given in Sections 6 and 7. It should be noted that it suffices to prove the statement in the special case when A is the identity matrix I. The general statement then follows by the observation $U^A = A \circ U^I$ and the fact that if $\det A \neq 0$, $x \to Ax$ is obviously an homeomorphism.

Similarly, the choice of the cell of periodicity $Q = (0,1) \times (0,1)$, made in Theorem 2.1 is not essential. Again the change of coordinates, this time in the independent variables, by

$$x \to \begin{pmatrix} 1 & \alpha \\ 0 & \beta \end{pmatrix} x, \quad x \in R^2 \quad , \tag{2.1}$$

with $\beta > 0$, allows one to recover any admissible cell of periodicity.

Before proceeding to the proof of Theorem 2.1, we need to recall a few facts about the geometry of solutions of divergence form elliptic equations in two variables (we shall mainly refer to [2]).

If Ω is simply connected and $u \in W^{1,2}_{\mathrm{loc}}(\Omega, R)$ is a weak solution to

$$\mathrm{div}(\sigma \nabla u) = 0 \quad \text{in} \quad \Omega \quad , \tag{2.2}$$

then there exists, and it is uniquely determined up to an additive constant, a function $\tilde{u} \in W^{1,2}_{\mathrm{loc}}(\Omega, R)$ such that

$$\nabla \tilde{u} = J\sigma \nabla u \quad \text{a.e.} \quad x \in \Omega \tag{2.3}$$

where

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad .$$

Such a function is called the *stream function* associated to u, and we have, in the weak sense

$$\operatorname{div}(\tfrac{\sigma}{\det \sigma}\nabla \tilde{u}) = 0 \quad \text{in} \quad \Omega \quad .$$

With the usual identification between R^2 and C we set

$$f = u + i\tilde{u} \quad , \quad z = x_1 + ix_2 \quad . \tag{2.4}$$

Then we have

$$f_{\bar{z}} = \mu f_z + \nu \bar{f}_z \quad \text{a.e.} \quad \text{in} \quad \Omega \tag{2.5}$$

where the coefficients $\nu, \mu \in L^\infty(\Omega)$ only depend (explicitly) on σ and satisfy

$$\mid \nu \mid + \mid \mu \mid \leq \tfrac{K-1}{K+1} < 1 \quad \text{a.e.} \quad \text{in} \quad \Omega \quad .$$

More details can be found in [2]. In particular, for any domain $D \subset\subset \Omega$,

$$f \in W^{1,2}(D, C) \quad \text{and} \quad \mid f_{\bar{z}} \mid \leq \tfrac{K-1}{K+1} \mid f_z \mid \quad \text{a.e.} \quad \text{in} \quad D \quad . \tag{2.6}$$

By definition, any f satisfying (2.6) is said a K-quasiregular mapping. It is well known that on any compact subset of Ω, f has the representation

$$f = F \circ \chi \tag{2.7}$$

where χ is K-quasiconformal and F is holomorphic, (see for instance [12] and [41]). Consequently, letting $h = \Re F$, $\tilde{h} = \Im F$, we obtain the representations

$$u = h \circ \chi \quad , \quad \tilde{u} = \tilde{h} \circ \chi \quad . \tag{2.8}$$

Notice that we also obtain that the notion of stream function is invariant under quasiconformal changes of coordinates. That is, if u solves (2.2) and $\chi : \Omega \to R^2$ is a quasiconformal mapping, then, clearly, $v = u \circ \chi^{-1}$ is a solution to

$$\operatorname{div}(\tau \nabla v) = 0 \quad \text{in} \quad G = \chi(\Omega) \quad , \tag{2.9}$$

where $\tau = T_\chi \sigma$ is defined by

$$T_\chi \sigma = \frac{D\chi \sigma D\chi^T}{\det D\chi} \circ \chi^{-1} \quad . \tag{2.10}$$

We note that the stream function $\tilde{v}$ associated to v via (2.9) is given by $\tilde{v} = \tilde{u} \circ \chi^{-1}$. Let us recall that by convention, the Jacobian matrix of a mapping $\chi = (\alpha, \beta)$ is defined by

$$D\chi = \begin{pmatrix} \alpha_{x_1} & \alpha_{x_2} \\ \beta_{x_1} & \beta_{x_2} \end{pmatrix} \quad . \tag{2.11}$$

Returning now to (2.8), we shall say that $z_0 \in \Omega$ is a *geometric critical point* if $\nabla h(\chi(z_0)) = 0$.

We observe that geometric critical points are isolated and furthermore that, in a small neighborhood of any geometric critical point z_0, the level set $\{u = u(z_0)\}$ is composed by $I + 1$ simple arcs whose pairwise intersection consists of $\{z_0\}$ only. Here $I = I(z_0, u)$ is the positive integer given by the multiplicity of the zero of $\partial_z h$ at the point $\chi(z_0)$. Such a number is called the *geometric index* of u at z_0. Now let $D \subset\subset \Omega$, be an open set such that ∂D contains no geometric critical points. We denote by $I(D, u)$ the sum of the geometric indices of the geometric critical points z_k of u within D. Observe, in particular, that, when u is smooth and ∂D is piecewise regular, the index $I(D, u)$ can be computed by the following contour integral

$$I(D, u) = -\frac{1}{2\pi} \int_{\partial D} \mathrm{d}(\arg \nabla u) \quad . \tag{2.12}$$

The following stability result for the geometric index will be used in the sequel.

Lemma 2.1 Let $\{\sigma_m\} \subset L^\infty(\Omega, \mathcal{M}^s_K)$ and let $u_m \in W^{1,2}_{\mathrm{loc}}(\Omega, R)$ be weak solutions to

$$\mathrm{div}(\sigma_m \nabla u_m) = 0 \quad \text{in} \quad \Omega \quad , \tag{2.13}$$

such that $u_m \to u$ in $W^{1,2}_{\mathrm{loc}}(\Omega)$ where u is a non constant solution to (2.2). Given any $D \subset\subset \Omega$ such that ∂D contains no geometric critical points of u, then, for m sufficiently large, ∂D contains no geometric critical points of u_m and we have

$$\lim_{m \to \infty} I(D, u_m) = I(D, u) \quad . \tag{2.14}$$

Proof. See Proposition 2.6 in [2]. $\square$

Lemma 2.1 will be used in conjunction with the following regularization argument.

Lemma 2.2 Let $\sigma \in L^\infty_\sharp(R^2, \mathcal{M}^s_K)$, let $A \in \mathcal{M}_+$ and let $U \in W^{1,2}_{\mathrm{loc}}(R^2, R^2)$ be the σ-harmonic mapping such that $U \in W^{1,2}_{\sharp,A}(R^2, R^2)$ and $U(0) = 0$. There exists a sequence $\{\sigma_m\} \subset L^\infty_\sharp(R^2, \mathcal{M}^s) \cap C^\infty(R^2, \mathcal{M}^s)$ such that $\sigma_m \to \sigma$ in $L^p_{\mathrm{loc}}(R^2, \mathcal{M}^s)$ for every $p < \infty$ and almost everywhere. Setting U_m to be the σ_m-harmonic mapping with $U_m \in W^{1,2}_{\sharp,A}(R^2, R^2)$ and $U_m(0) = 0$, we have $U_m \to U$ in $W^{1,2}_{\mathrm{loc}}(R^2, R^2)$.

Proof It follows the standard procedure of constructing regularized solutions by first mollifying the coefficients. □

We shall also repeatedly make use of the following simple fact.

Lemma 2.3 Let $\sigma \in L^\infty(\Omega, \mathcal{M}^s_K)$ and let $u \in W^{1,2}_{\mathrm{loc}}(\Omega, R)$ be σ-harmonic. If $x^0 \in \Omega$ is not a geometric critical point of u, then there exists a neighborhood $D \subset \Omega$ of x^0 on which the level lines $\{u = t\}$ of u are simple arcs and $\tilde{u}$ is strictly monotone along them.

Proof If x^0 is not a geometric critical point, then in the representation (2.7) we have $F'(\chi(x^0)) \neq 0$, therefore F is locally invertible and hence near x^0, f is a quasiconformal homeomorphism. Consequently, near $f(x^0)$ we have

$$u \circ f^{-1} = x_1 \quad , \quad \tilde{u} \circ f^{-1} = x_2 \quad . \ \square$$

The proof of Theorem 2.1, given below, will require the following three propositions in which the following conventions will be used. Given u_1 and u_2 solutions to (2.2), we will fix their stream functions $\tilde{u_1}, \tilde{u_2}$ by prescribing $\tilde{u_1}(0) = \tilde{u_2}(0) = 0$ and we set $\tilde{U} = (\tilde{u_1}, \tilde{u_2})$. For any fixed $\xi \in R^2$ with $|\xi| = 1$, we set

$$u = \langle \xi, U \rangle = \xi_1 u_1 + \xi_2 u_2 \quad , \quad \tilde{u} = \langle \xi, \tilde{U} \rangle = \xi_1 \tilde{u}_1 + \xi_2 \tilde{u}_2 \quad , \quad f = u + i\tilde{u} \quad . \qquad (2.15)$$

Clearly, u, $\tilde{u}$ and f depend on ξ but we will not keep track of this dependence in the notation. We stress here that for any choice of ξ the vector u defined in (2.15) is also a solution to (2.2) in R^2 and $\tilde{u}$ is its associated stream function. Hence f is K-quasiregular on C.

Proposition 2.1 Let Ω be a connected open set in R^2, let $\sigma \in L^\infty(\Omega, \mathcal{M}^s_K)$ and $U \in W^{1,2}_{\mathrm{loc}}(\Omega, R^2)$ be σ-harmonic. If for every ξ, $|\xi| = 1$, f is univalent , then U is univalent.

Proof Suppose by contradiction that there exist distinct points $x^1, x^2 \in \Omega$ such that $U(x^1) = U(x^2)$. Then, for each ξ, $u(x^1) = u(x^2)$. Hence one can find ξ such that

$$\xi_1(\tilde{u}_1(x^1) - \tilde{u}_1(x^2)) + \xi_2(\tilde{u}_2(x^1) - \tilde{u}_2(x^2)) = 0 \quad .$$

For this choice of ξ, f is manifestly not injective. $\square$

Proposition 2.2 Let the hypotheses of Theorem 2.1 be satisfied and let u be defined by (2.15). For every ξ, $|\xi|=1$, u has no geometric critical points in R^2.

Proof In view of Lemma 2.1 and Lemma 2.2, there is no loss of generality in assuming, in addition, that σ has C^∞ entries. By elliptic regularity theory, this implies that u is smooth. For $r>0$, we set $Q_r=(-r,r)\times(-r,r)$. By assumption each component of $\nabla u-\xi$ is periodic and therefore so is each component of ∇u since ξ is constant. Now, since u is not identically constant, it has finitely many critical points in, say, Q_3. Hence, we may find $\bar{x}=(\bar{x_1},\bar{x_2})$ with $\bar{x_1},\bar{x_2}\in(0,1)$ such that u has no critical points on the line segments

$$l^v\equiv\{x_1=\bar{x}_1 \quad , \quad |x_2|<3\} \quad , \quad l^h\equiv\{x_2=\bar{x}_2 \quad , \quad |x_1|<3\} \quad .$$

By periodicity, u has no critical points also on the translated segments

$$l^v_\pm\equiv\{x_1=\bar{x}_1\pm 2 \quad , \quad |x_2|<3\} \quad , \quad l^h_\pm\equiv\{x_2=\bar{x}_2\pm 2 \quad , \quad |x_1|<3\} \quad .$$

Setting

$$Q_2(\bar{x})=\{(x_1,x_2) \quad | \quad |x_1-\bar{x}_1|\le 2 \quad , \quad |x_2-\bar{x}_2|\le 2\}$$

we have that $\partial Q_2(\bar{x})$ is contained in the union of the translate segments above and moreover, denoting the horizontal sides of $\partial Q_2(\bar{x})$ by $s^h_\pm$ and the vertical one by $s^v_\pm$, the periodicity implies

$$\int_{s^h_-} \ d\arg\nabla u=\int_{s^h_+} \ d\arg\nabla u \quad ; \quad \int_{v^h_-} \ d\arg\nabla u=\int_{v^h_+} \ d\arg\nabla u \quad .$$

Hence taking into accounts the orientation of $\partial Q_2(\bar{x})$, we readily obtain

$$I(Q_2(\bar{x}),u)=-\frac{1}{2\pi}\int_{\partial Q_2(\bar{x})} \ d\arg\nabla u=0 \quad .$$

Therefore, according to (2.12) u has no critical points inside $Q_2(\bar{x})$. By periodicity, it has no critical points at all. $\square$

Proposition 2.3 Let the hypotheses of Theorem 2.1 be satisfied and let f be defined by (2.15). Then for every ξ, $|\xi|=1$, f is univalent.

The proof of Proposition 2.3 requires the following Lemma.

Lemma 2.4 Let $L>0$ and let G be an open set contained in the strip $\{x\in R^2 \ | \ |x_2|<L\}$. Let

$$u\in\cap_{R>0}W^{1,2}(G\cap B_R;R)$$

be a weak solution of the Dirichlet problem

$$\begin{cases} \operatorname{div}(\sigma\nabla u) = 0 & \text{in} \quad G \\ u = 0 & \text{on} \quad \partial G \end{cases} ,$$

where $\sigma \in L^\infty(\Omega, \mathcal{M}^s_K)$. If $M = \| u \|_{L^\infty(G)} < \infty$, then $u \equiv 0$.
Proof of Lemma 2.4 For any fixed $R > 0$, let us set

$$\eta(x_1) = \begin{cases} 1 & \text{when} & & \mid x_1 \mid & \le R \\ \frac{2R-|x_1|}{R} & \text{when} & R \le & \mid x_1 \mid & \le 2R \\ 0 & \text{when} & 2R \le & \mid x_1 \mid & \end{cases} .$$

Using $\phi(x) = \eta^2(x_1)u(x) \in W_0^{1,2}(G)$ as a test function in the weak formulation of (2.12), we are led to the Caccioppoli inequality

$$\int_G \langle \sigma\nabla u, \nabla u\rangle \eta^2 \le 4 \int_G \langle \sigma\nabla \eta, \nabla \eta\rangle u^2$$

which implies

$$\int_{G\cap\{|x_1|<R\}} \langle \sigma\nabla u, \nabla u\rangle \le 4 \int_{G\cap\{R\le|x_1|\le 2R\}} \frac{\sigma_{11}}{R^2} u^2 \le 8LM^2K\frac{1}{R} \to 0 \quad \text{as} \quad R \to \infty \quad .$$

Hence $\mid \nabla u \mid \equiv 0$ and, consequently $u \equiv 0$. $\square$
Proof of Proposition 2.3. By Proposition 2.2, the level lines of u are composed by pairwise disjoint simple arcs, having limit points at infinity only. Let us show that each level line $\{u = t\}$ is connected.

By standard local boundedness estimates (see for instance [30]) and by the periodicity of $u - \langle \xi, x\rangle$, we have that, for any $L > 0$, there exists $M > 0$ such that on any strip

$$S_{r,L} = \{x \in R^2 \quad | \quad r - L \le \langle \xi, x\rangle \le r + L\} \quad , \quad r \in R \quad ,$$

we have

$$\mid u - \langle \xi, x\rangle \mid \le M \quad .$$

In particular, we have

$$u(x) \to \pm\infty \quad \text{uniformly} \quad \text{as} \quad \langle \xi, x\rangle \to \pm\infty \quad ,$$

respectively. Moreover, there exists $L > 0$ such that, for every $t \in R$, the level line $\{u = t\}$ is contained in the strip $S_{t,L}$ and also

$$\begin{array}{lll} u > t & \text{when} & \langle \xi, x \rangle > t + L \quad , \\ u < t & \text{when} & \langle \xi, x \rangle < t + L \quad . \end{array}$$

The proof will be completed once we have shown that every level line $\{u = t\}$ is connected. In fact by Lemma 2.3, we obtain that $\tilde{u}$ is strictly monotone on such level lines. Suppose, by contradiction, that the level line $\{u = t\}$ were disconnected. Then either $\{u > t\}$ or $\{u < t\}$ would be disconnected and it would have at least one component, say G, contained in $S_{t,L}$. Up to a rigid transformation of coordinates we can apply Lemma 2.4 to $u - t$ in G and obtain $u \equiv t$ in G. This is a contradiction because, in G, we have either $u > t$ or $u < t$. □

Proof of Theorem 2.1 We set A to be the identity. The general case follows by Remark 2.1. Propositions 2.3 and 2.1 imply that U is univalent. For the sake of completeness, we provide an elementary self-contained proof of the ontoness of U and of the continuity of U^{-1}. Let us begin by proving that U is onto. The arguments used in the proof of Proposition 2.3 show that there exists $L > 0$ such that any level line of u_1 is contained in some vertical strip of width $2L$. Similarly any level line of u_2 is contained in some horizontal strip of the same width. Moreover

$$u_i(x_1, x_2) \to \pm\infty \quad \text{uniformly} \quad \text{as} \quad x_i \to \pm\infty \quad i = 1, 2 \quad .$$

Therefore for every $t \in R$,

$$\inf_{\{u_1 = t\}} u_2 = -\infty \quad , \quad \sup_{\{u_1 = t\}} u_2 = +\infty \quad .$$

Hence, by continuity, $u_2|_{\{u_1 = t\}}$ attains to every real value and therefore U maps R^2 onto itself. Next we prove that U^{-1} is continuous. Fix an arbitrary convergent sequence $y^n \to y^0$ in R^2 and set $x^0 = U^{-1}(y^0), x^n = U^{-1}(y^n)$. We need to show that $x^n \to x^0$. Suppose, by contradiction that, up to subsequences

$$\begin{array}{lll} \text{either} & x^n \to x^1 \in R^2 & , \quad x^1 \neq x^0 \\ \text{or} & | x^n | \to \infty & . \end{array}$$

In the first case,

$$U(x^1) = \lim_{n \to \infty} U(x^n) = \lim_{n \to \infty} y^n = y^0 = U(x^0)$$

and the univalence would be violated. In the second case, since $U - x$ is periodic and bounded, we would have

$$| y^0 |= \lim_{n\to\infty} | y^n |=| \lim_{n\to\infty} U(x^n) |= +\infty \quad .$$

This is also a contradiction. □

Remark 2.3 Observe that if U is σ-harmonic and locally one-to-one on a connected open set Ω, then it is either preserving or reversing the orientation. In the sequel it will be convenient to assume that U is orientation preserving. This property can always be achieved, up to replacing u_2 with $-u_2$. Notice also that a possible way of interpreting such an orientation character is in terms of the monotonicity of u_2 along the level lines $\{u_1 = const.\}$. Namely, if, locally, u_2 is non-decreasing in the $\tilde{u}_1$-direction, then U is orientation preserving. If, else, u_2 is non-increasing in the $\tilde{u}_1$-direction, then U is orientation reversing. Let us now take U orientation preserving, $\xi = e_1$ and form locally $w = u_2 \circ f^{-1}$ where $f = (u_1, \tilde{u}_1)$. For every u_1, $w = w(u_1, \tilde{u}_1)$ is a non-decreasing function of $\tilde{u}_1$, that is

$$\tfrac{\partial w}{\partial \tilde{u}_1} \geq 0 \quad \text{for} \quad \text{a.e.} \quad (u_1, \tilde{u}_1) \quad .$$

Now, being f^{-1} locally defined as a quasiconformal mapping, by the chain rule and the null set invariance property

$$\frac{\partial w}{\partial \tilde{u}_1}(u_1(x), \tilde{u}_1(x)) = \frac{1}{\det Df(x)} \det DU(x)$$

for a. e. $x \in \Omega$. Therefore

$$\det DU(x) \geq 0 \quad \text{for} \quad \text{a.e.} \quad x \in \Omega \quad . \tag{2.16}$$

Theorem 3.1 in the next Section sharpens the above bound by showing, in particular, that the strict inequality holds almost everywhere in (2.16).

We give now two further results concerning univalence. The first one, Theorem 2.2, gives a localized interpretation of the results discussed above. We emphasize the fact that Theorem 2.2 below, while providing necessary and sufficient conditions for the local injectivity of a σ-harmonic mapping U, shows a curious parallelism with the standard inverse mapping theorem for differentiable mappings.

Theorem 2.2. Let $\Omega \subset R^2$ be a connected open set. Let $\sigma \in L^\infty(\Omega, \mathcal{M}^s_K)$ and let $U \in W^{1,2}_{\text{loc}}(\Omega, R^2)$ be a σ-harmonic mapping. We adopt the conventions established by (2.15). The following properties are equivalent.

(i) u has no geometric critical points for every ξ, $\mid \xi \mid = 1$,
(ii) f is locally one-to-one for every ξ, $\mid \xi \mid = 1$,
(iii) U is locally one-to-one,
(iv) $\tilde{U}$ is locally one-to-one;

Proof of (i) $\Rightarrow$ (ii) Suppose that for a given ξ, u has no geometric critical points, then locally $f = F \circ \chi$ where χ is quasiconformal and F is holomorphic with $F' \neq 0$. Thus f is locally one-to-one.

Proof of (ii) $\Rightarrow$ (iii) This is given by Proposition 2.2.

Proof of (iii) $\Rightarrow$ (i) Up to replacing U with RU, with R a suitable constant orthogonal matrix, we may assume without loss of generality that $\xi = e_1 = (1,0)$. Suppose, by contradiction, that $u = u_1$ possesses a geometric critical point x^0. Without loss of generality we can also assume $x^0 = u_1(x^0) = 0$. In a neighborhood of $x_0 = 0$, the level line $\{u_1 = 0\}$ is composed by $2(I+1)$ simple arcs departing from $x_0 = 0$, with $I > 0$. On each such arcs, u_2 is continuous and strictly monotone, either decreasing or increasing. We can select at least two of the above arcs, say β and γ, on which u_2 has the same type of monotonicity (say increasing) in the direction exiting from $x_0 = 0$. Consequently, u_2 is not one-to-one on $\beta \cup \gamma$, so contradicting the local injectivity of U.

Proof of (ii) $\Leftrightarrow$ (iv) Replacement of U by $\tilde{U}$, leads to replacement of f by $-if$. Therefore, the equivalence [(iii) $\Leftrightarrow$ (i)] implies [(ii) $\Leftrightarrow$ (iv)]. $\square$

The final theorem in this section provides an extension of the Radó Theorem (see [59], [35] and [17]) to σ-harmonic mappings.

Theorem 2.3 Let $\Omega \subset R^2$ be a bounded simply connected open set, whose boundary $\partial\Omega$ is a simple closed curve. Let $\Phi = (\phi_1, \phi_2)$, $\Phi : \partial\Omega \to R^2$ be a homeomorphism of $\partial\Omega$ onto a convex closed curve Γ and let D be the bounded convex domain bounded by Γ.

Let $\sigma \in L^\infty(\Omega, \mathcal{M}_K^s)$ and let $U \in W^{1,2}_{\mathrm{loc}}(\Omega, R^2) \cap C(\bar{\Omega}, R^2)$ be the σ-harmonic mapping whose components are the solutions of the Dirichlet problems

$$\begin{cases} \operatorname{div}(\sigma \nabla u_i) = 0 & \text{in} \quad \Omega \\ u_i = \phi_i & \text{on} \quad \partial\Omega \end{cases} , \quad i = 1,2 \quad .$$

Then U is a homeomorphism of $\bar{\Omega}$ onto $\bar{D}$.

Proof The proof follows the scheme initiated by Kneser ([35], see also [20]) for the case of harmonic mappings, and developed in [9] for σ-harmonic mappings with smooth σ. There is only one step which requires a new argument and it concerns the local univalence of U. We focus on this and omit the rest of the proof.

In view of Theorem 2.2, it suffices to show that, for every ξ, $|\xi|=1$, $u=\langle\xi,U\rangle$ has no geometric critical points.

Setting $\phi=\langle\xi,\Phi\rangle$, we have that u solves

$$\begin{cases} \operatorname{div}(\sigma\nabla u)=0 & \text{in} \quad \Omega \\ u=\phi & \text{on} \quad \partial\Omega \end{cases} .$$

By the convexity of Γ, for every ξ, $|\xi|=1$, ϕ is unimodal on $\partial\Omega$, that is $\partial\Omega$ may be split into two simple arcs on which ϕ is, alternatively, increasing and decreasing. By Theorem 2.7 in [7] (see also [4], Theorem 2.3 and Corollary 2.6) we have that the unimodality of ϕ implies that u has no geometric critical points in Ω. $\square$

3 The Jacobian of a σ-harmonic mapping and BMO

The main subject of this Section is the proof of part (ii) of Theorem A. This will follow by Theorem 3.1 as explained in Remark 3.1.

We recall that $\phi\in L^1_{\text{loc}}(R^2)$ belongs to $\text{BMO}(R^2)$ if

$$||\,\phi\,||_*=\sup_{Q\subset R^2}\left(\frac{1}{|\,Q\,|}\int_Q|\,\phi-\phi_Q\,|\right)<\infty$$

where Q is any square in R^2 and $\phi_Q=\frac{1}{|Q|}\int_Q\phi$. Given an open set $D\subset R^2$, $\text{BMO}(D)$ is defined as the space of functions $\phi\in L^1_{\text{loc}}(D)$ such that, when extended to zero outside D, belong to $\text{BMO}(R^2)$ and the norm $||\,\phi\,||_*$ is defined accordingly. We also recall that $(\text{BMO}(D),||\cdot||_*)$ is a Banach space. The main object of this section is the following.

Theorem 3.1 Let $U\in W^{1,2}_{\text{loc}}(\Omega,R^2)$ be a σ-harmonic mapping with $\sigma\in L^\infty(\Omega,\mathcal{M}^s_K)$ which is locally one-to-one and sense preserving. For every $D\subset\subset\Omega$ we have

$$\log(\det DU)\in\text{BMO}(D) \quad . \tag{3.1}$$

The proof of Theorem 3.1 needs a few preparatory results. It will presented at the end of this Section.

Remark 3.1 It is well known from the theory of Muckenhoupt weights ([19] and [26]) that, for a nonnegative function w, $\log w \in \mathrm{BMO}(R^2)$ if and only if there exists $\epsilon > 0$ and $C > 0$ such that $w^\epsilon \in A^2(R^2)$ that is

$$\left(\tfrac{1}{|Q|}\int_Q w^\epsilon\right)\left(\tfrac{1}{|Q|}\int_Q w^{-\epsilon}\right) \le C \quad , \tag{3.2}$$

for every square $Q \subset R^2$. Therefore, in the hypotheses of Theorem 3.1, for every $D \subset\subset \Omega$, there exists $\epsilon, C > 0$ such that

$$\left(\frac{1}{\mid Q \mid}\int_Q (\det DU)^\epsilon\right)\left(\frac{1}{\mid Q \mid}\int_Q (\det DU)^{-\epsilon}\right) \le C \tag{3.3}$$

for every square $Q \subset R^2$. In particular we deduce

$$\det DU > 0 \quad \text{a.e.} \quad \text{in} \quad \Omega \tag{3.4}$$

and also, setting U^A as defined in Theorem 2.1 with $\det A \neq 0$, one has

$$\det A \det DU^A > 0 \quad \text{a.e.} \quad \text{in} \quad R^2 \quad , \tag{3.5}$$

which, in conjunction with Theorem 2.1, completes the proof of Theorem A in the introduction.

We recall below two fundamental results (Theorems 3.2 and 3.3) which will be needed for a proof of Theorem 3.1.

Theorem 3.2 (Reimann) Let f be a quasiconformal mapping on the open set $D \subset R^2$, then for every $D' \subset\subset D$

$$\log\det Df \in \mathrm{BMO}(D')$$

Proof. See [60], Theorem 1 and Remark 2. □

Theorem 3.3 (Reimann) Let $f : D \to G$ be a quasiconformal mapping, $D, G \subset R^2$. For every $D' \subset\subset D$, there exists $C > 0$ such that

$$\| v \circ f \|_* \le C \| v \|_* \quad \text{for} \quad \text{every} \quad v \in \mathrm{BMO}(f(D')) \quad .$$

Proof. See [60], Theorem 4. □

The next theorem requires the notion of adjoint equation for a nondivergence elliptic operator. Let $G \subset R^2$ be an open set. Let $\{a_{ij}\} \in L^\infty(G, \mathcal{M}_K^s)$ and set

$$L = \textstyle\sum_{i,j=1}^2 a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} \quad .$$

We say that $v \in L^1_{\rm loc}(G)$ is a weak solution of the adjoint equation

$$L^* v = 0 \quad \text{in} \quad G \tag{3.6}$$

if

$$\textstyle\int_G vLu = 0 \quad \text{for} \quad \text{every} \quad u \in W_0^{2,2}(G) \quad .$$

Theorem 3.4 (Bauman-Fabes-Strook). For every $w \in L^2_{\rm loc}(G)$, $w \geq 0$, which is a weak solution of the adjoint equation (3.6) we have

$$\left(\frac{1}{\mid Q \mid} \int_Q w^2 \right)^{\frac{1}{2}} \leq C \frac{1}{\mid Q \mid} \int_Q w \tag{3.7}$$

for every square Q such that $2Q \subset G$. Here $C > 0$ only depends on K, the ellipticity of $\{a_{ij}\}$.

Proof This Theorem is a slight adaptation between [8], Theorem 3.3 and [23], Theorem 2.1. A proof is readily obtained by following the arguments in [23]. The only additional ingredient which is needed here, is the observation that, with no need of any smoothness assumption on the coefficients of L, for the special case when the dimension is two (which is of interest here), for any ball $B \subset G$ and any $f \in L^2(B)$ there exists and it is unique, the strong solution

$$u \in W^{2,2}(B) \cap W_0^{1,2}(B)$$

to the Dirichlet problem

$$\begin{cases} Lu = f & \text{in} \quad B, \\ u = 0 & \text{on} \quad \partial B, \end{cases}$$

see [64], Theorem 3. $\square$

Proof of Theorem 3.1 (Preparation) Let $U = (u_1, u_2)$ satisfy the hypotheses of Theorem 3.1 and let

$$f = u_1 + i\tilde{u}_1 \tag{3.8}$$

be the quasiregular mapping introduced in the previous section via (2.2)-(2.4). In view of Theorem 2.2, for every $x \in \Omega$, we can find a neighborhood D of x, $D \subset\subset \Omega$ such that $U|_D$ and $f|_D$ (i.e. the restrictions of U and f to D) are univalent. Therefore, for the proof of Theorem 3.1, it suffices to show that (3.1) holds for any sufficiently small $D \subset\subset \Omega$, such that $U|_D$ and $f|_D$ are univalent. We set

$$G = f|_D(D)$$

and $v : G \to R^2$ given by

$$V = U|_D \circ (f|_D)^{-1} \tag{3.9}$$

where, by definition $(f|_D)^{-1} : G \to D$. From now on, with a slight abuse of notation, we will drop the subscripts denoting restrictions to D. We have $DU = (DV \circ f)Df$, and hence

$$\log(\det DU) = \log(\det DV) \circ f + \log(\det Df) \quad . \tag{3.10}$$

In view of Theorems 3.2 and 3.3, the thesis will be proved as soon as we show that $\log(\det DV)$ belongs to BMO on compact subsets of G. The advantage in replacing U by V, lies in the observation that, in contrast with $\det DU$, $\det DV$ satisfies an equation of the type (3.6) for a suitable choice of the operator L^*.

In fact, letting v_1 and $\tilde{v}_1$ be the first component of V and its stream function respectively, we can compute

$$v_1(x) = u_1 \circ f^{-1}(x) = u_1 \circ (u_1 + i\tilde{u}_1)^{-1}(x) = x_1 \quad ,$$

$$\tilde{v}_1(x) = \tilde{u}_1 \circ f^{-1}(x) = \tilde{u}_1 \circ (u_1 + i\tilde{u}_1)^{-1}(x) = x_2 \quad . \tag{3.11}$$

Moreover, by definition (see (2.9) and (2.10)),

$$\nabla \tilde{v}_1 = J\tau \nabla v_1 \quad , \tag{3.12}$$

where

$$\tau = T_f \sigma = \frac{Df \sigma Df^T}{\det Df} \circ f^{-1} \quad . \tag{3.13}$$

Hence, using (3.11) and (3.12)

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \tau_{11} & \tau_{12} \\ \tau_{12} & \tau_{22} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad ,$$

that is

$$\tau = \begin{pmatrix} 1 & 0 \\ 0 & c \end{pmatrix} \tag{3.14}$$

where, by construction,

$$c = \det \tau = \det(\sigma \circ f^{-1}) \in L^\infty(G)$$

so that

$$K^{-2} \leq c \leq K^2 \tag{3.15}$$

almost everywhere.

Furthermore, by (3.11),

$$\det DV = \frac{\partial v_2}{\partial x_2} \in L^2(G) \quad .$$

Consequently, v_2 satisfies

$$\tfrac{\partial}{\partial x_1}[(v_2)_{x_1}] + \tfrac{\partial}{\partial x_2}[c(v_2)_{x_2}] = 0 \quad \text{weakly} \quad \text{in} \quad G \qquad .$$

Differentiating the equation above with respect to x_2, we see that $w = \det DV$ is a distributional solution of

$$(w)_{x_1x_1} + (cw)_{x_2x_2} = 0 \quad \text{in} \quad G \quad .$$

In other words, w is a weak solution of the adjoint equation

$$L^*w = 0 \quad \text{in} \quad G \tag{3.16}$$

where

$$L = \frac{\partial^2}{\partial x_1^2} + c\frac{\partial^2}{\partial x_2^2} \quad .$$

On use of (3.16) and (3.15) we may now apply Theorem 3.4.

We summarize the resulting statement below.

Proposition 3.5 For every cube Q such that $2Q \subset G$, we have

$$\left(\frac{1}{\mid Q \mid}\int_Q (\det DV)^2\right)^{\frac{1}{2}} \leq C\frac{1}{\mid Q \mid}\int_Q \det DV \quad ,$$

where $C > 0$ only depends on K.

Proof of Theorem 3.1 (Conclusion) A well known characterization of BMO in terms of the reverse Hölder inequality (see e.g. [26] Theorem 2.11 and Corollary 2.18), shows that Proposition 3.5 implies $\log(\det DV) \in BMO(G')$ for every $G' \subset\subset G$. Thus, possibly after replacing D with $D' = f^{-1}(G')$, we have, by (3.10) and Theorems 3.2 and 3.3 that $\log(\det DU) \in BMO(D)$. $\square$

4 σ-harmonic mappings and quasiconformality

We introduce the following class of conductivity matrices σ.
Definition 4.1 Let $\Omega \subset R^2$ be a connected open set. We write

$$\sigma \in \Sigma_{\mathrm{qc}}$$

if $\sigma \in L^\infty(\Omega, \mathcal{M}^s_K)$ for some $K \geq 1$ and, for every $x^0 \in \Omega$, there exists a neighborhood D, $x^0 \in D \subset\subset \Omega$ and a univalent σ-harmonic mapping $U : D \to R^2$ which is also quasiconformal.
Theorem 4.1 Given $\sigma \in L^\infty(\Omega, \mathcal{M}^s_K)$, the following conditions are equivalent
(i) $\sigma \in \Sigma_{\mathrm{qc}}$
(ii) for every $x^0 \in \Omega$, there exists a neighborhood D, with $x^0 \in D \subset\subset \Omega$, a quasiconformal mapping $\chi : D \to R^2$, and a mapping $\phi \in W^{1,\infty}(\chi(D); R^2)$ such that

$$T_\chi \sigma = JD\phi^T \quad \text{on} \quad \chi(D) \quad , \tag{4.1}$$

where the transformation T is defined according to (2.10).
(iii) for every $\Omega' \subset \Omega$ and for every locally univalent σ-harmonic mapping $U : \Omega' \to R^2$, U is quasiregular on the compact subsets of Ω'.
Proof of (i) $\Rightarrow$ (ii) Let us fix $x^0 \in \Omega$ and let D, $x^0 \in D \subset\subset \Omega$, $U : D \to R^2$ be such that U is σ-harmonic and quasiconformal. Let $G = U(D)$ and $\chi = U$. Let

$$\tau = T_\chi \sigma \tag{4.2}$$

(see (2.10)). We obtain $u_i \circ \chi^{-1} = x_i$, $i = 1, 2$ in G and also

$$\mathrm{div}(\tau \nabla(u_i \circ \chi^{-1})) = 0 \quad \text{in} \quad G, \quad i = 1, 2 \tag{4.3}$$

weakly. Therefore

$$\mathrm{div} \begin{pmatrix} \tau_{11} \\ \tau_{21} \end{pmatrix} = 0 \quad , \quad \mathrm{div} \begin{pmatrix} \tau_{12} \\ \tau_{22} \end{pmatrix} = 0 \quad \text{weakly} \quad \text{in} \quad G \quad . \tag{4.4}$$

That is there exist functions $p, q \in W^{1,\infty}(G, R)$ such that

$$\begin{pmatrix} \tau_{11} \\ \tau_{21} \end{pmatrix} = J \begin{pmatrix} p_{x_1} \\ p_{x_2} \end{pmatrix} \quad ; \quad \begin{pmatrix} \tau_{12} \\ \tau_{22} \end{pmatrix} = J \begin{pmatrix} q_{x_1} \\ q_{x_2} \end{pmatrix} \quad .$$

Hence setting $\phi = (p, q) : G \to R^2$,

$$\tau = JD\phi^T \tag{4.5}$$

which, in combination with (4.4), gives (4.3).
Proof of (ii) $\Rightarrow$ (iii) Let $U : \Omega' \to R^2$ be σ-harmonic, locally univalent and sense preserving. It suffices to show that for every $x^0 \in \Omega'$ we can find D, $x^0 \in D \subset\subset \Omega'$, such that $U|_D$ is quasiconformal. Let D be the neighborhood appearing in (ii), chosen small enough so that $U|_D$ is univalent. We shall show that, locally, $V = U \circ \chi^{-1}$ is quasiconformal.

Given $\tau = T_\chi \sigma$, we have that V is τ-harmonic and, by (4.1) $\tau = JD\phi^T$. Let us continue τ outside of G, by setting τ to be the identity and let us define

$$\tau_m = \rho_{h_m} * \tau$$

where ρ_h is the usual mollifying kernel and $\{h_m\}$ is a suitable infinitesimal sequence. Fixing $G' \subset\subset G$ and m sufficiently large

$$\tau_m = J(D(\rho_{h_m}) * \phi))^T \quad \text{in} \quad G' \quad .$$

Therefore, setting $v_{m,i}$, $i = 1, 2$ as the weak solution to

$$\begin{cases} \operatorname{div}(\tau_m \nabla v_{m,i}) = 0 & \text{in} \quad G' \\ v_{m,i} = v_i & \text{on} \quad \partial G', \quad i = 1, 2 \end{cases} ,$$

we also have

$$L_m v_{m,i} = 0 \quad \text{in} \quad G', \quad i = 1, 2 \quad ,$$

where

$$L_m = \sum_{i,j=1}^{2} (\tau_m)_{ij} \frac{\partial^2}{\partial x_i \partial x_j}$$

is the nondivergence elliptic operator associated to τ_m. Consequently, since the τ_m are elliptic uniformly with respect to m, for every $G'' \subset\subset G'$ we obtain ([64], Theorem 1, Lemma 10)

$$|| v_{m,i} ||_{C^{1,\alpha}(G'')} + || v_{m,i} ||_{W^{2,2}(G'')} \leq C || v_{m,i} ||_{W^{1,2}(G')} \leq C || v_i ||_{W^{1,2}(G)}$$

where α, C are independent of m. Hence, at the possible expenses of passing to a subsequence,

$$v_{m,i} \to v_i \quad \text{in} \quad C^{1,\alpha}(G'') \cap W^{2,2}_{\mathrm{loc}}(G'') \quad \text{as} \quad m \to \infty$$

and $v_i \in C^{1,\alpha}_{\mathrm{loc}}(G) \cap W^{2,2}_{\mathrm{loc}}(G)$, $i = 1, 2$ are strong solutions to

$$Lv_i = 0 \quad \text{in} \quad G$$

where

$$L = \sum_{i,j=1}^{2} (\tau)_{ij} \frac{\partial^2}{\partial x_i \partial x_j} \quad .$$

In particular, for any $G' \subset\subset G$, there exists $C > 0$ such that

$$| DV |^2 \leq C < \infty \quad \text{in} \quad G' \quad .$$

and, applying (i) of Theorem 2.2 to V, there exists $c > 0$ such that

$$\det DV \geq c > 0 \quad \text{in} \quad G' \quad .$$

Consequently V is quasiconformal in G'.

Proof of (iii) $\Rightarrow$ (i) It suffices to prove that for every $x^0 \in \Omega$, there exists a neighborhood D, $x^0 \in D \subset\subset \Omega$ and a univalent σ-harmonic mapping $U : D \to R^2$. This existence result may be achieved in several ways. (See [9]). We give a proof here. Let $D \subset\subset \Omega$ be a square centered at x^0. Up to a dilation, we may assume that its sides have length one. Let us continue the restriction of σ to D periodically. Theorem 2.1. provides us with the requested univalent σ-harmonic mapping U. $\square$

The following corollary of Theorem 4.1 provides some additional equivalent characterization of Σ_{qc}. We continue the numbering of the statements in Theorem 4.1.

Corollary 4.1 Given $\sigma \in L^\infty(\Omega, \mathcal{M}^s_K)$, σ belongs to Σ_{qc} if and only if one of the following two statements holds

(iv) for every $x^0 \in \Omega$ there exists a neighborhood D, $x^0 \in D \subset\subset \Omega$ and two quasiconformal mappings $\chi, \psi : D \to R^2$ such that

$$\sigma = J((D\chi)^{-1} D\psi)^T \quad \text{in} \quad D$$

(v)

$$\frac{\sigma}{\det \sigma} \in \Sigma_{\mathrm{qc}} \quad .$$

Proof of (iv) Algebraic manipulation shows that (iv) is equivalent to (ii) through the relation $\Psi = \phi \circ \chi$.

Proof of (v) In view of the equivalence (iii) $\Leftrightarrow$ (iv) in Theorem 2.2, it suffices to observe that a σ-harmonic mapping U is locally quasiregular if and only if $\tilde{U}$ is such. $\square$

Unfortunately, the above equivalent characterizations are not very useful in practice to check whether a given σ belongs to Σ_{qc}. The next theorem provides a sufficient condition (which is not necessary as shown in the Examples 5.2, 5.3 of Section 5, see also Remark 5.2) but it is easier to check.

Theorem 4.2 Let $\sigma \in L^\infty(\Omega, \mathcal{M}^s_K)$. If for every $D \subset\subset \Omega$, there exists $\alpha > 0$ such that $\det \sigma \in C^\alpha(D, R)$ then $\sigma \in \Sigma_{\mathrm{qc}}$.

Proof Let $\Omega' \subset \Omega$ and let $U = (u_1, u_2) : \Omega' \to R^2$ be a σ-harmonic, locally univalent and sense preserving mapping. Let $f = u_1 + i\tilde{u}_1$ and let $D \subset\subset \Omega'$ be any subdomain on which f is univalent. The calculation used to prove Proposition 3.5, shows that the components v_1 and v_2 of $V = U \circ f^{-1}$ are the weak solutions to

$$\operatorname{div}(\tau \nabla v_i) = 0 \quad \text{in} \quad G = f(D) \quad , \quad i = 1, 2$$

where τ is given by (3.13). By (3.14)

$$\tau = \begin{pmatrix} 1 & 0 \\ 0 & c \end{pmatrix}$$

with

$$c = \det \tau = (\det \sigma) \circ f^{-1} \quad .$$

Therefore $\tau \in C^\beta(G, \mathcal{M}^s)$ for some $\beta > 0$.

Consequently $V \in C^{1,\beta}_{\mathrm{loc}}(G)$ (see for instance [30]) and, as already pointed out in the proof of (ii) $\Rightarrow$ (iii) in Theorem 4.1, we deduce that V is locally quasiregular. Hence also $U = V \circ f$ is such and condition (iii) of Theorem 4.1 is satisfied. $\square$

Remark 4.1 The characterizations (ii), (iv) of Σ_{qc} are given in terms of representations of σ which have a non-symmetric structure. This observation suggests to extend the present results to the case of nonsymmetric matrices σ. In fact the theory of geometric critical points as outlined in Section 2 also applies to the non-symmetric case (see for instance [3] and consequently one obtains that various present results (for instance Theorem 2.1, Theorem 2.2 and Theorem 4.1) can be extended to the nonsymmetric case in straightforward manner. On the other hand, the generalization of Theorems 3.1 and 4.2 to the nonsymmetric case, do not appear equally easy and might deserve further investigation.

5 Examples

In this section we shall examine whether certain special conductivities belong to Σ_{qc}.
Example 5.1 Given $K > 1$, let us consider the following conductivity defined on R^2.

$$\sigma(x_1, x_2) = \begin{cases} K^{-1}I & \text{if} \quad x_1x_2 > 0 \quad , \\ KI & \text{if} \quad x_1x_2 < 0 \quad . \end{cases} \tag{5.1}$$

We shall show that $\sigma \notin \Sigma_{\mathrm{qc}}$, by constructing a σ-harmonic mapping $U = (u_1, u_2)$ which is not quasiconformal near the origin. Introducing polar coordinates $(x_1, x_2) = (r\cos\theta, r\sin\theta)$, we set

$$u_1(x_1, x_2) = \begin{cases} r^\lambda \cos[\lambda(\frac{\pi}{4} + \theta)] & , \quad -\frac{\pi}{4} \le \theta \le 0 \quad , \\ r^\lambda A \sin[\lambda(\frac{\pi}{4} - \theta)] & , \quad 0 \le \theta \le \frac{\pi}{4} \quad , \end{cases} \tag{5.2}$$

$$u_2(x_1, x_2) = \begin{cases} r^\mu \sin[\mu(\frac{\pi}{4} + \theta)] & , \quad -\frac{\pi}{4} \le \theta \le 0 \quad , \\ r^\mu B \cos[\mu(\frac{\pi}{4} - \theta)] & , \quad 0 \le \theta \le \frac{\pi}{4} \quad , \end{cases} \tag{5.3}$$

where λ, μ, A and B are positive constants which shall be chosen later on. By (5.2) and (5.3), we have defined u_1 and u_2 in the sector $-\pi/4 \le \theta \le \pi/4$. We extend them to all of the plane, by the following reflection rules

$$\begin{cases} u_1(r,\theta) = -u_1(r, \frac{\pi}{2} - \theta) \quad , \\ u_1(r,\theta) = u_1(r, -\frac{\pi}{2} - \theta) \quad , \end{cases} \tag{5.4}$$

$$\begin{cases} u_2(r,\theta) = u_2(r, \frac{\pi}{2} - \theta) \quad , \\ u_2(r,\theta) = -u_2(r, -\frac{\pi}{2} - \theta) \quad , \end{cases} \tag{5.5}$$

for every $r > 0$, $\theta \in R$.

One can verify that $U = (u_1, u_2)$ is σ-harmonic on all of the plane provided that the following transmission conditions are fulfilled:

$$\begin{cases} u_i(r, 0^+) = u_i(r, 0^-) & , \quad i = 1,2 \quad r > 0 \quad , \\ \frac{1}{K}\frac{\partial u_i}{\partial \theta}(r, 0^+) = K\frac{\partial u_i}{\partial \theta}(r, 0^-) & , \quad i = 1,2 \quad r > 0 \quad . \end{cases} \tag{5.6}$$

In fact, the corresponding transmission conditions on the remaining angles of discontinuity of σ, are automatically satisfied thanks to the reflection rules (5.4) and (5.5). Consequently we have that U is σ-harmonic if and only if λ, μ, A and B satisfy

$$\left\{ \begin{array}{lcll} \cos(\lambda\frac{\pi}{4}) & = & A\sin(\lambda\frac{\pi}{4}) & , \\ K\sin(\lambda\frac{\pi}{4}) & = & \frac{1}{K}A\cos(\lambda\frac{\pi}{4}) & , \\ \sin(\mu\frac{\pi}{4}) & = & B\cos(\mu\frac{\pi}{4}) & , \\ K\cos(\mu\frac{\pi}{4}) & = & \frac{1}{K}B\sin(\mu\frac{\pi}{4}) & . \end{array} \right. \tag{5.7}$$

A solution of (5.7) is found by choosing $A = B = K$ and λ, μ such that

$$\left\{ \begin{array}{ll} 0 < \lambda < 1 < \mu < 2 & , \\ \tan(\frac{\lambda\pi}{4}) = \frac{1}{K} & , \\ \tan(\frac{\mu\pi}{4}) = K & . \end{array} \right. \tag{5.8}$$

With the choices (5.8), we compute

$$\frac{\mathrm{tr}(DUDU^T)}{\det DU} \geq \mathrm{const.}r^{\lambda-\mu} \to +\infty \quad \text{as} \quad r \to 0^+ \quad .$$

Thus, we have shown that U is not quasiregular near the origin. Let us now prove that U is univalent in the unit ball $B_R(0)$, for any $R > 0$. In view of Theorem 2.3, it suffices to verify that the curve $\Phi : \partial B_R \to R^2$ given by the restriction of U to ∂B_R is a simple closed convex curve.

This verification can be performed by direct calculations, or else, it can be obtained, in a more general framework, by the arguments which shall be introduced in Proposition 5.1 later in this section.

Remark 5.1 Given σ as in (5.1), in view of (iii) of Theorem 4.1, if $\sigma' = \sigma$ in a neighborhood of the origin, then $\sigma' \notin \Sigma_{\mathrm{qc}}$.

In particular, we have that a periodic conductivity $\sigma_\sharp$ not belonging to Σ_{qc} is readily constructed by imposing $\sigma_\sharp(x_1, x_2) = \sigma(x_1, x_2)$ when $\mid x_1 \mid < 1/2\,, \ \mid x_2 \mid < 1/2$ and continuing it periodically on the rest of R^2.

Example 5.2 We consider here the family of conductivities σ of the form

$$\sigma(x_1, x_2) = \gamma(\theta)I$$

where $\gamma \in L^\infty(0, 2\pi)$, $K^{-1} \leq \gamma \leq K$ and we analyze the necessary and sufficient condition on γ to have $\sigma \in \Sigma_{\mathrm{qc}}$ near the origin.

Proposition 5.1 Let $0 = \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \cdots$ be the eigenvalues, repeated according to multiplicities of the periodic Sturm-Liouville problem

$$\begin{cases} (\gamma\phi')' + \lambda\gamma\phi = 0 & , \quad \theta \in (0, 2\pi) \quad , \\ \quad \phi(0^+) \;=\; \phi(2\pi^-) & , \\ (\gamma\phi')(0^+) = (\gamma\phi')(2\pi^-) & . \end{cases} \tag{5.9}$$

The conductivity $\sigma = \gamma(\theta)I \in \Sigma_{\mathrm{qc}}$ near the origin, if and only if

$$\lambda_1 = \lambda_2 \quad .$$

Proof Let $\phi_0, \phi_1, \cdots$, be a complete sequence of eigenfunctions corresponding to the eigenvalues $\lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \cdots$. We recall that, for the periodic eigenvalue problem (5.9), we have that $\lambda_0 = 0$ is a simple eigenvalue and that the multiplicity of the remaining eigenvalues is at most two, see [18] (Chapter 8, Theorem 3.1). Therefore we have $0 = \lambda_0 < \lambda_1 \leq \lambda_2 < \cdots$. We consider the σ-harmonic mapping

$$U = (r^{\sqrt{\lambda_1}}\phi_1(\theta) \ , \ r^{\sqrt{\lambda_2}}\phi_2(\theta))$$

and shall show that it is univalent near the origin. Moreover it is quasiconformal if and only if $\lambda_1 = \lambda_2$. It is well-known that ϕ_1 and ϕ_2 have exactly two zeroes on the unit circle and that the zeroes of ϕ_2 separate those of ϕ_1. The same holds for the absolutely continuous functions $\gamma\phi_1'$ and $\gamma\phi_2'$, see [18], Chapter 8, Theorem 1.1. It follows that $\Phi = U_{|\partial B_1}$ is a simple closed curve surrounding the origin. We shall prove that $U_{|B_1}$ is univalent, by applying Theorem 2.3 and showing that the curve Φ is convex. It is convenient to parameterize Φ by setting $t = t(\theta)$ where

$$t = \int_0^\theta \frac{d\tau}{\gamma(\tau)} \quad .$$

As a function of t, ϕ_1 and ϕ_2 are T-periodic where

$$T = \int_0^{2\pi} \frac{d\tau}{\gamma(\tau)} \quad ,$$

and we have

$$\ddot{\phi}_i + \lambda_i q \phi_i = 0 \quad , \quad i = 1, 2 \ , \tag{5.10}$$

where $q(t) = \gamma^2(\theta)$ and $t \to \theta(t)$ is regarded as the inverse function of $\theta \to t(\theta)$. We use the notation $\frac{d\phi}{dt} = \dot{\phi}$. We obtain that $\dot{\Phi}$ is absolutely continuous and T-periodic. Therefore, we can check the convexity of Φ by computing its curvature

$$\kappa = \frac{\ddot{\phi}_2\dot{\phi}_1 - \dot{\phi}_2\ddot{\phi}_1}{((\dot{\phi}_1)^2 + (\dot{\phi}_2)^2)^{\frac{3}{2}}}$$

The sign of κ is governed by the numerator $\alpha = (\ddot{\phi}_2\dot{\phi}_1 - \dot{\phi}_2\ddot{\phi}_1)$ of the latter expression. By (5.10)

$$\alpha = q(-\lambda_2\dot{\phi}_1\phi_2 + \lambda_1\dot{\phi}_2\phi_1) \quad .$$

Hence, if $\lambda_1 = \lambda_2$, then α/q is a constant multiple of the Wronskian determinant of the pair ϕ_1, ϕ_2, which is a nonzero constant.

If $\lambda_1 < \lambda_2$, then

$$\frac{d}{dt}\left(\frac{\alpha}{q}\right) = (\lambda_1 - \lambda_2)\dot{\phi}_1\dot{\phi}_2 \quad .$$

That is, the extremal points of α/q are exactly the critical points of ϕ_1 and ϕ_2. An inspection of the signs of α on such four points, based on the separation properties of ϕ_1 and ϕ_2, shows that α never changes its sign. Therefore Φ is convex.

We are left with the examination of the quasiconformality of U near the origin. We compute

$$\frac{\mathrm{tr}(DUDU^T)}{\det DU} = \frac{r^{(\sqrt{\lambda_1}-\sqrt{\lambda_2})}\gamma(\dot{\phi}_1^{\,2} + \phi_1^2) + r^{(\sqrt{\lambda_2}-\sqrt{\lambda_1})}\gamma(\dot{\phi}_2^{\,2} + \phi_2^2)}{\alpha}$$

and this quantity is bounded near the origin if and only if $\lambda_1 = \lambda_2$. $\square$

Remark 5.2 It is important to notice that both cases $\lambda_1 = \lambda_2$ and $\lambda_1 < \lambda_2$ can occur. For instance, if γ is $\pi/2$-periodic, then it can be checked that we can choose $\phi_2(\theta) = \phi_1(\theta - \pi/2)$ and hence $\lambda_1 = \lambda_2$. Conversely, Example 5.1, provides us with a case in which $\lambda_1 < \lambda_2$. Indeed, in that case, we have $\lambda_1 = \lambda^2 < \mu^2 = \lambda_2$.

Remark 5.3 By the theory developed by Uhlenbeck [68], we observe that among the set of coefficients γ, $K^{-1} \le \gamma \le K$ (endowed with some metric such as $L^2(0, 2\pi)$, or $C^l([0, 2\pi])$), we have that the property that $\lambda_1 < \lambda_2$ is generic. It is thus tempting to formulate the conjecture that, among the class of uniformly elliptic conductivities, the property $\sigma \in \Sigma_{\mathrm{qc}}$ is also generic. Let us stress that such a loose formulation of the conjecture is intentional, since the question of which topology should be placed on the class of conductivities appears as a central issue of the conjecture itself.

Remark 5.4 The analysis performed in Proposition 5.1 could be generalized also to certain anisotropic conductivities for which the separation of variables in polar coordinates is still possible. Namely we could consider as well conductivities with the following structure

$$\sigma = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \beta(\theta) & 0 \\ 0 & \gamma(\theta) \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

with $\beta, \gamma \in L^\infty(0, 2\pi)$, $K^{-1} \leq \beta, \gamma \leq K$.

In this case, the relevant Sturm-Liouville problem is

$$\begin{cases} (\gamma\phi')' + \lambda^2\beta\phi = 0 & , \quad \theta \in (0, 2\pi) \quad , \\ \phi(0^+) \;=\; \phi(2\pi^-) & , \\ (\gamma\phi')(0^+) = (\gamma\phi')(2\pi^-) & . \end{cases}$$

Example 5.3 By Theorem 4.2 we know that the Hölder regularity of $\det\sigma$ is a sufficient condition for $\sigma \in \Sigma_{\mathrm{qc}}$. The Example 5.2 (recall also Remark 5.2), shows that such condition is not necessary. Now we show that there exist conductivities in Σ_{qc} whose determinant has discontinuities of very general type.

We consider

$$\sigma(x_1, x_2) = \begin{pmatrix} h(x_1)k(x_2) & 0 \\ 0 & l(x_1)m(x_2) \end{pmatrix} \tag{5.11}$$

where $h, k, l, m \in L^\infty(R)$ satisfy $\sqrt{K^{-1}} \leq h, k, l, m \leq \sqrt{K}$. Let U be a univalent σ-harmonic mapping defined near the origin and let us set

$$\xi_1(x_1) = \int_0^{x_1} \frac{dt}{h(t)} \quad , \quad \xi_2(x_2) = \int_0^{x_2} \frac{dt}{m(t)} \quad .$$

It is readily verified (by arguments analogous to those used in the proof of Theorem 4.1) that, up to the bilipschitz mapping $\chi : (x_1, x_2) \to (\xi_1, \xi_2)$, the components of U satisfy almost everywhere

$$m(x_2)k(x_2)\frac{\partial^2 u_i}{\partial \xi_1^2} + l(x_1)h(x_1)\frac{\partial^2 u_i}{\partial \xi_2^2} = 0 \quad , \quad i = 1, 2 \quad .$$

Hence, in the ξ-coordinates, u_1 and u_2 satisfy a uniformly elliptic equation in nondivergence form. Consequently $\xi \to U$ is a $C^{1,\alpha}$ mapping and hence quasiconformal. Observe that, in this example,

$$T_\chi \sigma = \begin{pmatrix} \beta(\xi_2) & 0 \\ 0 & \gamma(\xi_1) \end{pmatrix}$$

where

$$\beta(\xi_2) = m(x_2)k(x_2) \quad , \quad \gamma(\xi_1) = l(x_1)h(x_1)$$

and therefore $T_\chi \sigma$ manifestly satisfies (ii) in Theorem 4.1.

6 Generating quasiregular mappings from σ-harmonic mappings

Given an open set Ω, $K \geq 1$ and $\sigma \in L^\infty(\Omega, \mathcal{M}_K^s)$, any σ-harmonic *function* u generates a K-quasiregular mapping $f = u + i\tilde{u}$ via conjugation with its stream function $\tilde{u}$ (see (2.3) and (2.4)). In this section we show that any σ-harmonic *mapping* U which is sense preserving (i.e. such that $\det DU \geq 0$ almost everywhere), generates a one parameter family of quasiregular mappings. In particular one element of this family is exactly K-quasiregular. We recall that typically, σ-harmonic mappings can be taken to be sense preserving by Remark 2.3. We will see later in the Example 6.1, that the classical way to generate K-quasiregular mappings from a given σ-harmonic function u is a special case of our construction.

We will also see in Section 7, that the family of mappings introduced in the present section, has a central role in questions concerning homogenized coefficients.

We denote by

$$\sigma_1(x) \leq \sigma_2(x) \tag{6.1}$$

the eigenvalues of σ, at the point $x \in \Omega$. Let $U = (u_1, u_2) \in W^{1,2}_{\text{loc}}(\Omega, R^2)$ be σ-harmonic and let $\tilde{U} = (\tilde{u_1}, \tilde{u_2})$ be the mapping whose components are the stream functions of u_1 and u_2 respectively (see (2.3)). For $\lambda > 0$ we define

$$\phi_{U,\lambda} = \lambda U + J\tilde{U} \tag{6.2}$$

and

$$k_\lambda(x) = \max\left(\frac{\sigma_2(x)}{\lambda}, \frac{\lambda}{\sigma_1(x)}\right) \quad , \quad K_\lambda = || \, k_\lambda(x) \, ||_{L^\infty(\Omega)} \quad . \tag{6.3}$$

We denote by $\mathcal{D}_{U,\lambda}(x)$ the dilatation quotient of $\phi_{U,\lambda}$, see (1.18).

Proposition 6.1 Let $\sigma \in L^\infty(\Omega, \mathcal{M}^s_K)$ and let $U \in W^{1,2}_{\rm loc}(\Omega, R^2)$ be σ-harmonic.

i) If $\det DU \geq 0$ almost everywhere in Ω, then, for any $\lambda > 0$, $\phi_{U,\lambda}$ is K_λ-quasiregular. More precisely

$$\mathcal{D}_{U,\lambda}(x) \leq k_\lambda(x) \quad \text{almost everywhere}$$

and hence

$$\| \mathcal{D}_{U,\lambda} \|_{L^\infty(\Omega)} \leq K_\lambda \quad . \tag{6.4}$$

In particular $\phi_{U,1}$ is K-quasiregular.

ii) If, in addition, U is locally univalent, then, for every $\lambda > 0$, $\phi_{U,\lambda}$ is locally univalent.

Proposition 6.2 Let $\sigma \in L^\infty_\sharp(R^2, \mathcal{M}^s_K)$ and let $A \in \mathcal{M}_+$. If $U^A \in W^{1,2}_{\sharp,A}(R^2, R^2)$ (see (1.2)) is a σ-harmonic mapping, then, for every $\lambda > 0$, $\phi_{U^A,\lambda} = \lambda U^A + J\tilde{U}^A$ is, in addition, an homeomorphism of R^2 onto itself and therefore a K_λ-quasiconformal mapping.

Remark 6.1 An immediate corollary of Propositions 6.1 and 6.2 is that under the assumptions of Proposition 6.2, $\phi_{U^A,1}$ is actually K-quasiconformal.

Remark 6.2 At the end of the Section, (Example 6.3) we show that one can construct an example in which $\| \mathcal{D}_{U,\lambda} \|_{L^\infty(\Omega)} = K > 1$ for every $\lambda > 0$. Therefore, in general, one cannot choose λ so that ϕ_λ is conformal and, moreover, the inequality in (6.4) cannot be improved for $\lambda = 1$.

In the sequel ${\rm tr}F$ and ${\rm Adj}F$ denote the trace and the adjugate of a matrix F respectively. We recall that, by definition, for two by two matrices ${\rm Adj}F = JFJ^T$ and, also that $F{\rm Adj}F^T = (\det F)\mathrm{I}$. The proof of Proposition 6.1 is based upon a simple algebraic fact.

Lemma 6.1 Let $A \in \mathcal{M}$ and $S \in \mathcal{M}^s$. For $\lambda \in R$, set

$$B = \lambda A + {\rm Adj}(AS) \quad . \tag{6.5}$$

Then

$$(\det S - \lambda^2)A = -\lambda B + {\rm Adj}(BS) \quad , \tag{6.6}$$

$$\det B = \det A(\det S + \lambda^2) + \lambda {\rm tr}(ASA^T) \quad , \tag{6.7}$$

$$(\det S - \lambda^2)^2 \det A = \det B(\det S + \lambda^2) - \lambda {\rm tr}(BSB^T) \quad . \tag{6.8}$$

Proof of Lemma 6.1 To verify (6.6), we compute its right hand side according to definition (6.5):

$$-\lambda B+\mathrm{Adj}(BS)=-\lambda^2 A-\lambda\mathrm{Adj}(AS)+\mathrm{Adj}(\lambda AS)+\mathrm{Adj}[\mathrm{Adj}(AS)S]=$$

$$-\lambda^2 A-\lambda\mathrm{Adj}(AS)+\lambda\mathrm{Adj}(AS)+(AS)\mathrm{Adj}S=-\lambda^2 A+A\det S \quad .$$

This proves (6.6). To prove (6.7) and (6.8), we use the identity

$$\det(F+G)=\det F+\det G+\mathrm{tr}[F(\mathrm{Adj}G)^T]$$

which holds for any pair of two by two matrices F and G. The calculation is omitted. $\square$

Proof of i) of Proposition 6.1 We fix $x\in\Omega$. The strategy is to apply Lemma 6.1 with the following choices: $S=\sigma$ and $A=DU$, with U a σ-harmonic and sense-preserving mapping. By (2.3) $D\tilde{U}=DU\sigma J^T$ and therefore $\mathrm{Adj}(DU\sigma)=JD\tilde{U}$ which, by (6.2), implies

$$D\phi_{U,\lambda}=\lambda DU+JD\tilde{U}=\lambda DU+\mathrm{Adj}(DU\sigma) \quad .$$

We set $B=D\phi_{U,\lambda}$, take $\lambda>0$ and then apply Lemma 6.1. Since U is sense preserving, (6.8) implies

$$\det D\phi_{U,\lambda}(\det\sigma+\lambda^2)-\lambda\mathrm{tr}(D\phi_{U,\lambda}\sigma D\phi_{U,\lambda}^T)\geq 0 \quad . \tag{6.9}$$

Set $0\leq a_\lambda^1\leq a_\lambda^2$ to be the singular values of $D\phi_{U,\lambda}$ (i.e. the eigenvalues of the square root of the matrix $D\phi_{U,\lambda}^T D\phi_{U,\lambda}$).

Using (1.19), we see that $\phi_{U,\lambda}$ is L-quasiregular if and only if

$$\mathcal{D}_{U,\lambda}=\frac{a_\lambda^2}{a_\lambda^1}\leq L \quad . \tag{6.10}$$

We write (6.9) in these new variables. We have,

$$a_\lambda^1 a_\lambda^2(\det\sigma+\lambda^2)=\det D\phi_{U,\lambda}(\det\sigma+\lambda^2)\geq$$

$$\lambda\mathrm{tr}(D\phi_{U,\lambda}\sigma D\phi_{U,\lambda}^T)\geq\lambda(\sigma_2(a_\lambda^1)^2+\sigma_1(a_\lambda^2)^2) \quad . \tag{6.11}$$

The latter inequality, follows by the so-called Von-Neumann theorem (saying that the $\mathrm{tr}(D\phi_{U,\lambda}\sigma D\phi_{U,\lambda}^T)$ is minimized when σ and $D\phi_{U,\lambda}^T D\phi_{U,\lambda}$ are simultaneously diagonal and their eigenvalues are ordered with opposite monotonicity). Recalling (6.10), we regard (6.11) as an inequality in the variable $\mathcal{D}_{U,\lambda}$ and we obtain

$$\mathcal{D}_{U,\lambda}(x) \le k_\lambda(x) \tag{6.12}$$

with k_λ defined in (6.3). This implies (6.4). Moreover, since $K^{-1} \le \sigma_1 \le \sigma_2 \le K$,

$$k_\lambda(x) \le K \max\left(\frac{1}{\lambda}, \lambda\right) \quad , \tag{6.13}$$

hence $\phi_{U,1}$ is K-quasiregular.

Proof of ii) of Proposition 6.1 Let us, temporarily, assume in addition that $\sigma \in C^\infty(\Omega, \mathcal{M}^s)$. By (i) of Theorem 2.2 we have that $\det DU > 0$ everywhere. Consequently by (6.8), we obtain

$$\det D\phi_{U,\lambda} = \det(\lambda DU + \mathrm{Adj}(DU\sigma)) =$$
$$(\det \sigma + \lambda^2) \det DU + \mathrm{tr}(DU\sigma DU^T) > 0 \text{ everywhere} \quad .$$

Now we remove the smoothness assumption. Let $\{\sigma_m\}$ be a sequence of mollified conductivities, such that, for any $p \ge 1$, $\sigma_m \to \sigma$ in L^p_{loc}. Fix $x^0 \in \Omega$. Since U is locally univalent, there exists $\rho > 0$ such that setting $G = U^{-1}(B(U(x^0), \rho))$, U is one to one on a neighborhood of G.

Let U_m be the σ_m-harmonic mapping in G such that $U_m|_{\partial G} = U|_{\partial G}$. By Theorem 2.3 each U_m is univalent in G, moreover, we have $U_m \to U$ in $W^{1,2}_{\mathrm{loc}}(G, R^2)$. Consequently

$$\phi_{U_m,\lambda} \to \phi_{U,\lambda} \text{ in } W^{1,2}_{\mathrm{loc}}(G, R^2) \quad .$$

By the above arguments, for every m, $\phi_{U_m,\lambda}$ is locally univalent. Using Lemma 2.1 and the equivalence (i) $\Leftrightarrow$ (iii) of Theorem 2.2, we obtain ii). $\square$

Proof of Proposition 6.2 By (3.5), $\forall A \in \mathcal{M}_+$, U^A is sense preserving. Hence, by Proposition 6.1, $\phi_{U^A,\lambda}$ is quasiregular. Therefore it is enough to show that $\forall A \in \mathcal{M}_+$, $\phi_{U^A,\lambda}$ is an homeomorphism. Let us outline the strategy of the proof first. We will show that there exists $F_\lambda \in \mathcal{M}_+$ and $C_\lambda \in L^\infty_\sharp(R^2, \mathcal{M}^s_{K_\lambda})$ such that $\phi_{U^A,\lambda} \in W^{1,2}_{\sharp,F_\lambda}(R^2, R^2)$ and it is a C_λ-harmonic mapping. Then, by Theorem 2.1, it is an homeomorphism.

We fix $A \in \mathcal{M}_+$ and $\lambda > 0$ and, for short, we write $\phi_\lambda = \phi_{U^A,\lambda}$. We set

$$G_\lambda(x) = \begin{cases} \frac{D\phi_\lambda^T(x) D\phi_\lambda(x)}{\det D\phi_\lambda(x)} & \text{if } \det D\phi_\lambda(x) \ne 0 \\ I & \text{if } \det D\phi_\lambda(x) = 0 \quad . \end{cases} \tag{6.14}$$

By Proposition 6.1, G_λ defines a measurable field of matrices which is symmetric with $\det G_\lambda = 1$ almost everywhere. Moreover, by (6.4), one has

$$K_\lambda^{-1} I \le G_\lambda \le K_\lambda I$$

almost everywhere. In other words $G_\lambda \in L^\infty_\sharp(R^2, \mathcal{M}^s_{K_\lambda})$. Note, for future reference that this clearly implies

$$G_\lambda^{-1} \in L^\infty_\sharp(R^2, \mathcal{M}^s_{K_\lambda}) \quad . \tag{6.15}$$

By construction ϕ_λ satisfies the Beltrami equation

$$D\phi_\lambda^T D\phi_\lambda = G_\lambda \det D\phi_\lambda \tag{6.16}$$

which we rewrite as

$$G_\lambda^{-1} D\phi_\lambda^T = \mathrm{Adj} D\phi_\lambda^T$$

which implies

$$\mathrm{Div}(G_\lambda^{-1} D\phi_\lambda^T) = 0 \quad . \tag{6.17}$$

Observe that we obtained the same type of equation through a different calculation, in § 1.3, see in particular (1.22). So far we have never used the fact that $U^A \in W^{1,2}_{\sharp,A}$. Now we need this assumption. Indeed, we observe that $\tilde{U^A}$ can be decomposed as a sum of an affine term and a periodic one. More precisely, one has

$$\tilde{U^A} \in W^{1,2}_{\sharp,B} \quad , \quad B = J^T \int_Q \mathrm{Adj}(DU^A \sigma) \quad .$$

In the language introduced in Section 1.1, we have

$$A\sigma_{\mathrm{hom}} = JB \quad , \tag{6.18}$$

where σ_{hom} is the homogenized conductivity see (1.6). The only property that is needed here is that σ_{hom} is a constant, symmetric and positive definite matrix.

Therefore setting $C_\lambda = G_\lambda^{-1}$ and using (6.15) and (6.18), one has that ϕ_λ is C_λ-harmonic and also $\phi_\lambda - F_\lambda x \in W^{1,2}_\sharp(R^2, R^2)$, where $F_\lambda = \lambda A + JB$. In other words, ϕ_λ is a solution to (1.5) when σ and A are replaced by C_λ and F_λ respectively.

In view of Theorem 2.1, to conclude the proof we need to show that $F_\lambda \in \mathcal{M}_+$. Indeed, using (6.7) and the formulas below it, one easily checks that

$$\det F_\lambda = (\lambda^2 + \det \sigma_{\mathrm{hom}}) \det A + \lambda \mathrm{tr}(A\sigma_{\mathrm{hom}} A^T) > 0$$

because both terms in the sum are such. $\square$

We conclude this section with three examples.

Example 6.1 The classical way to construct the quasiregular map $f = u + i\tilde{u}$ from a given σ-harmonic function u and the new way of generating quasiregular mappings explained in this section are actually related in a simple fashion. Indeed, given the σ-harmonic function u, set $U = (u, 0)$ and $\phi_1 = U + J\tilde{U}$. Then it is easy to see that $\phi_{U,1} = (u, \tilde{u})$ so that $\phi_{U,1} = f$ up to the identification between R^2 and C. Clearly, U satisfies $\det DU \geq 0$, in fact $\det DU = 0$ almost everywhere. Therefore, the classical way can be seen as a very special case of the new one.

Example 6.2 If σ has constant determinant ($\det\sigma = d^2$) almost everywhere on a measurable set E, by (6.9) for any σ-harmonic mapping U,

$$\det D\phi_{U,\lambda}(\det\sigma + \lambda^2) - \lambda\mathrm{tr}(D\phi_{U,\lambda}\sigma D\phi^T_{U,\lambda}) \geq 0 \quad , \quad \text{almost everywhere in } E \quad .$$

In particular, since $\forall B \in \mathcal{M}$

$$2\sqrt{\det\sigma}\det B - \mathrm{tr}(B\sigma B^T) \leq 0 \quad ,$$

one has that

$$2d\det D\phi_{U,d} = \mathrm{tr}(D\phi_{U,d}\sigma D\phi^T_{U,d}) \quad , \quad \text{almost everywhere in } E \quad .$$

Therefore $\phi_{U,d}$ is K-quasiregular on E and satisfies the Beltrami equation

$$D\phi^T_{U,d}D\phi_{U,d} = \left(\frac{\sigma}{\sqrt{\det\sigma}}\right)^{-1}\det D\phi_{U,d} \quad , \quad \text{almost everywhere in } E \quad .$$

If σ is the identity on E, $\phi_{U,d} = \phi_{U,1}$ is therefore holomorphic in the interior of E.

Example 6.3 This example justifies Remark 6.2. Set Ω to be a ball centered at the origin and of radius one. Define for $x \neq 0$,

$$\sigma(x) = K^{-1}n\otimes n + Kt\otimes t \quad , \quad n = \frac{x}{\mid x\mid} \quad , \quad t = Jn \quad ; \quad U = x\mid x\mid^{K-1} \quad .$$

Notice that such a conductivity σ can be viewed as special case of those treated in Remark 5.4.

One can check that U is σ-harmonic in Ω and it is univalent. A calculation shows that

$$DU(x) = \mid x\mid^{K-1}(Kn\otimes n + t\otimes n) \quad , \quad DU(x)\sigma(x) = \mid x\mid^{K-1}(n\otimes n + Kt\otimes t)$$

and hence

$$D\phi_{U,\lambda}(x) = (\lambda+1) \mid x \mid^{K-1} (Kn \otimes n + t \otimes t) \quad .$$

Therefore, it is readily verified that the dilatation is K for any λ.

7 Applications to homogenization

Theorem B stated in the introduction is contained in the following Theorem 7.1. Given $\sigma \in L^\infty_\sharp(R^2, \mathcal{M}^s_K)$, we set

$$d_m = \operatorname*{ess\,inf}_{x\in Q} \sqrt{\det \sigma} \quad . \tag{7.1}$$

Theorem 7.1 Let $K > 1$ be given, if $\sigma \in L^\infty_\sharp(R^2, \mathcal{M}^s_K)$, then the homogenized conductivity σ_{hom} satisfies

$$\det \sigma_{\text{hom}} \geq d_m^2 \tag{7.2}$$

and, for every $\lambda \in (-d_m, d_m)$ and every $A \in \mathcal{M}$

$$\frac{\operatorname{tr}(A\sigma_{\text{hom}}A^T) - 2\lambda \det A}{\det \sigma_{\text{hom}} - \lambda^2} =$$

$$\inf_{U\in W^{1,2}_{\sharp,A}(R^2;R^2)} \frac{1}{\mid Q \mid} \int_Q \frac{\operatorname{tr}[DU(y)\sigma(y)DU(y)^T] - 2\lambda \det DU(y)}{\det \sigma(y) - \lambda^2} dy \quad . \tag{7.3}$$

Moreover the minimizer of (7.3) is uniquely determined up to an additive constant vector and is given by

$$\phi_{U^{B_\lambda},\lambda} = \lambda U^{B_\lambda} + J\tilde{U}^{B_\lambda} \quad , \tag{7.4}$$

where U^{B_λ} is the solution to (1.5) when A is replaced with

$$B_\lambda = \frac{-\lambda A + \operatorname{Adj}(A\sigma_{\text{hom}})}{\det \sigma_{\text{hom}} - \lambda^2} \tag{7.5}$$

Remark 7.1 The variational principle (7.3) is similar to that proved in [7].

We show that Theorem 7.1 carries more information in the next Corollary 7.1. To state the result, we need to recall the notation introduced in (1.10), (1.11). We also recall that by (1.12), for every $\lambda \in [0, s]$,

$$m(\sigma_{\text{hom}}, \lambda) \neq \emptyset \quad . \tag{7.6}$$

Corollary 7.1 Under the same assumptions as in Theorem 7.1,

$$\det \sigma_{\text{hom}} > \lambda^2 \tag{7.7}$$

and for every $\lambda \in (0, d_m)$ and every $A \in m(\sigma_{\text{hom}}, \lambda)$, the homogenized conductivity satisfies

$$\frac{\operatorname{tr}(A\sigma_{\text{hom}}A^T) - 2\lambda \det A}{\det \sigma_{\text{hom}} - \lambda^2} =$$

$$\inf_{U \in W(\sigma,\lambda,A)} \frac{1}{\mid Q \mid} \int_Q \frac{\operatorname{tr}[DU(y)\sigma(y)DU(y)^T] - 2\lambda \det DU(y)}{\det \sigma(y) - \lambda^2} dy \quad . \tag{7.8}$$

Remark 7.2 We shall show (Lemma 7.4) that for any choice of $\sigma \in L^\infty_\sharp(R^2, \mathcal{M}^s_K)$, $\lambda \in [0, d_m]$ and $A \in m(\sigma_{\text{hom}}, \lambda)$, the mapping $\phi_{U^{B_\lambda},\lambda}$ defined in (7.4) belongs to $W(\sigma, \lambda, A)$. In particular, the latter space is never empty.

Remark 7.3 The main point of Corollary 7.1 is that if $A \in m(\sigma_{\text{hom}}, \lambda)$, then the minimizers $\phi_{U^{B_\lambda},\lambda}$ of (7.8) are automatically K_λ quasiconformal. (See (6.4)). Indeed, one can check that if $A \in m(\sigma_{\text{hom}}, \lambda)$, then $\det B_\lambda > 0$ and hence, by (3.5), $\det DU^{B_\lambda} > 0$ almost everywhere. Therefore, by Propositions 6.1 and 6.2, $\phi_{U^{B_\lambda},\lambda}$ is K_λ-quasiconformal.

We review below the definition and the basic properties of the orthogonal splitting of two by two matrices into their conformal and anticonformal parts. Set

$$M_+ = \tfrac{1}{2}(M + \text{Adj}M) \quad M_- = \tfrac{1}{2}(M - \text{Adj}M) \tag{7.9}$$

and write the usual definitions

$$\mathcal{H}^+ \equiv \{M \in \mathcal{M} \ : \ M - \text{Adj}M = 0\} \,, \mathcal{H}^- \equiv \{M \in \mathcal{M} \ : \ M + \text{Adj}M = 0\} \,. \tag{7.10}$$

Then one easily checks that $\forall M \in \mathcal{M}$, $M = M_+ + M_-$ and that the decomposition is unique and orthogonal in the sense that

$$A \in \mathcal{H}^+ \ , \ \ B \in \mathcal{H}^- \ \ \Rightarrow \ \ \operatorname{tr}(AB^T) = 0 \ \ . \tag{7.11}$$

We write $\forall M \in \mathcal{M}$, $\mid M \mid^2 = \operatorname{tr}(MM^T)$. Then $\forall M \in \mathcal{M}$

$$\begin{aligned} &\mid M \mid^2 = \operatorname{tr}(MM^T) = \operatorname{tr}(M_+M_+^T) + \operatorname{tr}(M_-M_-^T) = \mid M_+ \mid^2 + \mid M_- \mid^2 \quad , \\ &2\det M = \operatorname{tr}(M_+M_+^T) - \operatorname{tr}(M_-M_-^T) = \mid M_+ \mid^2 - \mid M_- \mid^2 \quad , \\ &\text{Adj}M = M_+ - M_- \quad . \end{aligned} \tag{7.12}$$

We state now four lemmas needed to prove the results of the present section. The first two are essentially of algebraic nature.

Lemma 7.1 For $F, H \in \mathcal{M}$, $\lambda \in R$ and $S \in \mathcal{M}^s$ we define

$$f(F,S,\lambda) = \mathrm{tr}(FSF^T) + 2\lambda \det F \tag{7.13}$$

and

$$f^*(H,S,\lambda) = \sup_{F\in\mathcal{M}} [2\mathrm{tr}(HF^T) - f(F,S,\lambda)] \quad . \tag{7.14}$$

Then, as a function of the first variable, f is strictly convex if and only if $\lambda^2 < \det S$, convex but not strictly convex if and only if $\lambda^2 = \det S$.

Moreover, the explicit expression of f^* is given by

$$f^*(H,S,\lambda) =$$

$$\begin{cases} \frac{\mathrm{tr}(H\mathrm{Adj}SH^T) - 2\lambda \det H}{\det S - \lambda^2} & \text{if} \quad |\lambda| < \sqrt{\det S} \\ \frac{\mathrm{tr}[H\mathrm{Adj}SH^T]}{2\det S} & \text{if} \quad \lambda = \sqrt{\det S} \quad \text{and} \quad HS^{-\frac{1}{2}} \in \mathcal{H}^+ \\ \frac{\mathrm{tr}[H\mathrm{Adj}SH^T]}{2\det S} & \text{if} \quad \lambda = -\sqrt{\det S} \quad \text{and} \quad HS^{-\frac{1}{2}} \in \mathcal{H}^- \\ +\infty & \text{otherwise} \quad . \end{cases} \tag{7.15}$$

Lemma 7.2 For $F \in \mathcal{M}$, $\lambda \in [0, \sqrt{\det S}]$, $S \in \mathcal{M}^s$ and $H \in m(S,\lambda)$ (see (1.10)), we define

$$f^{*,+}(H,S,\lambda) = \sup_{F\in\mathcal{M}_+} [2\mathrm{tr}(HF^T) - f(F,S,\lambda)] \quad , \tag{7.16}$$

where $f(F,S,\lambda)$ is defined in (7.13). Then

$$f^{*,+}(H,S,\lambda) = \begin{cases} \frac{\mathrm{tr}(H\mathrm{Adj}SH^T) - 2\lambda \det H}{\det S - \lambda^2} & \text{if} \quad |\lambda| < \sqrt{\det S} \\ \frac{\mathrm{tr}[H\mathrm{Adj}SH^T]}{2\det S} & \text{if} \quad \lambda = \sqrt{\det S} \quad \text{and} \quad HS^{-\frac{1}{2}} \in \mathcal{H}^+ \ . \end{cases} \tag{7.17}$$

The next two lemmas use part (ii) of Theorem A, namely (3.5).

Lemma 7.3 Under the assumptions of Theorem 7.1, $\det \sigma_{\mathrm{hom}} \geq d_m^2$. Moreover setting Q_m as in (1.13) we have

$$|Q_m| < 1 \Rightarrow \sqrt{\det \sigma_{\mathrm{hom}}} > d_m \quad , \tag{7.18}$$

$$|Q_m| = 1 \Rightarrow \sqrt{\det \sigma_{\mathrm{hom}}} = d_m \quad .$$

To state the next result, recall (7.4) and (1.11).

Lemma 7.4 Let $\sigma \in L^\infty_\sharp(R^2, \mathcal{M}^s_K)$, $\lambda \in [0, d_m]$ and $A \in m(\sigma_{\rm hom}, \lambda)$ be given and let $\phi_{U^{B_\lambda},\lambda}$ be defined by (7.4). Then: i) $\phi_{U^{B_\lambda},\lambda} \in W(\sigma, \lambda, A)$ and ii)

$$\frac{1}{\mid Q \mid}\int_Q \frac{\mathrm{tr}(D\phi_{U^{B_\lambda},\lambda}(y)\sigma(y)D\phi_{U^{B_\lambda},\lambda}(y)^T) - 2\lambda \det D\phi_{U^{B_\lambda},\lambda}(y)}{\det\sigma - \lambda^2}dy =$$

$$\mathrm{tr}(B_\lambda \sigma_{\rm hom} B_\lambda^T) + 2\lambda \det B_\lambda = \frac{\mathrm{tr}(A\sigma_{\rm hom}A^T) - 2\lambda \det A}{\det \sigma_{\rm hom} - \lambda^2} \quad . \tag{7.19}$$

In particular, for any choice of $\sigma \in L^\infty_\sharp(R^2, \mathcal{M}^s_K)$, $\lambda \in [0, d_m]$ and $A \in m(\sigma_{\rm hom}, \lambda)$,

$$W(\sigma, \lambda, A) \neq \emptyset \quad . \tag{7.20}$$

The above lemmas will be proved later.

Proof of Theorem 7.1, given Lemma 7.1, 7.3 and 7.4 By Lemma 7.3, (7.2) holds. A possible proof of (7.3) follows the argument in [7]. Here we give a different and conceptually more direct proof. First of all we note that, for fixed λ, the variational principle (7.3) has a unique minimizer up to an additive constant. Indeed, by Lemma 7.1 applied for $S = \mathrm{Adj}\sigma$, $\forall \lambda \in (-d_m, d_m)$ the function

$$A \to \frac{\mathrm{tr}(A\sigma(y)A^T) - 2\lambda \det A}{\det\sigma(y) - \lambda^2}$$

is the polar of a strictly convex function and therefore itself a convex function.

Next we show that the minimizers satisfy (7.4). Indeed the minimizers satisfy the following Euler-Lagrange equations

$$\begin{cases} \mathrm{Div}\left[\frac{[D\Psi_A(y)\sigma(y)]^T - \lambda \mathrm{Adj}[D\Psi_A(y)]^T}{\det\sigma(y) - \lambda^2}\right] = 0 \quad \text{in} \quad R^2 \\ \Psi_A \in W^{1,2}_{\sharp,A}(R^2, R^2) \qquad , \end{cases} \tag{7.21}$$

in the weak sense. In two dimensions, (7.21) is equivalent to the following conditions. There exists $B_0 \in \mathcal{M}$ and there exist $U_\sharp \in W^{1,2}_\sharp(R^2, R^2)$ such that

$$\frac{D\Psi_A(y)\sigma(y) - \lambda \mathrm{Adj}(D\Psi_A(y))}{\det\sigma(y) - \lambda^2} = \mathrm{Adj}(DU_\sharp + B_0) \quad , \quad \text{almost everywhere} \tag{7.22}$$

or equivalently, setting $U_{B_0} = U_\sharp + B_0 x$ and taking Adj to both sides of (7.22), we have

$$DU_{B_0}(y) = \frac{-\lambda D\Psi_A(y) + \mathrm{Adj}(D\Psi_A(y)\sigma(y))}{\det\sigma(y) - \lambda^2} \quad , \quad \text{almost everywhere} \quad . \tag{7.23}$$

Using Lemma 6.1, and the definition of $D\Psi_A$, (7.23) can be written as follows. There exists $B_0 \in \mathcal{M}$ and there exist $U_\sharp \in W^{1,2}_\sharp(R^2, R^2)$ such that $U_{B_0} = U_\sharp + B_0 x$ satisfies

$$\lambda DU_{B_0}(x) + \mathrm{Adj}[DU_{B_0}\sigma(x)] = D\Psi_A(x) \quad . \tag{7.24}$$

One can solve (7.24) with respect to U_{B_0} if and only if the left hand side is the differential of a vector field in the suitable space i.e. if and only if

$$\begin{cases} \mathrm{Div}[\sigma(x)DU^T_{B_0}(x)] = 0 & \text{in} \quad R^2 \\ U_{B_0} \in W^{1,2}_{\sharp,B_0}(R^2, R^2) & \end{cases} \tag{7.25}$$

in the weak sense. Note that (7.25) uniquely determines U_{B_0} in terms of B_0 up to an additive constant vector. From (7.25) and the definition (1.5), we see that

$$U_{B_0} = U^{B_0} \quad .$$

Using (7.24) and (7.25) we conclude

$$\Psi_A(x) = \lambda U^{B_0} + J\tilde{U}^{B_0} + \xi \quad , \tag{7.26}$$

for some constant vector ξ. In view of (7.26) and recalling (7.4) and (7.5), to prove the second part of Theorem 7.1, we are left with showing that $B_0 = B_\lambda$ as defined in (7.5). Indeed, integration of both sides of (7.24) yields

$$A = \int_Q \{\lambda DU^{B_0} + \mathrm{Adj}[DU^{B_0}\sigma(x)]\}dx = \lambda B_0 + \mathrm{Adj}\int_Q [DU^{B_0}\sigma(x)]dx \quad .$$

The latter combined with (1.6) yields

$$A = \lambda B_0 + \mathrm{Adj}(B_0\sigma_{\mathrm{hom}}) \quad . \tag{7.27}$$

We now solve for B_0 in terms of A in (7.27). By (6.6), $B_0 = B_\lambda$ (as defined in (7.5)). Finally, evaluation of the right hand side of (7.3) via Lemma 7.4, gives equality with its left hand side. □

Proof of Corollary 7.1, given Lemma 7.1, 7.3 and 7.4 Lemma 7.3 implies (7.7). By assumption, $A \in m(\sigma_{\text{hom}}, \lambda)$ which, by Proposition 6.1 and (7.5), implies that $\det B_\lambda > 0$. By Theorem 7.1, the right hand side of (7.3) is minimized by a mapping satisfying (7.4). By Proposition 6.2 and the fact that $\det B_\lambda > 0$, such a mapping is quasiconformal and, by Proposition 6.1, it belongs to $W(\sigma, \lambda, A)$. $\square$

The next Corollary complements Theorem C (see the Introduction), with the analysis of the cases $\mid Q_m \mid = 0, 1$ not considered in the hypothesis (1.14). We refer to the notation introduced in (1.13).

Corollary 7.2

(i) If $\mid Q_m \mid = 0$, then (1.15) holds and, for every $A \in m(\sigma_{\text{hom}}, d_m)$,

$$\frac{\operatorname{tr}(A\sigma_{\text{hom}}A^T) - 2d_m \det A}{\det \sigma_{\text{hom}} - d_m^2} =$$

$$\inf_{U \in W(\sigma, d_m, A)} \int_Q \frac{\operatorname{tr}[DU(y)\sigma(y)DU(y)^T] - 2d_m \det DU(y)}{\det \sigma(y) - d_m^2} dy \quad .$$

(ii) If $\mid Q_m \mid = 1$, then $\det \sigma_{\text{hom}} = d_m^2$ and for every $A \in m(\sigma_{\text{hom}}, d_m)$,

$$\frac{\operatorname{tr}(A\sigma_{\text{hom}}A^T)}{2\det \sigma_{\text{hom}}} = \inf_{U \in W(\sigma, d_m, A)} \int_Q \frac{\operatorname{tr}[DU(y)\sigma(y)DU(y)^T]}{2\det \sigma} \, dy \quad .$$

Proof of Theorem C, given Lemma 7.1, 7.2, 7.3 and 7.4. By Lemma 7.3 $\det \sigma_{\text{hom}} > d_m^2$. By Lemma 7.4, the set $W(\sigma, d_m, A)$ is non empty (so that (1.16) makes sense) and the left hand side of (1.16) is smaller than or equal to its right hand side. Hence it is enough to prove that the left hand side of (1.16) is greater than or equal to its right hand side. For $F \in \mathcal{M}$ we set

$$g(F) = \operatorname{tr}(F\sigma F^T) + 2d_m \det F \tag{7.28}$$

and then define

$$g^{*,+}(H) = \sup_{F \in \mathcal{M}_+} [2\operatorname{tr}(HF^T) - g(F)] \quad , \quad H \in \mathcal{M} \quad . \tag{7.29}$$

By (7.29),

$$\forall F \in \mathcal{M}_+ \quad , \quad \forall H \in \mathcal{M} \ , \ g(F) + g^{*,+}(H) \geq 2\operatorname{tr}(HF^T) \quad . \tag{7.30}$$

The notation reflects the obvious resemblance between $g^{*,+}$ and the polar of the function g denoted as usual by g^*.

Choose two arbitrary constant matrices $A \in \mathcal{M}$ and $B \in \mathcal{M}_+$ and set $F = DU$, $H = \mathrm{Adj}DV$ with $U \in W^{1,2}_{\sharp,B}(R^2,R^2)$ (see (1.2)) and satisfying $\det DU > 0$ almost everywhere and $V \in W^{1,2}_{\sharp,A}(R^2,R^2)$. Then use (7.30) and integrate over Q. We obtain

$$\int_Q g(DU)dx \geq 2\int_Q \mathrm{tr}(\mathrm{Adj}(DV)DU^T)dx - \int_Q g^{*,+}(\mathrm{Adj}DV)dx =$$

$$2\mathrm{tr}(\mathrm{Adj}(A)B^T) - \int_Q g^{*,+}(\mathrm{Adj}DV)dx \quad . \tag{7.31}$$

The last equality follows integrating by parts. Now we apply Theorem 3.1 which implies

$$\forall B \in \mathcal{M}_+ \ , \quad \inf_{\{U \in W^{1,2}_{\sharp,B}(R^2,R^2)\ :\ \det DU>0 \text{ a.e. } x\in Q\}} \int_Q g(DU) = \mathrm{tr}(B\sigma_{\mathrm{hom}}B^T) + 2d_m \det B \ . \tag{7.32}$$

Using (7.31) and (7.32) we obtain

$$\forall A \in \mathcal{M} \quad , \quad \forall B \in \mathcal{M}_+ \quad , \quad \forall\ V \in W^{1,2}_{\sharp,A}(R^2,R^2) \quad ,$$

$$2\mathrm{tr}(\mathrm{Adj}(A)B^T) - \mathrm{tr}(B\sigma_{\mathrm{hom}}B^T) - 2d_m \det B \leq \int_Q g^{*,+}(\mathrm{Adj}DV)dx$$

which implies

$$\forall A \in \mathcal{M} \quad , \quad \sup_{B\in\mathcal{M}_+} 2\mathrm{tr}(\mathrm{Adj}(A)B^T) - \mathrm{tr}(B\sigma_{\mathrm{hom}}B^T) - 2d_m \det B \leq$$

$$\inf_{V\in W^{1,2}_{\sharp,A}(R^2,R^2)} \int_Q g^{*,+}(\mathrm{Adj}DV)dx \quad . \tag{7.33}$$

By (1.13), (1.14), Lemma 7.3 and Lemma 7.1, the function $B \to \mathrm{tr}(B\sigma_{\mathrm{hom}}B^T) + 2d_m \det B$ is strictly convex and therefore the maximum over $B \in \mathcal{M}$ in the left hand side of (7.33) is given by the value of the polar function of $\mathrm{tr}(B\sigma_{\mathrm{hom}}B^T) + 2d_m \det B$. Using Lemma 6.1, one checks that the optimal $B \in \mathcal{M}_+ \Leftrightarrow A \in m(\sigma_{\mathrm{hom}}, d_m)$. Therefore in (7.33) the sup over $B \in \mathcal{M}_+$ coincides with that taken over $B \in \mathcal{M}$.

The latter is easily calculated using again Lemma 7.1. The previous remarks and (7.33) imply

$$\frac{\mathrm{tr}(\mathrm{Adj}(A)\mathrm{Adj}(\sigma_{\mathrm{hom}})\mathrm{Adj}(A^T)) - 2d_m \det(\mathrm{Adj}A)}{\det \sigma_{\mathrm{hom}} - d_m^2} =$$

$$\frac{\mathrm{tr}(A\sigma_{\mathrm{hom}}A^T) - 2d_m \det A}{\det \sigma_{\mathrm{hom}} - d_m^2} \leq \inf_{V \in W^{1,2}_{\sharp,A}(R^2,R^2)} \int_Q g^{*,+}(\mathrm{Adj}DV)dx \quad . \tag{7.34}$$

To conclude the proof, we need to check that the right hand sides of (7.34) and (1.16) are the same. This follows by Lemma 7.2 upon noting that, by (7.17), (7.28) and (7.29) one has $g^{*,+}(H) = f^{*,+}(H,\sigma,d_m)$. □

Sketch of the proof of Corollary 7.2 Case (i). This needs only minor modifications to the previous argument and will be omitted.
Case (ii). By Lemma 7.3, $\det \sigma_{\mathrm{hom}} = d_m^2$. To obtain the second statement, we modify the argument above after formula (7.33). The left hand side in (7.33) is convex but no longer strictly convex in the variable B. Therefore one has to apply to the left hand side of the inequality the same argument used for the right hand side, taking $d_m = \sqrt{\det \sigma_{\mathrm{hom}}}$. The algebra is identical and it is omitted. □

Proof of Lemma 7.1 Let us fix $\lambda \in R$ and $S \in \mathcal{M}^s_+$. We compute the gradient and the Hessian of the function $F \to p(F) = 2\mathrm{tr}(HF^T) - f(F,S,\lambda)$:

$$Dp(F) = 2(H - FS - \lambda \mathrm{AdjF}) \quad , \tag{7.35}$$

$$Hp(F) = -2\begin{pmatrix} S & \lambda J^T \\ \lambda J & S \end{pmatrix} \quad , \tag{7.36}$$

where the four by four Hessian matrix of p has been written in two by two block form. It is easy to see that $H(p)$ is positive definite if and only if $\det S > \lambda^2$ and positive semidefinite if and only if $\det S = \lambda^2$. This proves the first part. To prove (7.15), consider

$$p(F) = 2\mathrm{tr}(HF^T) - \mathrm{tr}(FSF^T) - 2\frac{\lambda}{\sqrt{\det S}} \det(FS^{\frac{1}{2}}) \quad .$$

Setting

$$A = FS^{\frac{1}{2}} \quad , \quad s = \sqrt{\det S} \tag{7.37}$$

and writing $\forall M \in \mathcal{M}$, $\mid M \mid^2 = \mathrm{tr}(MM^T)$, one has

$$p(F) = \tilde{p}(A) = 2\mathrm{tr}(HS^{-\frac{1}{2}}A^T) - \mid A \mid^2 -2\ \frac{\lambda}{s} \det A \ \ . \tag{7.38}$$

We split A and $HS^{-\frac{1}{2}}$ into their conformal and anticonformal parts and use the properties reviewed earlier in this section. It is convenient to set

$$B = HS^{-\frac{1}{2}} \ \ . \tag{7.39}$$

Using this formalism, (7.9), (7.10), (7.11) and (7.12), one has

$$p(F) = \tilde{p}(A) =$$

$$2\mathrm{tr}(B_+A_+^T) + 2\mathrm{tr}(B_-A_-^T) - \mid A_+ \mid^2 - \mid A_- \mid^2 -2\frac{\lambda}{s}(\det A_+ + \det A_-) =$$

$$2\mathrm{tr}(B_+A_+^T) + 2\mathrm{tr}(B_-A_-^T) - \mid A_+ \mid^2 - \mid A_- \mid^2 -2\frac{\lambda}{s}(\mid A_+ \mid^2 - \mid A_- \mid^2) \ \ ,$$

hence

$$\tilde{p}(A) = 2\mathrm{tr}(B_+A_+^T) - \mid A_+ \mid^2 \left(1 + \frac{\lambda}{s}\right) + 2\mathrm{tr}(B_-A_-^T) - \mid A_- \mid^2 \left(1 - \frac{\lambda}{s}\right) \ \ . \tag{7.40}$$

Now we consider several cases.
Case 1: $(\mid \lambda \mid < s)$. By (7.40), $\tilde{p}(A)$ is strictly convex and it is maximized at the unique stationary point A^{opt} which satisfies

$$A_+^{\mathrm{opt}} = \frac{s}{s+\lambda}B_+ \ \ , \ \ A_-^{\mathrm{opt}} = \frac{s}{s-\lambda}B_- \ \ . \tag{7.41}$$

The value of the maximum is displayed in (7.15), first line.
Case 2: $(\lambda = s)$. By (7.40), $\tilde{p}(A)$ is still convex but not strictly convex. Indeed

$$\tilde{p}(A) = 2\mathrm{tr}(B_+A_+^T) - 2 \mid A_+ \mid^2 + 2\mathrm{tr}(B_-A_-^T) \ \ . \tag{7.42}$$

From (7.42) one sees that necessary and sufficient condition for the supremum of $\tilde{p}$ to be finite is $\mathrm{tr}(B_-A_-^T) = 0$ for all A. This holds if and only if $B_- = 0$, which, by

(7.9) and (7.10) is equivalent to $B \in \mathcal{H}^+$. In view of (7.39), the latter is equivalent to $HS^{-1/2} \in \mathcal{H}^+$. The optimal A in this case is not unique, but each of them satisfies

$$A_+^{\mathrm{opt}} = \frac{1}{2} B_+ \quad . \tag{7.43}$$

The anticonformal part of A^{opt} is arbitrary. The value of $\tilde{p}$ at the optimal A's is instead uniquely determined and displayed in (7.15), second line.
Case 3: ($\lambda > s$). By assumption $\lambda = s(1+2\alpha^2)$ for some $\alpha \neq 0$. By (7.40),

$$\tilde{p}(A) = 2\mathrm{tr}(B_+ A_+^T) - 2 \mid A_+ \mid^2 (1+\alpha^2) + 2\mathrm{tr}(B_- A_-^T) + 2 \mid A_- \mid^2 \alpha^2 \quad . \tag{7.44}$$

Therefore (7.44) is unbounded in the anticonformal part, consequently $f^* = +\infty$.

The remaining cases, $\lambda = -s$ and $\lambda < -s$, are similar to Case 2 and Case 3 respectively and will be omitted. $\square$

Proof of Lemma 7.2 Let us first rephrase the hypothesis $H \in m(S, \lambda)$ in terms of the new variable B (see (7.39)). One has

$$\begin{array}{l} H \in m(S,\lambda) \Leftrightarrow (s-\lambda)^2 \mid B_+ \mid^2 - (s+\lambda)^2 \mid B_- \mid^2 > 0 \quad \text{if} \quad \lambda \in (0,s) \\ H \in m(S,s) \Leftrightarrow B_- = 0 \quad . \end{array} \tag{7.45}$$

Now we proceed to analyze two cases.
Case 1: ($0 \leq \lambda < s$). Recall, that the optimal A for the unconstrained problem (relative to Case 1 of Lemma 7.1), satisfies (7.41). It follows that it satisfies

$$\det A^{\mathrm{opt}} = \det A_+^{\mathrm{opt}} + \det A_-^{\mathrm{opt}} = \left(\frac{s}{s+\lambda}\right)^2 \mid B_+ \mid^2 - \left(\frac{s}{s-\lambda}\right)^2 \mid B_- \mid^2 \quad .$$

Therefore, by (7.45), $A^{\mathrm{opt}} \in \mathcal{M}_+$ if and only if $H \in m(S,\lambda)$. This shows that A^{opt} as defined in (7.41) is also optimal for the constrained problem because the constraint is automatically satisfied. Note that the hypothesis $H \in m(S,\lambda)$ is not only sufficient but also necessary to have $\det A^{\mathrm{opt}} > 0$.
Case 2: ($\lambda = s$). By (7.45), we have $B_+ = 0$. Hence by (7.40), we obtain

$$\tilde{p}(A) = 2\mathrm{tr}(B_+ A_+^T) - 2 \mid A_+ \mid^2 \quad . \tag{7.46}$$

The function in (7.46) ought to be maximized over the open set $\mathcal{M}_+$. If the supremum is achieved at an interior point, then, this point must be stationary and then one easily verifies the corresponding value of $\tilde{p}$ is given by (7.17), second line. If the supremum were only achieved at some $\tilde{A}$ belonging to the closure of $\mathcal{M}_+$, then $\det \tilde{A} = 0$ which can be written as

$$|\tilde{A}_+|^2 = |\tilde{A}_-|^2 \quad . \tag{7.47}$$

However, (7.46) does not depend on A_-. Therefore the constraint (7.47) can be always satisfied and therefore it is irrelevant and the supremum of $\tilde{p}$ is not (strictly) increased by it. □

Proof of Lemma 7.3 Our starting point is formula (1.6). We set $A \in \mathcal{M}_+$. Applying (1.17) to $\tilde{U}^A$ and recalling that $|Q| = 1$, one has

$$\int_Q \det D\tilde{U} = \det A \det \sigma_{\text{hom}} = \int_Q \det \sigma(y) \det DU^A(y) dy =$$

$$\int_{Q_m} d_m^2 \det DU^A(y) dy + \int_{Q \backslash Q_m} \det \sigma(y) \det DU^A(y) dy \quad .$$

Therefore applying (1.17) to U^A,

$$\det A (\det \sigma_{\text{hom}} - d_m^2) = \int_{Q \backslash Q_m} (\det \sigma(y) - d_m^2) \det DU^A(y) dy \quad .$$

By (3.5) and (1.13), the integrand of the right hand side is strictly positive almost everywhere. Therefore, the right hand side is nonnegative and hence $\det \sigma(y) - d_m^2 \geq 0$. Now there are two cases. If $|Q_m| = 1$, then obviously $\det \sigma_{\text{hom}} = d_m^2$. Otherwise, the measure of $Q \backslash Q_m$ is strictly positive, hence the right hand side in the above formula is strictly positive and therefore so is its left hand side. □

Proof of Lemma 7.4 . We will check that for any choice of $\sigma \in L^\infty_\sharp(R^2, \mathcal{M}^s_K)$, $\lambda \in [0, d_m]$ and $A \in m(\sigma_{\text{hom}}, \lambda)$, the mapping $\phi_{U^{B_\lambda}, \lambda} \in W(\sigma, \lambda, A)$. By (7.4) and (2.3), we have

$$D\phi_{U^{B_\lambda}, \lambda} = \lambda DU^{B_\lambda} + DJ\tilde{U}^{B_\lambda} = \lambda DU^{B_\lambda} + \text{Adj}(DU^{B_\lambda} \sigma)$$

and by (6.8),

$$(\det \sigma - \lambda^2)^2 \det DU^{B_\lambda} = \det D\phi_{U^{B_\lambda}, \lambda} (\det \sigma + \lambda^2) - \lambda \text{tr}(D\phi_{U^{B_\lambda}, \lambda} \sigma D\phi^T_{U^{B_\lambda}, \lambda}) \quad . \tag{7.48}$$

By (3.5), $\det B_\lambda > 0 \Rightarrow \det DU^{B_\lambda} > 0$ almost everywhere and, by (7.5) and (6.8),

$$\det B_\lambda > 0 \Leftrightarrow \det A (\det \sigma_{\text{hom}} + \lambda^2) - \lambda \text{tr}(A \sigma_{\text{hom}} A^T) > 0 \Leftrightarrow A \in m(\sigma_{\text{hom}}, \lambda) \quad .$$

Finally it easy to check that, by construction, $\int_Q D\phi_{U^{B_{\lambda},\lambda}} = A$. Using (7.48) it follows that $\phi_{U^{B_{\lambda},\lambda}} \in W(\sigma, \lambda, A)$. (Note that for $\lambda = d_m$, both sides of (7.48) vanish consistently with the definition of $W(\sigma, d_m, A)$ given in (1.10), (1.11)). This establishes part i). Part ii) is a calculation following the same lines and it is omitted. Finally (7.20) is an immediate consequence of part i). □

8 Threshold exponents and area formulas

8.1 Threshold exponents

We give a complete (and affirmative) solution to two conjectures due to G. W. Milton [50]. The conjectures have been discussed in detail in a paper by Leonetti and Nesi, [42]. They can be, roughly, stated as follows. If $\sigma \in L^\infty(\Omega, \mathcal{M}_K^s)$, under reasonable boundary conditions, any σ-harmonic *function* satisfies the property that $\mid \nabla u \mid$ and $\mid \nabla u \mid^{-1}$ belong to some *precise* family of L^p spaces. The original conjectures were posed in a rather general framework and were formulated in any dimensions. We focus here on dimension two. In [42], the authors were able to treat Dirichlet or Neumann boundary conditions relying on previous work by Alessandrini and Magnanini [2]. However, a more satisfactory statement should include periodic boundary conditions. Now we fill that gap.

Given a vector $\xi \in R^2$, $\mid \xi \mid = 1$, let u be a solution to the problem

$$\begin{cases} \operatorname{div}(\sigma(x)\nabla u) = 0 & , \quad x \in Q \\ u - \langle \xi, x \rangle \in W_\sharp^{1,2}(R^2, R) & . \end{cases} \tag{8.1}$$

Note that, according to the notation introduced in (2.15), $u = \langle \xi, U^I \rangle$ where U^I is the σ-harmonic mapping solving (1.5) when $A = I$. Set

$$p_K = \frac{2K}{K-1} \quad \text{and} \quad q_K = \frac{2}{K-1} \quad ,$$

these are in fact the *threshold exponents* introduced by Milton. We shall show that

$$\forall p < p_K \ , \quad \mid \nabla u \mid \in L^p(Q) \quad \text{and} \quad \forall q < q_K \ , \quad \mid \nabla u \mid^{-1} \in L^q(Q) \quad .$$

In fact more is true and the precise statement requires the notion of the weak-L^p spaces of Marcinkiewicz (see, for instance [39]). We recall that a measurable function

f belongs to $L^p_{\text{weak}}(Q)$, $1 < p < \infty$, if and only if there exists a constant $c > 0$ such that, for every measurable set $E \subset Q$, one has

$$\int_E | f | \leq c | E |^{1-\frac{1}{p}} \quad .$$

Theorem 8.1 For every $\xi \in R^2$, $| \xi |= 1$, we have

$$| \nabla u | \in L^{p_K}_{\text{weak}}(Q) \quad , \tag{8.2}$$

$$| \nabla u |^{-1} \in L^{q_K}_{\text{weak}}(Q) \quad . \tag{8.3}$$

Sketch of the proof. Formula (8.2) follows immediately from Theorem 1 in [42], which provides an optimal form of the local higher integrability property of first derivatives, as obtained by Bojarski [13], [14] for quasiregular mappings, and by Meyers, [49], for solutions of elliptic equations in divergence form in any space dimension. Formula (8.3) follows from Theorems 2 and 3 in [42] and the observation that the set of geometric critical point is empty in this case because of Proposition 2.2. □

8.2 A first area formula: geometric interpretation for functions of σ_{hom}.

This is a continuation of the analysis developed in [42]. Here we give the periodic version of that result.

Let $\sigma \in L^\infty_\sharp(R^2, \mathcal{M}^s_K)$. Set $\xi \in R^2$, $| \xi |= 1$. Let u be a solution to (8.1) and let $\tilde{u}$, f be defined according to (2.3), (2.15). As explained in Section 2, f is K-quasiregular. Therefore (see § 1.3 and Section 6) there exists $G \in L^\infty_\sharp(R^2, \mathcal{M}^s_K)$ such that $\det G = 1$ almost everywhere and such that $f \in W^{1,2}_{\sharp,A}(R^2, R^2)$ is a weak solution to

$$\operatorname{Div}(G(x)Df^T(x)) = 0 \quad \text{in} \quad R^2$$

where A is the two by two matrix which has $\int_Q \nabla u = \xi$ on the first column and $\int_Q \nabla \tilde{u} = J\sigma_{\text{hom}}\xi$ on the second one. The only not obvious statement is that $\int_Q \nabla \tilde{u} = J\sigma_{\text{hom}}\xi$ and this is a consequence of (2.3), of the definition of σ_{hom} and of an integration by parts. Clearly

$$\det A = \int_Q \det Df(x)dx = \int_Q \langle J\nabla u, \nabla \tilde{u} \rangle = \langle J\xi, J\sigma_{\text{hom}}\xi \rangle = \langle \xi, \sigma_{\text{hom}}\xi \rangle > 0$$

and therefore, by Theorem 2.1, f is univalent and hence quasiconformal. It follows that

$$\langle \sigma_{\text{hom}}\xi, \xi\rangle = \mid f(Q) \mid \tag{8.4}$$

and therefore the left hand side represents the *area* of $f(Q)$. This observation is similar to the one made in in the final part of [42], except that it now applies directly to σ_{hom}.

8.3 A second area formula: geometric interpretation for $\det \sigma_{\text{hom}}$ in the two-phase problem.

We explore the properties of σ-harmonic mapping U is some special case. We begin with a preliminary result which is of independent interest.

Proposition 8.1 Let $\sigma \in L^\infty(\Omega, \mathcal{M}_K^s)$ (or $\sigma \in L^\infty_\sharp(R^2, \mathcal{M}_K^s)$). Let U be a σ-harmonic mapping which is sense preserving and univalent ($\det A \neq 0$ and set $U = U^A$ the σ-harmonic mapping defined by (1.5)). Then the change of variable formula holds: for any measurable set $E \subset \Omega$ ($E \subset Q$) and for any function $f \in L^1(U(\Omega), R)$ ($f \in L^1(U(Q), R)$)

$$\int_E f(U(x)) \mid \det DU(x) \mid dx = \int_{U(E)} f(y)dy \quad . \tag{8.5}$$

In particular the area formula holds.

Sketch of the proof This is a corollary of the Radó-Reicheldorfer theorem, see for instance the book by Giaquinta, Modica e Souček [28], Theorem 2, p. 223. Indeed, our mapping $U \in W^{1,p}_{\text{loc}}$ for some $p > 2$. This follows by N. Meyers' theorem [49]. Therefore it satisfies the Lusin property (see [28] Theorem 3, p. 223) and therefore the (generalized) change of variable formula applies (see [28] (4), p. 219). It remains to check that the Banach indicatrix function is one almost everywhere in our case. This can be achieved as follows. First we note that U is differentiable almost everywhere and therefore approximately differentiable almost everywhere (see [28] Theorem 5, p. 200). Hence, the generalized Banach indicatrix function can be interpreted in the classical sense. Therefore it is identically one by the injectivity of the mapping U. $\Box$

Now we use Proposition 8.1 in a special but interesting case. Assume that $\sigma \in L^\infty_\sharp(R^2, \mathcal{M}_K^s)$ and that, in addition, $\det \sigma$ assumes only two distinct values say d_1^2 and d_2^2. For instance if σ is isotropic (i.e. proportional to the identity) at any point then we are back to the two-phase problem presented in the Introduction. If σ is not isotropic, the next simplest example is when each eigenvalue of σ takes only two values, usually called principal conductivities of the basic crystals. This is often referred to as the two-polycrystal problem because of its physical interpretation. The corresponding G-closure problem has been solved only in the context of unconstrained volume fraction, [25]. Substantial progress has been made more recently in the case when the volume fraction is prescribed. However a complete understanding is yet not available. Let us give this problem a geometric interpretation.

Our starting points are formulas (1.5) and (1.6). Arguing as in the proof of Lemma 7.3, we obtain

$$\det A \det \sigma_{\text{hom}} = d_1^2 \det A + (d_2^2 - d_1^2) \int_{Q_2} \det DU^A(x) dx$$

where Q_2 denotes the set where $\det \sigma = d_2^2$.

Therefore if $d_1 = d_2$, $\det \sigma_{\text{hom}} = d_1^2$, while, if $d_1 \neq d_2$ and $\det A > 0$

$$\frac{\det \sigma_{\text{hom}} - d_1^2}{d_2^2 - d_1^2} = \frac{\int_{Q_2} \det DU^A(x) dx}{\det A} = \frac{| U^A(Q_2) |}{\det A} = \frac{| U^A(Q_2) |}{| U^A(Q) |} \quad . \tag{8.6}$$

The final equality is a consequence of Proposition 8.1.

Hence, the minimization (maximization) of $\det \sigma_{\text{hom}}$ is *equivalent* to the problem of finding the "best" σ-harmonic mapping with respect to the following criterion: minimize (maximize) with respect to the microgeometry the (relative) area of $U^A(Q_2)$.

Unfortunately, the area distortion properties of generic σ-harmonic mappings are not very nice as suggested by the examples of Section 5. For this reason, at present the only bound which can be deduced directly by (8.6) is

$$\min(d_1^2, d_2^2) \leq \det \sigma_{\text{hom}} \leq \max(d_1^2, d_2^2) \quad .$$

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